Introduction to Pauli Geometric Algebra

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Pauli Algebra

Geometric Algebras

Geometric algebras (also called Clifford algebras) are used to endow physical spaces with a useful algebraic structure. By analyzing the physical system within this context, we can find alternate interpretations of the underlying physics. These can simplify computational problems in addition to giving us much more compact and clean notation. In most cases the results can be expressed in a coordinate free way, introducing an appropriate coordinate system only when necessary.

An algebra is constructed by providing a linear space with an additional binary operation called the *product* of the algebra. Although this product is usually non-commutative, it is distributive with respect to the linear space addition, and it is assumed to be associative for our case. With these rules, the idea of matrix multiplication immediately comes to mind. It will actually be useful to keep this picture in mind, as long as we conceive the algebra's sum and product in an abstract way. An additional essential condition for the algebra is closure with respect to its product, *i.e.* the complete algebra must contain all possible products of its elements. Again, in our matrix multiplication reference, this would imply choosing square matrices of fixed size: a product of two $n \times n$ matrices is again an $n \times n$ matrix – in addition to the fact that a linear combination of matrices is again a matrix.

Geometric algebras constitute a very specific instance of associative algebras. The constraint imposed on their structure allows us to give concrete geometric interpretations to both the elements and the operations within the algebra. In a sense this is the natural extension of the Cartesian conception of identifying geometry and algebra, and unifying them into a single structure. The geometric building blocks are points, vectors, oriented surfaces, and oriented volumes. The algebraic part relates them in a constructive way and allows us to unify both concepts and equations from different fields of physics.

Next section introduces the main concepts of geometric algebras, as well as the notation that we will use. In particular we define the algebra associated to the 3-dimensional space, known as the Pauli algebra.

Geometric Product of Vectors

We first want to build up the geometric algebra starting from a physical vector space $\mathbb{V}$ regarded as an underlying part of the larger linear space of the algebra $\mathcal{G}$. We also need to admit a metric defined by the usual dot product of two vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{V}$:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \tag{1}$$

in terms of their magnitudes a and b, and the angle θ between them, thus forcing us to include the real numbers $\mathbb{R}$ as a linear subspace. This in turn provides the Clifford algebra with a *graded* structure where the scalars have grade zero and the physical vector space $\mathbb{V}$ has grade one. We next find the elements of grade two, called *bivectors*, by forming the "wedge" $\wedge$ (or exterior or Grassmann) product of two vectors encoding the plane defined by them. Given that two collinear vectors do not form a plane,

$$\mathbf{a} \wedge \mathbf{a} = 0. \tag{2}$$

Applying this to the sum of two vectors $\mathbf{a}+\mathbf{b}$ and using distributivity, we obtain the general antisymmetry property of the wedge product:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \tag{3}$$

Furthermore, the area of the the parallelogram formed by the two vectors is $Area = ab\,|\sin\theta|$, so the bivector represents an *oriented* surface (see Fig.1).

Clifford's stroke of genius converted Schwartz's inequality (for both the dot and wedge products) to an equality by defining the geometric product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{V}$ as:

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{4}$$

and then building up the geometric algebra by demanding closure. This geometric product combines zero grade scalars with second grade bivectors with a resulting magnitude[1]:

$$\|\mathbf{a}\,\mathbf{b}\| = ab \tag{5}$$

Thus geometric algebras constrain the symmetric part of the product of two vectors to correspond to their dot product, as is evident in Eq.(4).

In order to close the algebra, we need to keep incorporating new multivectors of higher grade. The wedge product of two vectors gives a bivector, bivectors can now be wedged with another vector to produce a trivector, and so on. These additional structures represent oriented volumes (and hypervolumes) as illustrated in Fig.1 and will eventually close the algebra in a finite number of steps due to the antisymmetry property Eq.(3). With this geometric interpretation, the wedge product turns out to be associative. Because of their geometric liaison, multivectors are also very useful for interpreting the behavior of many familiar physical quantities as we will show below.

