Quadrupole Moment and Potential

by Dr. Colton, Physics 441 (last updated: 13 May 2024)

Introduction

Each of the terms in the multipole expansion, Griffiths Eq (3.95), can be separated into two pieces: one that describes the source (which does not depend on **r**), and one that describes how that source produces a potential (which does depend on **r**). For example, for the monopole term, $n = 0$, we have:

$$q = \int \rho(\mathbf{r}')d\tau' \tag{1}$$

$$V_{mono} = \frac{1}{4\pi\epsilon_0 r}\, q \tag{2}$$

Here ρ is the charge density function and $\mathbf{r}'$ represents the locations where the charges are (which varies over the integral). The net charge q can also be called the "monopole moment". All of the information about the charge distribution is contained in the calculation for (1). All of the information about the field point is contained in the calculation for (2).

Similarly, for the dipole term $n = 1$, we have:

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau' \tag{3}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0 r^2}\, \mathbf{p}\cdot\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0 r^2}\left(p_x\hat{r}_x + p_y\hat{r}_y + p_z\hat{r}_z\right) \tag{4}$$

Here **p** is the "dipole moment" with components p_x, p_y, and p_z. As usual, $\hat{\mathbf{r}}$ is a unit vector pointing towards the field point, and $\hat{r}_i$ for $i = x, y, z$ represents the Cartesian components of $\hat{\mathbf{r}}$. All of the information about the charge distribution is contained in the calculation for (3) and all of the information about the field point is contained in the calculation for (4).

Each successive term in the multipole expansion involves a larger and larger collection of numbers needed to specify the more-and-more complicated "moment" that describes the source, and each successive term falls off by an additional power of r.

Name	Symbol (i, j, etc. represent Cartesian coordinates)	Tensor rank	Number of components to specify the moment	r-dependence
monopole	q	0	1	$1/r$
dipole	$\mathbf{p}$ or p_i	1	3	$1/r^2$
quadrupole	$\overleftrightarrow{\mathbf{Q}}$ or Q_{ij}	2	9	$1/r^3$
octopole	O_{ijk}	3	27	$1/r^4$

These moments are called "tensors", which can be thought of as an extension of vectors. A "rank 0" tensor is a scalar, a "rank 1" tensor is a vector, a "rank 2" tensor can be represented by a matrix, and so forth.

Quadrupole moment and potential

Here are the two equations (source and field) for the quadrupole term.

$$Q_{ij} = \int \left(\frac{3}{2} r'_i r'_j - \frac{1}{2} r'^2 \delta_{ij}\right) \rho(\mathbf{r}') d\tau' \tag{5}$$

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \sum_{i,j} \hat{r}_i \hat{r}_j \, Q_{ij} \tag{6}$$

All of the information about the charge distribution is contained in the calculation for (5). All of the information about the field point is contained in the calculation for (6).

Q_{ij} is called the “quadrupole moment,” also the “quadrupole moment tensor.” It is a collection of nine numbers (i and j each can be x, y, z) which can be written as a 3×3 matrix, just like $\mathbf{p}$ can be written as a 3-element vector. Moreover, the formula in (5) is symmetric in reversing i and j, so there are actually only six independent matrix elements. In matrix form it can be written like this:

$$\mathbf{Q_{ij}} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix} \tag{7}$$

Once the quadrupole moment tensor is known, the potential can be determined via (6). Recall that $\hat{r}_i$ represents the three Cartesian components of the unit vector pointing to the field point, $\hat{\mathbf{r}}$.

The similarities between the quadrupole potential equation and the dipole potential equation can be made even more striking by rewriting the dot product in Eq (4) in terms of a summation:

$$V_{dip} = \frac{1}{4\pi\epsilon_0 r^2} \sum_{i} \hat{r}_i \, p_i \tag{8}$$

The derivation of (5) and (6) is given at the end of the handout.

