Uncertainty, Errors, and Noise in Experimental Measurements

> "... as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks ... it is the latter category that tend to be the difficult ones."
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> Donald Rumsfeld, responding to a question at a U.S. Department of Defense (DoD) news briefing, February 12, 2002. From Wikipedia article There are known knowns, accessed 2 February 2018.

In the text, Sections 6.8.1 to 6.8.5 (`Signal-to-Noise Ratio`, `Optimizing the Signal-to-Noise Ratio`, `The Lock-In Amplifier and Gated Integrator or Boxcar`, `Signal Averaging`, `Waveform Recovery`) all provide additional and complementary information to the discussion here. If you are interested in counting experiments (radiation or low-level photon counting) you may also want to look at Section 6.8.6 (`Coincidence and Time-Correlation Techniques`).

Consult the document `Statistics Review for Experimental Measurements` found on Learning Suite in `Content` ⇒ `LabVIEW Basics Course` ⇒ `Measurements, Uncertainties, and Noise` or on the course web page.

1.0 Introduction

Experimental measurements will inherently be somewhat uncertain. One of the challenges of experimental work is to minimize the uncertainty in our measurements. Several sources of uncertainty are discussed below as well as some ways to reduce the effect of those sources.

Another challenge is to properly report the uncertainty in measurements as correctly as possible. There are three ways uncertainty or estimated error may be reported. The value of the gravitational constant, G, will be used to illustrate them.

1. The value of the gravitational constant, G, has been through a progression of values:
   - Henry Cavendish: $6.754 \times 10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$.
   - 1973 CODATA-recommended value: $6.6720(41) \times 10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$
   - 1986 CODATA-recommended value: $6.67259(85) \times 10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$
   - 1998 CODATA-recommended value: $6.673(10) \times 10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$

- 2002 CODATA-recommended value: $6.6742(10) \times 10^{-11}\,\mathrm{m^3/(kg \cdot s^2)}$
- 2006 CODATA-recommended value: $6.67428(67) \times 10^{-11}\,\mathrm{m^3/(kg \cdot s^2)}$
- 2010 CODATA-recommended value: $6.67384(80) \times 10^{-11}\,\mathrm{m^3/(kg \cdot s^2)}$
- 2014 CODATA-recommended value: $6.67408(31) \times 10^{-11}\,\mathrm{m^3/(kg \cdot s^2)}$

2. The value of the gravitational constant, G, is given as $6.67408(31) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ (the 2014 CODATA recommended value). This is an expression of the officially accepted value of this constant. In this case the "31" in parentheses means that the uncertainty in the last two digits is 31 ($6.67408 \pm 0.00031 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$).
3. The value of G could be given as $(6.6741 \pm 0.0003) \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$. This form is traditionally used where the listed uncertainty is indicating an interval corresponding to a high level of confidence (usually a statement that about 68% of a random set of measurements of the value will fall within the given range).

   Remember that the value of the uncertainty in a measurement or result of a calculation is also somewhat uncertain. It is common practice to give the uncertainty with two digits of precision if the first digit is a "1" or a "2" and one digit of precision otherwise.
4. In some cases, it is adequate to indicate the uncertainty by the number of significant digits given in the value. The value of G would be given as $6.6741 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$. With the stated uncertainty in the value, it may be appropriate to write it as $6.674 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$. In these cases the least significant digit given is uncertain. That is, the uncertainty is some fraction of the least significant digit given. This form is usually inadequate for work where the values given are the crux of the work.

For further information on uncertainties, see "Evaluation of measurement data – Guide to the expression of uncertainty in measurement", Joint Committee for Guides in Metrology 100:2008. This document can be found at `http://www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf`. This is a *very* detailed document describing determining and reporting uncertainty in measurements. NIST (the National Institute of Standards and Technology) has a shortened version of this document at `https://www.nist.gov/pml/nist-technical-note-1297`.

