Lecture 20: Fri, 22 Feb 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. Group velocity is given by:

a. $\frac{k}{\omega}$

b. $\frac{\Delta k}{\Delta \omega}$

c. $\frac{\omega}{k}$

d. $\frac{\Delta \omega}{\Delta k}$ $\frac{d\omega}{dk}$

Q2. True/False: Group velocity tracks the speed of the "ripples" resulting from the superposition of multiple waves.

Q3. The intensity of the field, time-averaged over rapid oscillations, may be expressed approximately as which of the following?

a. $I(\vec{r},t) \cong \frac{n\varepsilon_0 c}{2}\left|E(\vec{r},t)\right|^2$

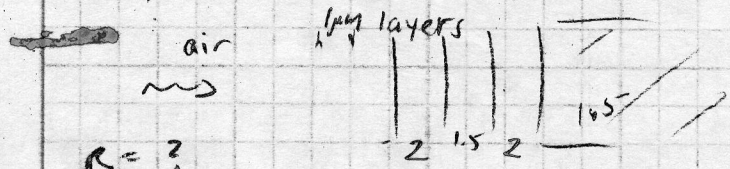
b. $I(\vec{r},t) \cong n\varepsilon_0 c\left|E(\vec{r},t)\right|^2$

Example: normal incidence $\theta = 0°$

$\beta_j = k_j l_j \cos\theta_j$

$= \frac{2\pi}{\lambda}(1)$ in microns

div. by n



$R = ?$

$$A = \frac{1}{2 n_0 \cos\theta_0}\begin{pmatrix} n_0\cos\theta_0 & \\ n_0 & -\cos\theta_0 \end{pmatrix}\left[\prod_{j=1}^{N}\begin{pmatrix} \cos\beta_j & -i\frac{\sin\beta_j\cos\theta_j}{n_j} \\ -i n_j \frac{\sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix}\right]\begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

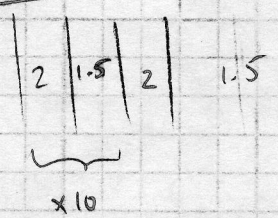
$$= \frac{1}{2(1)(1)}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \cos 2\pi/\lambda & -i\frac{\sin 2\pi/\lambda}{2} \\ -i2\sin 2\pi/\lambda & \cos\frac{2\pi}{\lambda} \end{pmatrix}\begin{pmatrix} \cos\frac{2\pi}{\lambda} & -i\sin 2\pi/\lambda/1.5 \\ -i1.5\sin\frac{2\pi}{\lambda} & \cos\frac{2\pi}{\lambda} \end{pmatrix}$$

$$\times\begin{pmatrix} \cos\frac{2\pi}{\lambda} & -i\sin 2\pi/\lambda/2 \\ -i2\sin\frac{4\pi}{\lambda} & \cos 2\pi/\lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1.5 & 0 \end{pmatrix}$$

Matrix Multi.

done with Mathematica, $R = |r|^2$ plotted

<u>Different structure</u>



Again, done w/ Mathem.

Note: Use, <u>MatrixPower</u> command

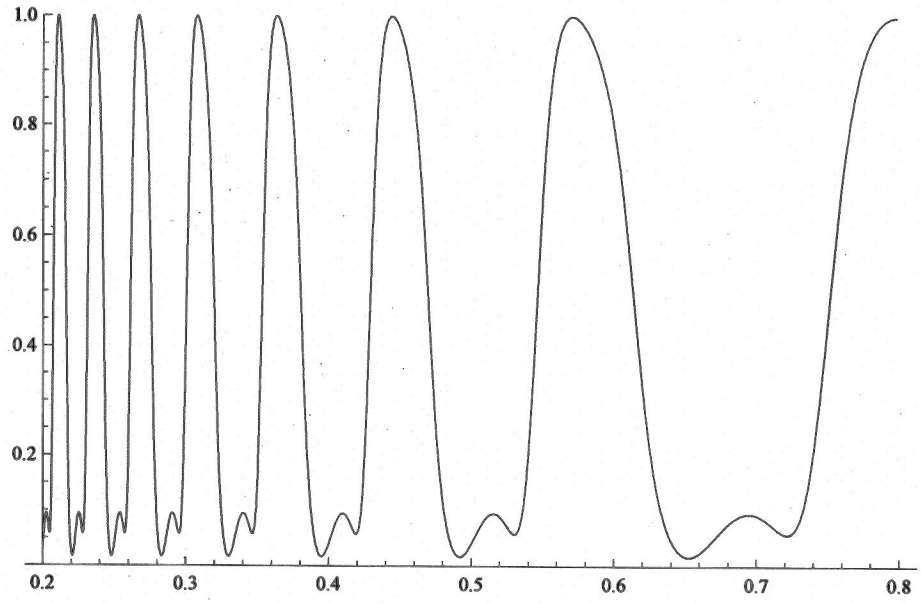
Or use PtW's tips[19] - section 6.9 for raising matrices to a power using Sylvester's theorem