Extensions

The next term of the multipole expansion will give rise to an “octopole” potential and an octopole moment O_{ijk} (O for octopole) as a rank 3 tensor which will contain 27 components in a 3×3×3 array of numbers. The potential will involve a 27 term summation. Recognizing the patterns, we can write them as the following:

$$O_{ijk} = \int \left(\frac{5}{2} r'_i r'_j r'_k - \frac{3}{2} [stuff]\right) \rho(\mathbf{r}') d\tau' \tag{9}$$

$$V_{oct}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^4} \sum_{i,j,k} \hat{r}_i \hat{r}_j \hat{r}_k \, O_{ijk} \tag{10}$$

For the sake of completeness the “stuff” in Eq (16) is the following: $r'^2\left(r'_i \delta_{jk} + r'_j \delta_{ik} + r'_k \delta_{ij}\right)$, but you don’t need to know that for this class.

Disclaimer

Equations (5) and (6) are given in Griffiths 5th edition problem 3.57 and 4th edition problem 3.52. By contrast, the 3rd edition (in problem 3.45) defines Q_{ij} without the factor of 1/2 in the two terms, choosing instead to include an extra 1/2 in the equation for V_{quad}. A check of the internet reveals some continued disagreement on this, but most sources actually seem to favor the 3rd edition version of the equations. However, I personally prefer the 5th and 4th edition equations because it makes the Legendre polynomials more apparent so that's what I've used in this handout.

Derivation of quadrupole moment and potential

Setting $n = 2$ in Griffiths equation (3.95) and recognizing that $P_2(\cos\alpha) = \frac{3}{2}\cos^2\alpha - \frac{1}{2}$, we obtain the quadrupole term:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' \tag{11}$$

Here, α is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

Working with the two terms inside the integral, we have:

<u>*First term*</u>

$$\frac{3}{2}r'^2\cos^2\alpha = \frac{3}{2}(r'\cos\alpha)^2 \tag{12}$$

Note that $r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$ which can be written as a summation, $\sum_i \hat{r}_i \cdot r'_i$, where like before $\hat{r}_i$ are the Cartesian components of $\hat{\mathbf{r}}$.

Because we have two different $r'\cos\alpha$ terms multiplied together, we'll have two summations. We can write the second one as being over j.

$$\frac{3}{2}r'^2\cos^2\alpha = \frac{3}{2}\sum_{i,j}\hat{r}_i r'_i \hat{r}_j r'_j \tag{13}$$

<u>*Second term*</u>

$$\frac{1}{2}r'^2 = \frac{1}{2}r'^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) \tag{14}$$

This is true since $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ is just equal to one. We can then write $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ as a summation, $\sum_i \hat{r}_i\cdot\hat{r}_i$, and turn it into a double summation using the Kronecker delta function δ_{ij}: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sum_{i,j}\hat{r}_i\hat{r}_j\delta_{ij}$. Thus we have

$$\frac{1}{2}r'^2 = \sum_{i,j}\frac{1}{2}r'^2\hat{r}_i\hat{r}_j\delta_{ij} \tag{15}$$

<u>Piecing together</u>

Interchanging the order of the integral and the summation, (11) turns into the following:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\sum_{i,j}\int\left(\frac{3}{2}\hat{r}_i r'_i \hat{r}_j r'_j - \frac{1}{2}r'^2\hat{r}_i\hat{r}_j\delta_{ij}\right)\rho(\mathbf{r}')d\tau' \tag{16}$$

Next I'll pull out $\hat{r}_i\hat{r}_j$ from each terms and put in front of the integral because the integral is over the primed coordinates:

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \sum_{i,j} \hat{r}_i \hat{r}_j \left\{ \int \left(\frac{3}{2} r_i' r_j' - \frac{1}{2} r'^2 \delta_{ij} \right) \rho(\mathbf{r}') d\tau' \right\} \tag{17}$$

The stuff in the curly braces no longer has any **r** dependence! That was exactly our goal—we have separated out the source information. Equation (11) can now be written as two separate equations, namely equations (5) and (6).