2.0 Some sources of error and uncertainty in measurements

To reduce the uncertainty in your measurements, it is useful to understand some of the common sources of error and uncertainty in those measurements. There are two significant sources that are present in almost every measurement – instrument precision and accuracy, and electronic noise. These will be discussed below, but this is not an exhaustive list of sources of uncertainty.

2.1 Limitations in measurement precision

Even if you have a perfectly accurate measurement device (voltmeter, ohmmeter, etc.) there are limits to the precision with which you can make a measurement. For instance, if you have what is known as a $3\frac{1}{2}$-digit voltmeter where the first digit can be either a 1 or a 0 followed by three digits, your precision is limited by the number of digits available. If you measure a voltage of 1.250 V using this meter, the actual value could fall anywhere within the range 1.250 ± 0.0005 V simply due to the number of available digits.

In the document Statistics Review for Experimental Measurements, it is shown that the uncertainty in the above value can be more precisely given under the assumption that any value within the given range is equally probable. In that case, the uncertainty is given by $\sigma = c/\sqrt{12}$ where c is the interval between successive values of the least significant digit (0.001 in this case). This results in a value with uncertainty of 1.250 ± 0.0003 V.

You can always buy a more precise instrument such as a Keysight model 3458A $8\frac{1}{2}$-digit Digital Multimeter (starting at $9750), a Keithley 2002 Series $8\frac{1}{2}$-digit multimeter (about $6800), or a National Instruments PXIe-4081 $7\frac{1}{2}$-digit multimeter (about $3690.00). You can also use techniques such as dithering to get a little more precision. But there will always be limits to the precision with which you can make a measurement. In any case, the uncertainty in your measurement will be close to one-half of the last significant digit in the value.

This uncertainty will be present in any digital measuring device. There will be a similar uncertainty if you are reading the value from an analog device such as a pressure gauge or a d'Arsonval voltmeter. The uncertainty will then involve your estimate of the precision with which you can read the device.

2.2 Limitations in measurement accuracy

There may be an added uncertainty in your measured value due to the calibration of your instrument. Instruments often include in their specifications the required interval for calibration to maintain the expected accuracy of the device. In critical measurements, regularly scheduled calibration of instruments is a requirement.

The specification sheet for a measuring device will often have information on the absolute accuracy of the device and an expectation of the drift after calibration. This can become critical to your results if high accuracy measurements are expected.

Remember that a measurement may be very precise (*i.e.*, successive measurements are very close together) but may not be very accurate (*i.e.*, very close to the correct value) if the instrument hasn't been recently calibrated.

2.3 Uncertainty due to electrical noise

The presence of electrical noise is a common source of uncertainty. This noise may be due to external influences (*e.g.*, radio interference, pickup of the 60-Hz AC supply voltage) or circuit design problems. This can be reduced by proper electrical design (look up "Ground Loop" for an important example) and by shielding techniques (using shielded cables and metal cabinets for electrostatic and electromagnetic shielding). If it is a circuit design problem, you will have to fix the circuit.

2.3.1 Johnson-Nyquist noise

There are also several sources that are inherent to electronic circuits. One common source is known as Johnson-Nyquist noise or just Johnson noise. This noise results from thermal motions of the charge carriers in a conductor and is independent of applied voltages or currents. It is also independent of frequency (true up to a few THz), so it is considered "white noise." It can show up as a noise voltage signal on any resistor or capacitor. It is often given in terms of a power spectral density or mean square voltage variance per Hz of circuit bandwidth:

$$\overline{v_n^2} \;=\; 4\,k_B\,T\,R$$

with SI units of V^2/Hz, where k_B is Boltzmann's constant ($1.38 \times 10^{-23}\,\mathrm{J/K}$), T is the temperature of the resistor in K, and R is the resistance in Ω.