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \ldots$$

Forgot n_j in β_j's

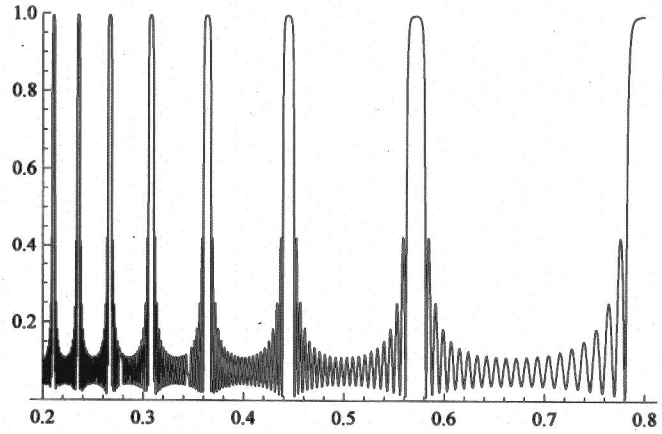
```
In[1]:= a[l_] =
        1/2 {{1, 1}, {1, -1}} . {{Cos[2 Pi / l], -I / 2 Sin[2 Pi / l]}, {-2 I Sin[2 Pi / l], Cos[2 Pi / l]}} .
          {{Cos[2 Pi / l], -I / 1.5 Sin[2 Pi / l]}, {-1.5 Sin[2 Pi / l], Cos[2 Pi / l]}} .
          {{Cos[2 Pi / l], -I / 2 Sin[2 Pi / l]}, {-2 I Sin[2 Pi / l], Cos[2 Pi / l]}} . {{1, 0}, {1.5, 0}};

In[2]:= r[l_] = a[l][[2, 1]] / a[l][[1, 1]];

In[3]:= Plot[Abs[r[l]]^2, {l, .2, .8}, PlotRange → {0, 1}]
```

Out[3]=



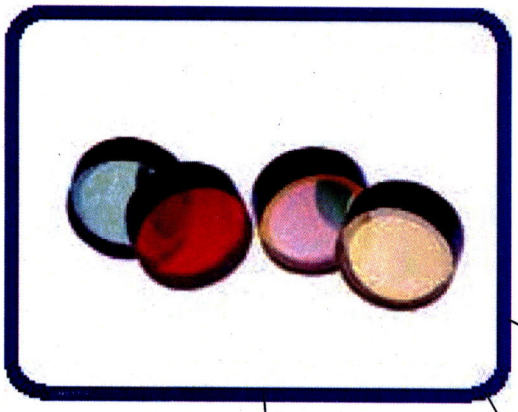

```
In[24]:= a2[l_] = 1/2 {{1, 1}, {1, -1}} .
           MatrixPower[({{Cos[2 Pi / l], -I / 2 Sin[2 Pi / l]}, {-2 I Sin[2 Pi / l], Cos[2 Pi / l]}} .
              {{Cos[2 Pi / l], -I / 1.5 Sin[2 Pi / l]}, {-1.5 I Sin[2 Pi / l], Cos[2 Pi / l]}}), 10] .
           {{Cos[2 Pi / l], -I / 2 Sin[2 Pi / l]}, {-2 I Sin[2 Pi / l], Cos[2 Pi / l]}} . {{1, 0}, {1.5, 0}};

In[25]:= r2[l_] = a2[l][[2, 1]] / a2[l][[1, 1]];

In[26]:= R2[l_] = Abs[r2[l]]^2;

In[27]:= Plot[Abs[r2[l]]^2, {l, .2, .8}, PlotRange → {0, 1}]
```