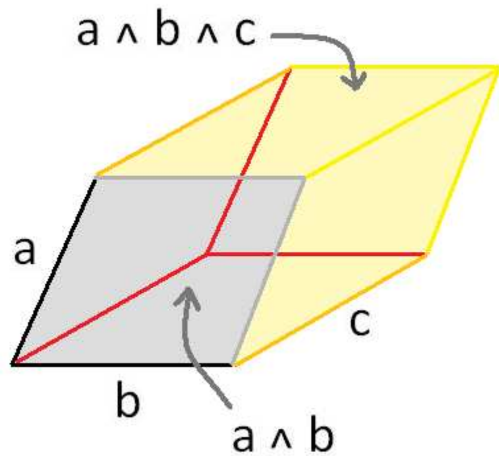

Figure 1: The wedge product of two vectors **a**, **b** is an oriented area, while the wedge of **a** , **b** , **c** is an oriented volume.

Geometric Algebra in 3-d

Our main example is the Clifford algebra $\mathcal{G}_3$ associated to the three dimensional Euclidean space $\mathbb{V} = \mathbb{R}^3$. This Pauli algebra[2] is eight dimensional and consists of linear combinations of multivectors of grades zero to three, *i.e.* scalars, 3-d vectors, 3-d bivectors, and 1-d trivectors (also called pseudoscalars). The basis element of the real line $\mathbb{R}$ is the number 1. For $\mathbb{R}^3$ we choose an orthonormal basis, namely three orthogonal unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The advantage of using orthonormality is that we can rely on the more versatile Clifford product in order to construct the subsequent multivector bases. For instance, the bivector basis element $\mathbf{e}_1 \wedge \mathbf{e}_2$ turns out to be the same as the product $\mathbf{e}_1\mathbf{e}_2$ in this case. The eight basis elements of the Pauli algebra are classified by grades in the following Table:

Table I

Grade	Basis	“Complex form”	Space
0	1	1	$\mathbb{R}$
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbb{R}^3$
2	$\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$	$i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3$	$i\mathbb{R}^3$
3	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	i	$i\mathbb{R}$

Notice that the three resulting unit bivectors square to -1 instead of 1. This follows from their antisymmetry as illustrated with the first one:

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\wedge\mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1 \tag{6}$$

and hence $(\mathbf{e_1 e_2})^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -(\mathbf{e}_1\mathbf{e}_1)(\mathbf{e}_2\mathbf{e}_2) = -1$.

This is also true for the pseudoscalar (unit trivector):

$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1 \tag{7}$$

It can also be appreciated from Fig.1 that this trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ represents the (right handed) oriented unit cube. At the same time we can use this property, Eq.(7), together with the fact that it commutes with all the basis elements, to identify it with the imaginary unit i (in an algebraic sense). This is the actual meaning of the third column in Table I.

In summary, we can take i as the basis element of the trivectors, and $\{i\mathbf{e}_\mathrm{k}\}$ as the basis of the bivectors for the Pauli algebra. In other words, every vector $\mathbf{a} \in \mathbb{R}^3$ has a corresponding dual bivector $\mathcal{A} = i\mathbf{a}$, and *vice versa*, $\mathbf{a} = -i\mathcal{A}$. This duality associates a vector $\mathbf{a}$ normal to the surface defined by the bivector $\mathcal{A}$ in a natural way.

For the present example $\mathcal{G}_3$ the duality property is expressed as:

$$\mathbf{a} \wedge \mathbf{b} = i\mathbf{a} \times \mathbf{b} \tag{8}$$

in terms of the unit pseudoscalar i. Two main features distinguish Grassmann's wedge product from Gibbs's cross product:

a) the $\wedge$ is well defined for any number of dimensions (as well as for pseudo-Euclidean spaces), and

b) the $\wedge$ is associative while the $\times$ fulfills Jacobi's identity.

For the particular case of the Pauli algebra we can thus rewrite the defining Eq.(4) as:

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \tag{9}$$

in terms of the usual dot and cross products between two 3-dimensional vectors.