If you have a circuit with a bandwidth Δf that is something other than 1 Hz you can find the root mean square (RMS) noise voltage across a resistor by

$$v_n \;=\; \sqrt{4\,k_B\,T\,R\,\Delta f}\,.$$

The thermal noise in an RC filter circuit has a particularly simple form:

$$v_n \;=\; \sqrt{\frac{k_B\,T}{C}}\,.$$

The resistance in the filter isn't in the equation because it adds noise to the signal but simultaneously decreases the bandwidth due to the filtering effect.

There are three ways to reduce this noise: reduce the temperature, reduce the resistance, or reduce the circuit bandwidth (changing k_B isn't really an option).

2.3.2 Shot noise

In low-level signals, shot noise can be a problem. It is caused by the discrete nature of signal carriers (such as electrons in a circuit or photons in an optical experiment). For example, the arrival of individual photons or the conduction of individual electrons follow Poisson statistics as will be discussed below. In these cases, the statistical nature of the signal means that the standard deviation in a signal consisting of N photons or electrons is $\sqrt{N}$.

Often we aren't counting individual electrons but if we are trying to measure a small current shot noise can become a significant problem. In general, if a current I is flowing through a conductor the RMS fluctuations in that current are given by

$$\sigma_i \quad = \quad \sqrt{2\,q\,I\,\Delta f}$$

where q is the charge on an electron and Δf is the detection circuit bandwidth in Hz.

In an average circuit the way to reduce the importance of shot noise is to increase the current (or arrival rate of photons) so that the noise is a smaller fraction of the signal or to reduce the circuit bandwidth.

2.3.3 1/f noise

The third common noise is merely referred to as "1/f noise" or "pink noise." This refers to the typical spectrum of the noise – the power in the noise is roughly proportional to $1/f$, so it is often only important at low frequencies when it exceeds the magnitude of the Johnson-Nyquist noise. This is also sometimes referred to as "flicker noise."

The source of this noise is not definite. It shows up in meteorological data, E-M radiation from some astronomical bodies, almost all electronic circuits, heart beat rhythms, statistics of DNA sequences, financial data, most natural images, and almost all musical melodies. There is no known lower frequency bound in pink noise – it seems to be everywhere. It is best avoided by operating at a sufficiently high frequency (*e.g.*, using a high-pass filter to remove low-frequency components from the signal).

A rough example of 1/f noise is shown in Figure 1. The noise spectrum shown in the figure falls off slower than 1/f but illustrates the way the noise behaves with frequency.

3.0 A summary of relevant statistics

Estimating the uncertainty in a particular measurement often requires an understanding of the statistical nature of the measurement. The nature of a given measurement

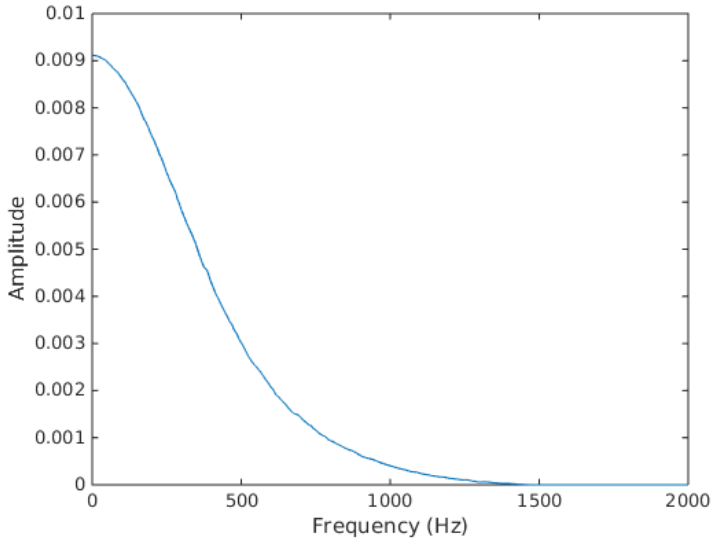

Figure 1: The trace from an HP 8590L Spectrum Analyzer with no input signal. The bandwidth of the spectrum analyzer was set to 300 Hz, the minimum available. The trace is distorted somewhat by the large bandwidth relative to the frequency width of the noise signal. At very low frequency the amplitude levels off because the integration time is too short to represent the signal accurately. By 1500 Hz the amplitude is approximately equal to the noise floor of the spectrum analyzer (1.0×10^{-9}).