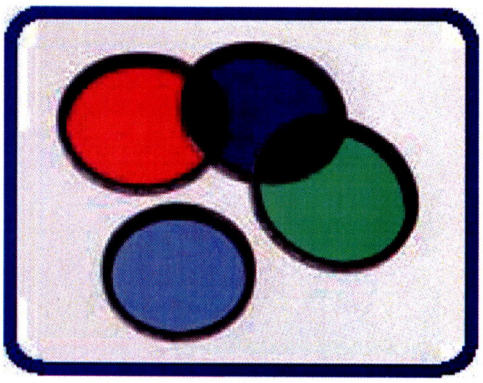
Out[27]=

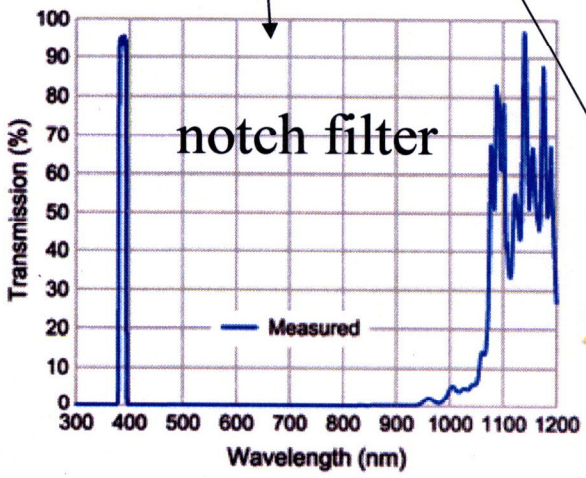


Wavelength selective filters, mirrors

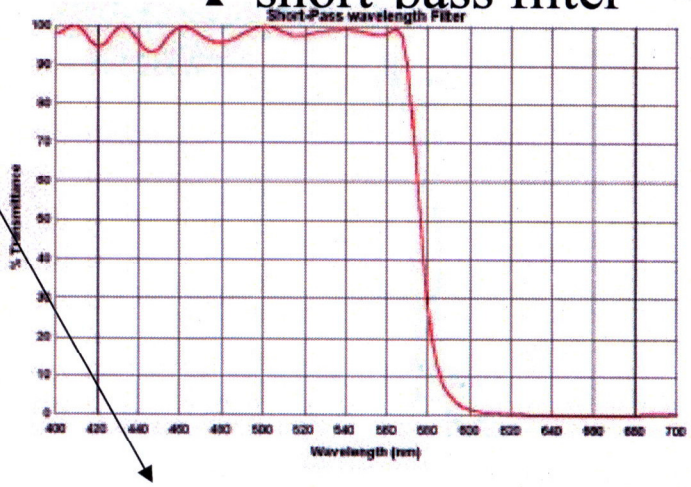
Interference filters vs absorption filters