Example: The Inertia Tensor

Angular Velocity and Angular Momentum Vectors

The classical mechanics formula for the angular momentum vector in terms of the mass m, the position vector $\mathbf{r}$, and the velocity $\mathbf{v}$ is $\mathbf{L} = m\,\mathbf{r} \times \mathbf{v}$, with corresponding dual bivector $\mathcal{L} = i\mathbf{L}$. For any particle rotating with angular velocity $\boldsymbol{\omega}$ the tangential velocity is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Consider a rigid body rotating about some axis (see Fig.2). Each particle will have the same angular velocity $\boldsymbol{\omega}$. Using Eq.(8) the total angular momentum of the rigid body can be written in terms of the Clifford product in Eq.(9) by summing over all the particles $k = 1 : N$,

$$\begin{aligned}
\mathbf{L} &= \sum_k^N m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) \\
&= -i \sum_k^N m_k \mathbf{r}_k \wedge (\boldsymbol{\omega} \times \mathbf{r}_k) \\
&= -i \sum_k^N m_k \mathbf{r}_k \, (\boldsymbol{\omega} \times \mathbf{r}_k) \\
&= \sum_k^N m_k \mathbf{r}_k \, (\mathbf{r}_k \wedge \boldsymbol{\omega}) \\
&= \int \mathbf{r}(\mathbf{r} \wedge \boldsymbol{\omega}) dm \qquad (10)
\end{aligned}$$

where the last step is an abstraction from a finite number of point particles to a mass distribution over the rigid body.

The inertia tensor plays the role of the mass (tensor) for rotational motion: the angular momentum vector $\mathbf{L}$ is obtained as the (scalar) product of the inertia tensor $\mathcal{I}$ with the angular velocity vector $\boldsymbol{\omega}$. We can describe $\mathcal{I}$ is a "one-slot" machine sending vectors to vectors, so its matrix representation has two indices. In other words, it is a linear mapping $\mathcal{I} : \mathbb{R}_3 \rightarrow \mathbb{R}_3$, and hence $\mathbf{L} = \mathcal{I}(\boldsymbol{\omega})$.

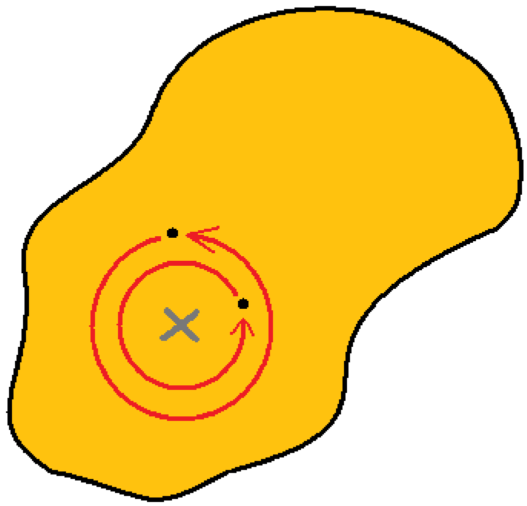

Figure 2: All points in a rigid body rotate about the rotation axis (here, indicated by an x) at the same angular velocity, $\boldsymbol{\omega}$.

Moment of Inertia for an Axisymmetric Body

Eq.(10) defines the inertia tensor as a linear function *i.e.* given any vector $\mathbf{A}$ the image vector $\mathbf{B}$ is given by:

$$\mathbf{B} = \mathcal{I}(\mathbf{A}) = \int \mathbf{r}\,(\mathbf{r} \wedge \mathbf{A}) dm \tag{11}$$

and the matrix elements I_{kl} with respect to the given orthonormal basis $\{\mathbf{e}_k\}$ can be extracted as the projection:

$$I_{kl} = \mathbf{e}_k \cdot \mathcal{I}(\mathbf{e}_l). \tag{12}$$

This 3x3 matrix $\{I_{kl}\}$ is symmetric and includes the moments and products of inertia with respect to the original basis.

Let us now look at a concrete simple example. Using Eq.(10), it is straightforward to find the inertia tensor for a rotating rod and write it in a coordinate-free way. Consider a thin rod of length a extending from $-a/2$ to $a/2$ and rotating about an arbitrary axis passing through its center (see Fig.3).