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will depend on the type of measurement and the type of error sources that are present.

The following is a summary of results from the document Statistics Review for Experimental Measurements. You should consult that document for details on the definitions and derivations of the materials in this section.

As discussed above, whenever we make a measurement there is usually a range of possible values that have a finite probability of occurring. Sometimes this range is very small and sometimes it is very large. In either case, the probabilities for the possible values can be used to provide an estimate of the uncertainty in our measurement. The probability of a given value occurring is usually represented by a distribution function. Since the distribution function represents a probability it must be properly normalized:

$$\sum_s P(s) \;=\; 1$$

$$\int P(s)\,\mathrm{d}s \;=\; 1$$

where $P(s)$ is a distribution function. The sum (the first equation) represents the normalization of a distribution over discrete values and s must go over the entire possible range of values. The integral (the second equation) is for a distribution over a continuous set of values and s must go over the entire range of values.

Once we know the expected distribution function for a measurement we can determine what the expected or average value from a collection of measurements will be and the variance in that group. The expected uncertainty in the measurements is usually related to the standard deviation of the measurements which is the square root of the variance. If our experiment involves only discrete possible values, such as non-negative integers, then we can calculate the expected value from the equation

$$\bar{s} \;=\; \sum_s s\,P(s)$$

where the sum is over all possible values of s. With some effort it can be shown that the variance, denoted by σ^2 and defined by $\sigma^2 = \overline{(s-\bar{s})^2}$, is given by

$$\sigma^2 \;=\; \sum_s s^2 P(s) - \bar{s}^2\,.$$

If the experiment involves continuous possible values, you replace the sums by integrals

$$\begin{aligned}\bar{s} &= \int_{s_1}^{s_2} s\,P(s)\,\mathrm{d}s \\ \sigma^2 &= \int_{s_1}^{s_2} s^2 P(s)\,\mathrm{d}s - \bar{s}^2\,.\end{aligned}$$

It is important to remember that this discussion of distributions assumes that the individual measurements are completely independent and uncorrelated so that any variations between successive measurements are random.

3.1 Binomial distribution

Suppose we measure something that can either succeed or fail on any given trial (there can be only two possible results) such as flipping a coin. If we denote the probability

that a given trial succeeds by the variable p, the probability of failure by $q = 1 - p$, and complete n trials, then the probability of having s successes in those n trials is given by the binomial distribution

$$\begin{aligned} P(s \text{ successes in } n \text{ trials}) &= B_{n,p}(s) \\ &= \frac{n!}{s!(n-s)!} p^s q^{n-s} \,. \end{aligned}$$

The average value is

$$\bar{s} = n\,p$$

and the variance is

$$\sigma^2 = n\,p\,(1-p)\,.$$

3.2 Poisson distribution

The Poisson distribution is associated with counting randomly occurring events that have an average rate of occurrence over a particular interval (time, distance, area, etc.). Examples of this type of measurement would be measuring the rate of decay of a radioactive sample or counting photons in a low-level optical experiment. It can be shown that

$$P_\mu(n) = \mathrm{e}^{-\mu} \frac{\mu^n}{n!}$$

where $P_\mu(n)$ represents the probability that exactly n events occur in the specified interval.

The mean value is given by

$$\bar{n} = \mu$$

and the variance is given by

$$\sigma^2 = \mu\,.$$

This is a somewhat surprising, but very useful, result for this distribution since the variance is the same as the expected value.