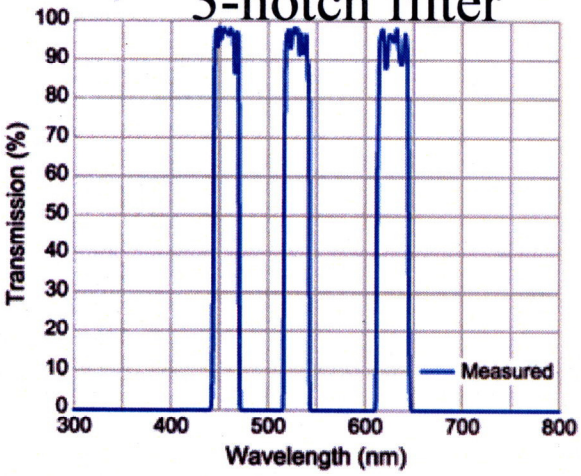


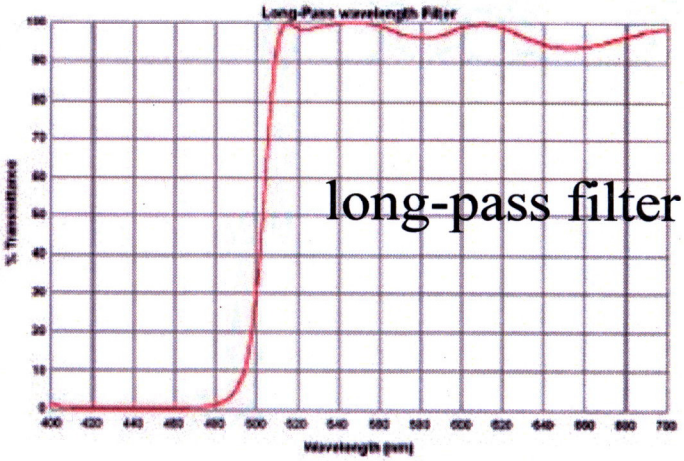


short-pass filter



3-notch filter





long-pass filter

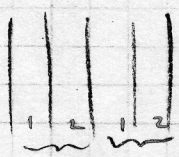
Stack Mis

Best example: an infinite stack designed to give high reflectivity for one λ

choose thicknesses so $\beta_j = \frac{\pi}{2}$ for each layer

$$\left(\beta = k\ell\cos\theta = \frac{2\pi}{\lambda}\ell\right)$$

$$\prod M = \prod \begin{pmatrix} \cancel{\cos\beta_j} & -i\sin\beta_j/n_j \\ -in_j\frac{\sin\theta_j}{\cos\theta_j} & \cancel{\cos\beta_j} \end{pmatrix}$$



(in pairs)

$$= \prod_{\text{every other}} \underbrace{\begin{pmatrix} 0 & -i\cos\theta_1/n_1 \\ -i\frac{n_1}{\cos\theta_1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\cos\theta_2/n_2 \\ -i\frac{n_2}{\cos\theta_2} & 0 \end{pmatrix}}_{\begin{pmatrix} -\frac{n_2}{n_1}\frac{\cos\theta_2}{\cos\theta_1} & 0 \\ 0 & -\frac{n_1}{n_2}\frac{\cos\theta_1}{\cos\theta_2} \end{pmatrix}}$$

$$= \begin{pmatrix} \left(-\frac{n_2}{n_1}\frac{\cos\theta_2}{\cos\theta_1}\right)^q & \\ & \left(-\frac{n_1}{n_2}\frac{\cos\theta_1}{\cos\theta_2}\right)^q \end{pmatrix} \qquad q = \#\text{ bilayers}$$

$\theta = 0$

$n_0 = 1$

$$A = \frac{1}{2n_0\cos\theta_0}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} \left(-\frac{n_2}{n_1}\right)^q & \\ & \left(-\frac{n_1}{n_2}\right)^q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n_{N+1} & 0 \end{pmatrix}}_{\begin{pmatrix} \left(-\frac{n_2}{n_1}\right)^q & 0 \\ n_{N+1}\left(-\frac{n_1}{n_2}\right)^q & 0 \end{pmatrix}}$$

$$A = \frac{1}{2}\begin{pmatrix} \left(-\frac{n_2}{n_1}\right)^q + n_{N+1}\left(-\frac{n_1}{n_2}\right)^q & 0 \\ \left(-\frac{n_2}{n_1}\right)^q - n_{N+1}\left(-\frac{n_1}{n_2}\right)^q & 0 \end{pmatrix}$$

$$r_p = \frac{a_{21}}{a_{11}} = \frac{\left(-\frac{n_2}{n_1}\right)^q - n_{N+1}\left(-\frac{n_1}{n_2}\right)^2}{\left(-\frac{n_2}{n_1}\right)^2 + n_{N+1}\left(-\frac{n_1}{n_2}\right)^q}$$

$n_1 > n_2$, $q \to \infty$

$$r_1 = \frac{-n_{N+1}\left(-\frac{n_1}{n_2}\right)^q}{+n_{N+1}\left(-\frac{n_1}{n_2}\right)^q} = -1 \longrightarrow R = 1 \text{ 100\% reflected}$$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^q$ = Sylvester's Thm may be helpful.

stick $\cos\theta$ back in $\rightarrow r_1$ still -1

TOPS. 35502

Chapter 7 Superposition of quasi parallel plane waves

Recall extra credit problem how can $n \neq 1$?

