Lecture 32: Fri, 21 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. Huygen's principle uses

a. paraxial light
b. ray methods
c. wavelets

Q2. Babinet's principle relies on

a. linear superposition
b. scalar approximation
c. small angle approximation
d. upset students

Q3. $\nabla^2 \vec{E}(\vec{r}) + k^2 \vec{E}(\vec{r}) = 0$ is the _____ equation

a. Fraunhofer
b. Fresnel
c. Helmholtz
d. scalar
e. wave

Ch. 10: Diffraction

Back to wave eqn: $\nabla^2 \vec{E} = \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2}$

rect: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

spherical? $\vec{E} = \vec{E}(r,\theta,\phi)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \vec{E}}{\partial r}\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \vec{E}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \vec{E}}{\partial \phi^2}}_{=0}$$

simplest case: $E = E(r)$ only $\rightarrow$

If not vector, $f = \frac{A}{r}\cos(kr - \omega t)$ works

If vector that must satisfy Maxwell

$\vec{E} = \frac{A}{r}\cos(kr - \omega t)\,\hat{\phi}$ doesn't work

} HW problem

But it comes close, especially for large r, so we'll use it

complex: $E = E_0 \frac{e^{ikr}}{r}$

focus on scalar magnitude

$\rightarrow$ scalar vs vector problems discussed in Append. 10.A

really we're looking at

$$E = r_0 E_0 \left(1 - \frac{i}{kr}\right)\frac{e^{ikr}}{r}\cos\theta$$

which for large r gives

$$\vec{E} = -\hat{\phi}\, r_0 E_0 \frac{e^{ikr}}{r}\sin\theta$$

extra $\sin\theta$!

we'll handle by saying

travelling in z-dir

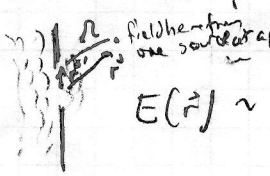
$\rightarrow$ then $\vec{E} \perp$ to z

and always $\theta \simeq \frac{\pi}{2}$

Where are we going?

Huygens: light as propagation of infinite number of spherical wavelets

Fig 10.1



$$E(\vec{r}) \sim \int_{aperture} E(\text{at aperture})\,\frac{e^{ik\Omega}}{\Omega}\,da'$$

$\vec{\Omega}' = \vec{r} - \vec{r}'$

$\Omega = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$

"guess"

called Fresnel-Kirchhoff if $-i/\lambda$ factor in front

book: $\Omega = R$

Can we derive this from wave equation?

$$\nabla^2 E - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \frac{\partial}{\partial t} \to -i\omega$$

$$\frac{\partial^2}{\partial t^2} \to -\omega^2$$

$$\frac{\omega^2}{v^2} = k^2 \qquad \frac{\omega}{k} = v$$

$$\nabla^2 \vec{E}(\vec{r})\cancel{e^{-i\omega t}} + k^2 \vec{E}(\vec{r})\cancel{e^{-i\omega t}} = 0$$

"Helmholtz eqn" or "vector Helmholtz eqn"

<u>big approx</u> ignore vectors

$$\nabla^2 E(\vec{r}) + k^2 E(\vec{r}) = 0$$

again, see Appendix A

previous spherical waves $E = E_0 \frac{e^{ikr}}{r}$

would be $\simeq E_0 \frac{e^{ikz}}{r}$ = (more generally) $f \cdot g$

$f \to f(r) \simeq \tilde{E}$, $g \to e^{ikz}$

approx: travelling in z-direction

$\vec{k}\cdot\vec{r} = kz$

$\tilde{E}$: amplitude function (when $z=0$)

Need to take some derivatives to plug trial soln of $\tilde{E}e^{ikz}$ into Helmholtz eqn

$$\frac{\partial}{\partial z}(fg) = fg' + f'g$$

$$\frac{\partial^2}{\partial z^2}(fg) = fg'' + f'g' + f'g' + gf''$$

$$= fg'' + gf'' + 2f'g'$$

$$= f(-k^2)g + f''g + 2f'(ik)g$$

$g' = \frac{\partial}{\partial z} e^{ikz} = (ik)g$

$g'' = -k^2 g$

Helmholtz eqn becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\tilde{E}e^{ikz} + k^2\tilde{E}e^{ikz} = 0$$

$$e^{ikz}\frac{\partial^2}{\partial x^2}\tilde{E} + e^{ikz}\frac{\partial^2}{\partial y^2}\tilde{E} + \left(\cancel{-k^2\tilde{E}e^{ikz}} + e^{ikz}\frac{\partial^2}{\partial z^2}\tilde{E} + 2ike^{ikz}\frac{\partial}{\partial z}\tilde{E}\right) + \cancel{k^2\tilde{E}e^{ikz}} = 0$$

cancel all e^{ikz}'s

$$\frac{\partial^2\tilde{E}}{\partial x^2} + \frac{\partial^2\tilde{E}}{\partial y^2} + \underbrace{\frac{\partial^2}{\partial z^2}\tilde{E}}_{\simeq 0} + 2ik\frac{\partial\tilde{E}}{\partial z} = 0$$

"paraxial wave approximation"

amplitude of E doesn't change much in z-direction (like plane wave)

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The or maybe "a" Soln to this eqn: (HW problem)

$$\tilde{E} = -\frac{i}{\lambda z}\iint_{\text{aperture}} \underbrace{\tilde{E}(x',y')}_{\text{field strength across aperture (if varying)}} e^{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]}\,dx'dy'$$

← "aperture function"

stop here in Fri

start here on Mon

do integral, then $E = \underbrace{\tilde{E}\, e^{ikz}}$

gives answer just like our original guess with $\rho = \left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}$

$$= z\left[1+\left(\frac{x-x'}{z}\right)^2+\left(\frac{y-y'}{z}\right)^2\right]^{1/2}$$

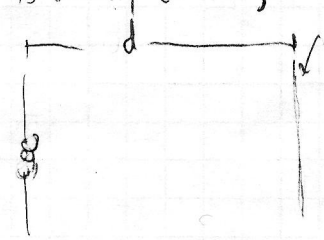
$$\simeq z\left(1+\frac{1}{2}\frac{(x-x')^2}{z^2}+\frac{1}{2}\frac{(y-y')^2}{z^2}+\dots\right)$$

but with proper $-\frac{i}{\lambda}$ out in front

just • keep zeroth order for denom. ($\frac{1}{z}$)

• keep first order for exponent (need that since small changes in x → big changes in e^{ikx})

To actually do integral



use $(x-x')^2 = x^2 - 2xx' + x'^2$

can be taken out of integral

(same for y)

$$\boxed{E = -\frac{i}{\lambda d}\, e^{ikd}\, e^{\frac{ik}{2d}(x^2+y^2)} \iint_{\text{aperture}} \tilde{E}(x',y')\, e^{i\frac{k}{2d}(x'^2+y'^2)}\, e^{-i\frac{k}{d}(xx'+yy')}\,dx'dy'}$$

"The Fresnel Approximation"

Babinet: if aperture =

(square lets light through circle blocks it)

then $\iint_{\square}(\ )\,da' - \iint_{\circ}(\ )\,da'$

Not so interesting?

diffraction from hair $\iint_{\text{hair}} = \iint_{\text{all space}} - \iint_{\text{slit}}$ → hair diffraction very similar to slit!

from ● (blocked circle) → diffraction like ○ (open circle) = (we'll see later)

bright spot in middle! "Poisson's spot"