Choosing s as the integration variable, $dm = m\,ds/a$, and $\mathbf{r} = s\hat{\mathbf{n}}$ in terms of the unit vector $\hat{\mathbf{n}}$ along the rod:

$$\begin{aligned}\mathcal{I}(\boldsymbol{\omega}) &= \int_{-a/2}^{a/2} s\hat{\mathbf{n}}\,(s\hat{\mathbf{n}} \wedge \boldsymbol{\omega})\,\frac{m\,ds}{a} \\ &= \frac{ma^2}{12}\hat{\mathbf{n}}\,(\hat{\mathbf{n}} \wedge \boldsymbol{\omega}) \end{aligned} \tag{13}$$

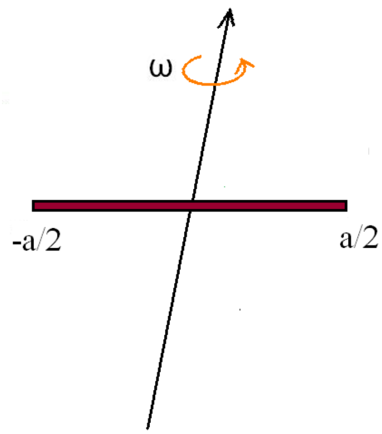

Figure 3: A thin rod of length a rotating about its center at angular velocity $\boldsymbol{\omega}$.

This result can be rewritten in terms of geometric products only, using $\hat{\mathbf{n}} \wedge \boldsymbol{\omega} = (\hat{\mathbf{n}}\boldsymbol{\omega} - \boldsymbol{\omega}\hat{\mathbf{n}})/2$ from Eq.(4). This yields a more symmetric form:

$$\mathcal{I}(\mathbf{A}) = \frac{ma^2}{24}(\mathbf{A} - \hat{\mathbf{n}}\mathbf{A}\hat{\mathbf{n}}). \tag{14}$$

The second term $\mathbf{A}' =$ -$\hat{\mathbf{n}}\mathbf{A}\hat{\mathbf{n}}$ has a simple geometric interpretation: $\mathbf{A}'$ corresponds to the vector $\mathbf{A}$ reflected with respect to the plane $i\hat{\mathbf{n}}$.

Axially Symmetric Case

Let us next consider the more general case of an axially symmetric body rotated about an arbitrary axis $\boldsymbol{\omega}$. Given that the inertia tensor is symmetric, it can be diagonalized with corresponding orthogonal eigenvectors. The eigenvalues $\{I_1, I_2, I_3\}$ are real numbers and represent the principal moments of inertia. Define $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ as the respective unit vectors along the principal axes of the rigid body, and assume that $\mathbf{f}_3$ is the symmetry axis, so that the two moments of inertia associated to the plane $i\mathbf{f}_3$ are equal, *i.e.* $I_1 = I_2$.

In the body-fixed basis (see Fig.4), the inertia tensor can be written in terms of the components of $\boldsymbol{\omega}$ as:

$$\begin{aligned} \mathcal{I}(\boldsymbol{\omega}) &= I_1\omega_1\mathbf{f}_1 + I_2\omega_2\mathbf{f}_2 + I_3\omega_3\mathbf{f}_3 \\ &= I_1(\omega_1\mathbf{f}_1 + \omega_2\mathbf{f}_2) + I_3\omega_3\mathbf{f}_3 \\ &= I_1\boldsymbol{\omega} + (I_3 - I_1)\omega_3\mathbf{f}_3. \end{aligned} \tag{15}$$

The last term contains the component of $\boldsymbol{\omega}$ along the symmetry axis and can be rewritten in terms of the geometric product:

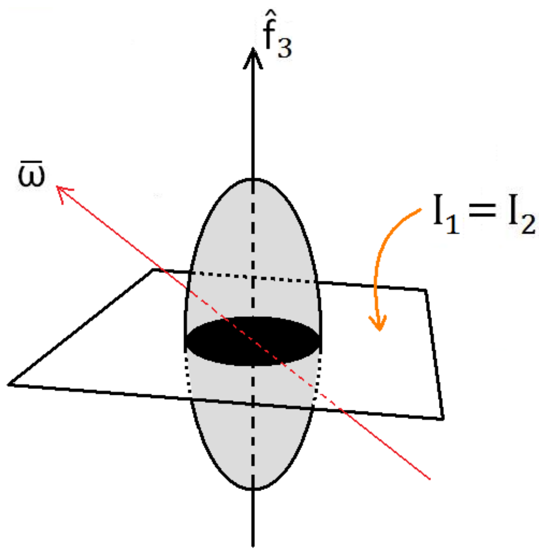

Figure 4: An axisymmetric rigid body rotated about the arbitrary axis $\boldsymbol{\omega}$. Due to the symmetry of the system $I_1 = I_2$.