→ to send info you must modulate waves

→ Fourier : modulate = superposition

$$\vec{E} = \sum_j \vec{E}_j e^{i(\vec{k}_j \cdot \vec{r} - \omega_j t)}$$

$\vec{k}_j$'s = "mostly parallel"

Poynting vector $S = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ real $\vec{E} + \vec{B}$ $\quad B = \frac{k}{\omega} \times E$

$$\frac{1}{\mu_0}\left(\sum_j \overset{\text{real}}{\vec{E}_j} e^{i(\vec{k}_j \cdot r - \omega_j t)}\right) \times \left(\sum_m \frac{\vec{k}_m}{\omega_m} \times \overset{\text{real}}{\vec{E}_m} e^{i(\vec{k}_m \cdot \vec{r} - \omega t)}\right)$$

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$= \frac{1}{\mu_0} \sum_{jm} \frac{\vec{k}_m}{\omega_m} \left[Re\left(E_j e^{i(k_j r - \omega_j t)}\right) \cdot Re\left(E_m e^{i(k_m r - \omega t)}\right)\right]$$

$$- Re\left(\vec{E}_m e^{i(k_m r - \omega_m t)}\right)\left[\underbrace{Re\left(E_j e^{i(k_j \cdot r - \omega_j t)}\right) \cdot \frac{\vec{k}_m}{\omega_m}}_{\text{mostly parallel, this } \simeq 0}\right]$$

all E's ⊥ to all k's

$\frac{\omega}{k} = \frac{c}{n}$

$\frac{1}{\mu_0} = \epsilon_0 c^2$

$$= \epsilon_0 c \sum_{jm} n_m Re\left(E_j e^{i(k_j r - \omega t)}\right) \cdot Re\left(E_m e^{i(k_m r - \omega t)}\right)$$

Approx : n_m mostly constant

$Re = \frac{A + A^*}{2}$

$$\left(\frac{E_j e^{i(k_j r - \omega t)} + E_j^* e^{-i(\ )}}{2}\right) \cdot \left(\frac{E_m e^{i(\ )} + E_m^* e^{-i(\ )}}{2}\right)$$

FOIL $F = E_j \cdot E_m e^{i((k_j + k_m) \cdot r - (\omega_j + \omega_m) t)}$

L = similar

$\underbrace{\qquad}_{\text{oscillates @ } \omega_j + \omega_m}$

time average = 0

$I = \langle S \rangle$

$\left.\begin{matrix} 0 \\ I \end{matrix}\right\}$ is zero when $j = m$ simplify them

$$I = \frac{\epsilon_0 c n}{4} \sum_j \left(Re(E_j \cdot E_j^*) + Re(E_j^* \cdot E_j) \right)$$

$Re\, A + Re\, B = Re(A + B)$

$\sum_j E_j = E$

$$= \boxed{\frac{n \epsilon_0 c}{2} Re\left(\underbrace{E \cdot E^*}_{\text{always real anyway}}\right)}$$

obvious? Magnitude of not I retains slowly varying time but not fast varying

Two interfering plane waves $\quad E_1 = E_0 e^{i(k_1 \cdot r - \omega t)}$

$\boxed{\text{video: beats}}$ $\quad E_2 = E_0 e^{i(k_2 r - \omega t)}$ $\quad$ same amp

$$I = \frac{n \epsilon_0 c}{2} E_{tot} E_{tot}^*$$

$$= \frac{n \epsilon_0 c}{2} E_0 \cdot E_0^* \underbrace{\left[e^{i(k_1 \cdot r - \omega t)} + e^{i(k_2 \cdot r - \omega t)}\right]\left[e^{-i(k_1 \cdot r - \omega t)} + e^{-i(k_2 r - \omega t)}\right]}_{2 + \underbrace{e^{i((k_2 - k_1)\cdot r - (\omega_2 - \omega_1)t)} + e^{-i(\quad)}}_{2\cos(\quad)}}$$

$$= n \epsilon_0 c \, \vec{E}_0 \cdot \vec{E}_0^* \left(1 + \cos(\Delta \vec{k} - \Delta\omega t)\right)$$

travels at $v_g = \underline{\underline{\frac{\Delta\omega}{\Delta k}}}$

in general $v_g = \underline{\underline{\frac{d\omega}{dk}}}$ is good expression if $\Delta\omega$ (bandwidth?) not too large

$\frac{\omega}{k} = \frac{c}{n}$ $\quad$ $\frac{d\omega}{dk}$ related to freq dependence of n

$k = \frac{1}{c}\omega n(\omega)$

$\frac{d\omega}{dk} = \frac{1}{dk/d\omega}$

Lecture 21: Mon, 25 Feb 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The Fourier transform method we will use most involves:

a. cos
b. sin
c. complex exponentials

Q2. Your book primarily uses which of the following notation to represent the Fourier transform of $f(t)$?