3.3 The uniform distribution

When any value within a particular range of values is equally likely we have what is called a uniform distribution with a distribution function given by

$$P(s) \quad = \quad \begin{cases} 1/c \text{ for } (\bar{s} - c/2) \leq s \leq (\bar{s} + c/2) \\ 0 \text{ otherwise.} \end{cases}$$

The denominator in P is a normalization so that the integral of $P(s)$ over all s is equal to one (1). It is apparent that the average value is $\bar{s}$ (you can do the integral if you would like). The variance is not as obvious (the derivation of this expression can be found in the document Statistics Review for Experimental Measurements):

$$\sigma^2 \quad = \quad \frac{c^2}{12}\,.$$

As an example, if a 3 1/2-digit voltmeter is used to measure a voltage and it reads 1.250 V, the actual value could lie anywhere within the range of 1.2495 to 1.2505 V with equal likelihood. If there is no reason to believe that any value in that range is more likely than any other value we have a continuous distribution. The average or expected value is then 1.250 V and the standard deviation is $\sigma = c/\sqrt{12} = 0.001/3.464 = 0.00029$ V since $c = 0.001$ V is the interval between successive least-significant digits on the meter. With modern digital equipment this uncertainty will always be present because we only record discrete values from a continuous signal.

If you are using the National Instruments USB-6221-BNC to measure a signal with the input range set to ± 1 V, the code width is about 32 μV, so any single measurement will have an uncertainty of approximately $\pm 16/\sqrt{3} = \pm 9.2\,\mu$V.

3.4 The normal or Gaussian distribution

According to the authors of the *Introductory Statistics* online text, "The normal, a continuous distribution, is the most important of all the distributions. It is widely used and even more widely abused. Its graph is bell-shaped. You see the bell curve in almost all disciplines. ... The normal distribution is extremely important, but it cannot be applied to everything in the real world." [1].

[1] OpenStax College, *Introductory Statistics*, OpenStax College, 19 September 2013, p. 361, available at http://cnx.org/content/col11562/latest/

The Gaussian distribution is given by

$$P(s) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(s-\bar{s})^2/(2\sigma^2)} \,.$$

Using the integral equations, it is easy to show

$$\bar{s} \;=\; \int_{-\infty}^{\infty} s\, P(s)\, \mathrm{d}s$$

and

$$\sigma^2 \;=\; \int_{-\infty}^{\infty} s^2\, P(s)\, \mathrm{d}s - \bar{s}\,.$$

Many measurements of continuous variables in physics exhibit a Gaussian distribution. If you are in doubt, you can collect a large number of measurements, bin the values, plot a histogram of the number of times the value fell within each of the bins, and compare this to a Gaussian curve.

If your measurements are truly Gaussian the meaning of σ is very specific. If you take many measurements, 68.3% of them will fall within the range of $(\bar{s} \pm \sigma)$, 95.4% will fall within the range of $(\bar{s} \pm 2\,\sigma)$, and 99.7% will fall in the range of $(\bar{s} \pm 3\,\sigma)$.

4.0 Propagating errors

Often a measured value will be used as a parameter in some calculation to give the desired result. For example, if you have a temperature transducer that produces a voltage related to the temperature, you apply some calibration equation to the voltage to arrive at the actual temperature. That equation may simply be a linear function, or it can involve exponentials or logarithms, depending on the type of transducer. It will be necessary to analyze how the uncertainty in the final temperature is related to any uncertainty in the measured voltage and uncertainties in the calibration parameters.