$$\begin{aligned}\omega_3 &= \boldsymbol{\omega} \cdot \mathbf{f}_3 \\ &= \frac{1}{2}(\mathbf{f}_3\boldsymbol{\omega} + \boldsymbol{\omega}\mathbf{f}_3). \end{aligned} \tag{16}$$

Substituting back in Eq.(15) we obtain the desired form for the inertia tensor for the case of an axisymmetric body[**?**]:

$$\mathcal{I}(\mathbf{A}) = \frac{1}{2}(I_1 + I_3)\mathbf{A} + \frac{1}{2}(I_3 - I_1)\mathbf{f}_3\mathbf{A}\mathbf{f}_3 \tag{17}$$

expressed as a linear transformation from vectors to vectors.

Written in this form the inertia tensor for any axisymmetric rigid body appears as a simple generalization of the much simpler case of a rotating rod. In what follows we will deal with the non-trivial case of bivector valued mappings of bivectors and will be able to find very similar algebraic expressions to the ones above, but with a different geometric interpretation.

Biforms in Physics: Mapping Bivectors to Bivectors

When we considered the inertia tensor above we used vectors to define the rotation axes. In order to generalize rotations to higher dimensions we need to define them with respect to a plane defined by a bivector. For the 3-d

Pauli algebra, we can still use Table I and let the vector $\boldsymbol{\omega}$ be replaced by the corresponding bivector

$$\Omega = i\boldsymbol{\omega} \tag{18}$$

which defines the plane perpendicular to the $\boldsymbol{\omega}$ axis. This can indeed be interpreted as an imaginary vector in a pure algebraic sense.

On the other hand, we also know that the angular momentum $\mathbf{L}$ behaves as a vector with respect to rotations but not with respect to inversions or reflections. We saw in section that angular momentum can also be correctly described as a bivector. We thus define the bivector

$$\mathcal{L} = \mathbf{r} \wedge \mathbf{p} = i\mathbf{r} \times \mathbf{p} = i\mathbf{L}. \tag{19}$$

For instance, for a particle moving in a central potential, the bivector $\mathcal{L}$ is conserved and hence defines the fixed plane of the orbit.

For the case of a rigid body we have to sum over all the particles (or integrate over the mass distribution) as in Eq. (10) :

$$\begin{aligned} \mathcal{L} &= i\int \mathbf{r}(\mathbf{r} \wedge \boldsymbol{\omega})dm \\ &= i\int \frac{1}{2}(r^2\boldsymbol{\omega} - \mathbf{r}\boldsymbol{\omega}\mathbf{r})dm \\ &= \int \frac{1}{2}(r^2\Omega - \mathbf{r}\Omega\mathbf{r})dm \end{aligned} \tag{20}$$

The inertia tensor now becomes a *biform*, *i.e.* a bivector valued linear transformation of bivectors. Thus the inertia tensor $\mathcal{I}$ is reinterpreted as mapping the plane defined by a bivector $\mathcal{B}$ to a new plane $\mathcal{C}$:

$$\mathcal{C} = \mathcal{I}(\mathcal{B}) = \int \frac{1}{2}(r^2\mathcal{B} - \mathbf{r}\mathcal{B}\mathbf{r})dm \tag{21}$$

This in turn leads directly to the analogue of Eq. (17):

$$\mathcal{L} = \mathcal{I}(\Omega) = \frac{I_1 + I_3}{2}\Omega + \frac{I_3 - I_1}{2}\mathbf{f}_3\Omega\mathbf{f}_3 \tag{22}$$

as a mapping from bivectors to bivectors for the axisymmetric case.

We would like to emphasize that, although this last equation looks almost identical to Eq. (17) they are conceptually different. While the latter refers to a second rank tensor mapping vectors to vectors, Eq. (22) is a biform, *i.e.* a bivector valued transformation of a bivector.

Bibliography

[1] This property also allows (non-vanishing) vectors to be invertible with respect to the Clifford product.

[2] The well known Pauli matrices form a representation of the basis elements of $\mathbb{R}^3$ in terms of traceless Hermitian 2×2 matrices.