a. $f(\omega)$
b. $\mathcal{F}\{f(t)\}$
c. $FT\{f(t)\}$ ← Colton
d. $g(t)$
e. $g(\omega)$

Q3. The magnitude squared of the Fourier transform is called the _____ spectrum (aside from some multiplicative constants).

a. amplitude
b. Fourier
c. frequency
d. power

(cont from last lecture)

Example:

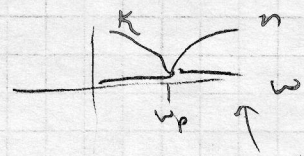
metal/plasma $\chi = \dfrac{\omega_p^2}{\cancel{\omega_0^2} - i\omega\gamma - \omega^2}$

$= -\dfrac{\omega_p^2}{\omega^2}$ if no damping

review

$n = \sqrt{1+\chi}$

$\tilde{n} = \sqrt{1 - \dfrac{\omega_p^2}{\omega^2}}$



$n<1$ when $\omega > \omega_p$

so $v_{phase} = \dfrac{\omega}{k} = \dfrac{c}{n}$ is larger than c

let $\omega_p = 10^8/s$

We have 2 plane waves superimposed,

$\omega_1 = 5 \cdot 10^8/s$

$\omega_2 = 6 \cdot 10^8/s$

What is velocity of group?

$v_g = \dfrac{\Delta\omega}{\Delta k}$

Method 1 $\dfrac{\omega}{k} = \dfrac{c}{n} \rightarrow \omega = \dfrac{c}{n} k$

$k = \omega \cdot \dfrac{n}{c}$

$k_1 = \dfrac{5 \cdot 10^8}{3 \cdot 10^8} \sqrt{1 - \left(\dfrac{10^8}{5 \cdot 10^8}\right)^2} = 1.6330$

$k_2 = \dfrac{6 \cdot 10^8}{3 \cdot 10^8} \sqrt{1 - \left(\dfrac{1}{6}\right)^2} = 1.9720$

$\dfrac{\Delta\omega}{\Delta k} = \dfrac{1 \cdot 10^8}{1.972 - 1.633} = 2.950 \cdot 10^8 m/s$

Method 2: $\dfrac{\Delta\omega}{\Delta k} \approx \dfrac{d\omega}{dk}$ since ω_1 close to ω_2

$k = \dfrac{1}{c} \omega n = \dfrac{1}{c} \omega \sqrt{1 - \dfrac{\omega_p^2}{\omega^2}}$

$\dfrac{dk}{d\omega} =$ (product rule)

$\dfrac{1}{c}\left[\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} + \omega \cdot \dfrac{1}{2\sqrt{\ }}(-2\omega_p^2 \omega^{-3})\right]$

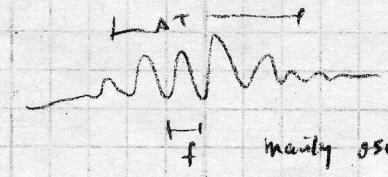
$= \dfrac{1}{c}\left(n + \dfrac{\omega_p^2}{n\omega^2}\right)$

$\dfrac{d\omega}{dk} = \dfrac{c}{n + \frac{\omega_p^2}{n\omega^2}} \times \dfrac{n}{n} = \dfrac{nc}{n^2 + \frac{\omega_p^2}{\omega^2}} = \underline{\underline{nc}}$

$\dfrac{d\omega}{dk} = \left[\sqrt{1 - \left(\dfrac{10^8}{5 \cdot 10^8}\right)^2}\right] c = 2.950 \cdot 10^8 m/s$

TOPS 35502

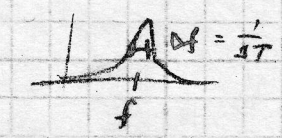
Fourier



mainly oscillating at f

but contains other frequency components

approx



do not like to label this E but we're stuck with it

FT (E(t))

$$\vec{E}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t}\, dt$$ FT complex

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$ inverse FT

real, but sometimes represented ~ complex form (like eg plane waves)

real & imag parts represent how many cosine + sine components make up $\vec{E}$, respectively

$\Downarrow$

if this is complex then $E(\omega)$ loses its double peak structure + only has positive peaks (with 2× amplitude)

Show viewgraphs

These additional frequencies are real. Two evidences

1. Excitation of forbidden energy levels

short pulse can induce transitions that long pulses cannot

2. Temporal dispersion → different frequency components travel at different velocities.
when pulse exits material, it is not as short!

TOPS 35502

Fourier Transforms of Gaussian Wave Packets

In[1]:= `f[t_, tau_, w0_] = E^(-t^2/tau^2) Cos[w0 t]`

Out[1]= $e^{-\frac{t^2}{\text{tau}^2}} \text{Cos}[t\, w0]$

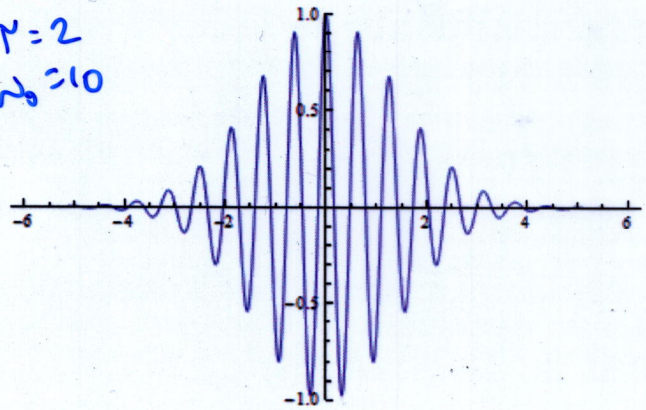
In[2]:= $g[w_, tau_, w0_] = \frac{\text{tau}}{2\,\text{Sqrt}[2]} \left(e^{-\frac{1}{4}\text{tau}^2 (w-w0)^2} + e^{-\frac{1}{4}\text{tau}^2 (w+w0)^2}\right)$

Out[2]= $\frac{\left(e^{-\frac{1}{4}\text{tau}^2 (w-w0)^2} + e^{-\frac{1}{4}\text{tau}^2 (w+w0)^2}\right) \text{tau}}{2\sqrt{2}}$

In[3]:= `Plot[f[t, 2, 10], {t, -6, 6}, PlotRange → {-1, 1}]`

$\tau = 2$
$w_0 = 10$

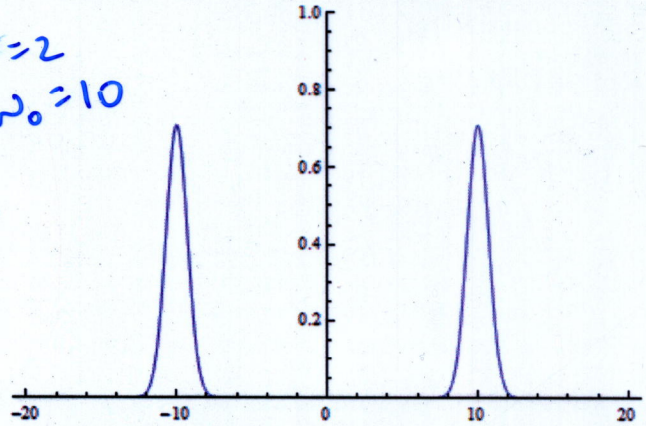
Out[3]=



In[4]:= `Plot[g[w, 2, 10], {w, -20, 20}, PlotRange → {0, 1}]`

$\tau = 2$
$w_0 = 10$

Out[4]=

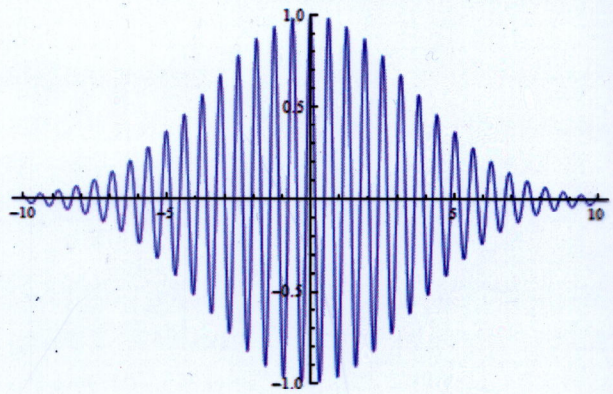


—what if the original function had $e^{iw_0 t}$ instead of $\cos w_0 t$?

$\tau = 5, w_0 = 10$

In[5]:= `Plot[f[t, 5, 10], {t, -10, 10}, PlotRange → {-1, 1}]`

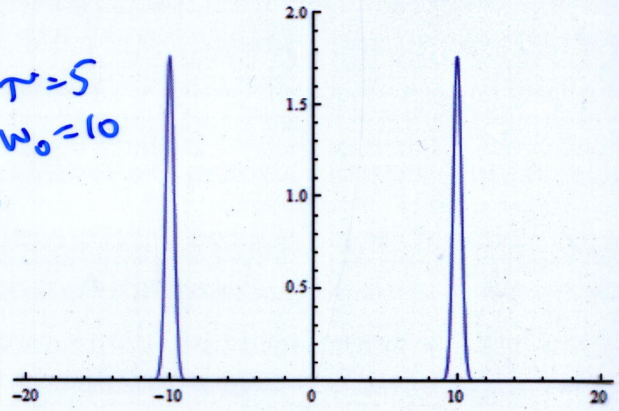
Out[5]=



In[6]:= `Plot[g[w, 5, 10], {w, -20, 20}, PlotRange → {0, 2}]`

$\tau = 5$
$w_0 = 10$

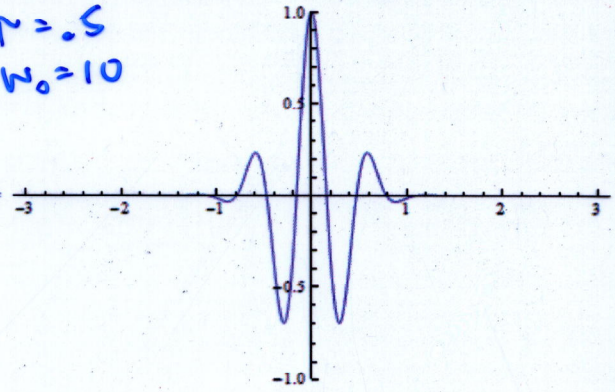
Out[6]=



In[7]:= `Plot[f[t, .5, 10], {t, -3, 3}, PlotRange → {-1, 1}]`

$\tau = .5$
$w_0 = 10$

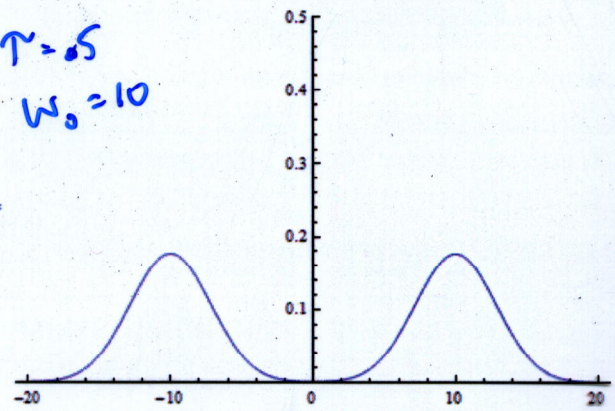
Out[7]=



In[8]:= `Plot[g[w, 0.5, 10], {w, -20, 20}, PlotRange → {0, .5}]`

$\tau = .5$
$w_0 = 10$

Out[8]=



Parseval's Theorem $\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \underbrace{\int_{-\infty}^{\infty} |f(t)|^2 dt}$

basically the intensity, if f = electric field

$$S = \frac{1}{\mu_0} E \times B = \frac{1}{\mu_0} \frac{|E|^2}{c/n} \rightarrow \langle S \rangle = \frac{1}{2} \cdot$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{2}\frac{n}{\mu_0 c} |E(\omega)|^2 d\omega}_{\text{how much of that energy is associated with frequency } \omega} = \underbrace{\int_{-\infty}^{\infty} \frac{1}{2}\frac{n}{\mu_0 c} |E(t)|^2 dt}_{\frac{\text{the total energy}}{\text{area}}}$$

$I = \frac{P}{\text{area}}$

$= \frac{\frac{\text{energy}}{dt}}{\text{area}}$

$I(\omega) = \frac{1}{2}\frac{n}{\mu_0 c} |E(\omega)|^2$

notation → not the Fourier transform of the intensity vs time

then $\int_{-\infty}^{\infty} I(\omega) d\omega = \int_{-\infty}^{\infty} I(t) dt$

Plot of $\frac{n}{2\mu_0 c} |E(\omega)|^2$ vs ω

called "Power spectrum"

or for simplicity just $|E(\omega)|^2$ often plotted

Interpretation in terms of photons

TOPS. 35502