The process of propagating errors through a calculation is fairly straightforward in many cases. For instance, if the value y is a function of the inputs x_1, x_2, ..., x_n

$$y \;=\; f(x_1, x_2, x_3, \ldots, x_n)\,,$$

and we know the uncertainties of the x_i are given by δ_i, then the uncertainty in y, δy, can be found by generating a first-order Taylor series expansion about the average value $\bar{y}$. Then, since $\delta_y = y - \bar{y}$ we can write

$$\delta_y = \sum_{i=1}^{n} \delta_i \left(\frac{\partial f}{\partial x_i} \right) .$$

In this form, the uncertainty is incorrectly estimated because it is not expected that all the error terms will simply sum (*i.e.*, some of them may have different signs, or they may all have the same sign). You can get a better estimate by considering the results from a random walk where the square of the distance covered is approximately equal to the sum of the squares of each of the individual steps. Then we have

$$\delta_y^2 = \sum_{i=1}^{n} \delta_i^2 \left(\frac{\partial f}{\partial x_i} \right)^2 .$$

We are assuming here that the variables x_i are independent – that is any change in one of them does not cause a change in any of the others. If they are not independent, there will be another term involving the cross correlation of the form

$$\sum_{i=1}^{n} \sum_{j=1(j\neq i)}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \delta_{ij}$$

where δ_{ij} is called the covariance of x_i and x_j and arises from correlations between those variables (a change in one of them causes a change in the other). If the δ_i values are uncorrelated (*e.g.*, δ_1 doesn't correlate in any way with δ_2 so $\delta_{12} = 0$), then we don't have the extra term in the equation.

There are also some pitfalls in this equation if the function f is nonlinear or the uncertainties δ_i are large in some sense. This is because the propagation equation is based on a first-order Taylor expansion of the equation for y under the assumption and we only kept the lowest order powers of δ_i.

5.0 Dealing with noise in measurements

One of the efforts in experimental measurements is to reduce the uncertainty as far as possible, so the measurements are as precise as possible.

The first step in reducing the uncertainty is to reduce the noise sources as far as possible. This means you have to be sure that your electronic circuitry is well designed; there are no ground loops, all electronic and electrical components are shielded, and all cables and leads are shielded.

5.1 Signal averaging to "remove" noise

Even after careful design, there will be some noise remaining on your signal. Most common electrical noise sources result in a random signal – common exceptions are RF noise and 60-Hz AC noise. If the noise is random and your signal of interest is "slowly varying" you can often use signal averaging to improve your measurement since the random noise signals will average to zero.

Using the standard process of error propagation reviewed above, it can be shown that averaging N measurements of the value y, where each sample has a standard deviation of σ_y (usually the sample standard deviation of the N samples that were averaged), will result in a value

$$\begin{aligned} \bar{y} &= \frac{1}{N}\sum_{i=1}^{N} y_i \\ \sigma_y &= \sqrt{\frac{\sum_{i=1}^{N}(y_i-\bar{y})^2}{N-1}} \end{aligned}$$

with standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma_y}{\sqrt{N}}\,.$$

See the document Statistics Review for Experimental Measurements for the derivation of this result.

Thus, averaging 100 samples will result in a factor of 10 reduction in the standard deviation of the resulting average. Unfortunately, the square-root dependence on N means that rather large values of N are required to significantly improve the standard deviation. For $N = 100$ we get a factor of 10 reduction, but it requires $N = 10000$ to get a factor of 100 reduction and $N = 10^6$ to get a factor of 1000 reduction.

An illustration of the effects of signal averaging can be seen in Figures 2 through 4. One million points were acquired from a measurement of the voltage across a $2.2\,\mathrm{k}\Omega$ resistor with a current of $100\,\mu\mathrm{A}$. The values were acquired with a National Instruments USB-6221-BNC data acquisition module. The channel was set to a range of $\pm 1\,\mathrm{V}$ with a resolution (code width) of $32.4\,\mu\mathrm{V}$ at a sample rate of $10,000$ samples/s. The average value is $0.214901\,\mathrm{V}$ and the standard deviation is $71.2\,\mu\mathrm{V}$. Figure 2 shows the histogram of the entire collection of data. The bin width was set to the device code width with the bins centered on the possible discrete values from the data acquisition module. A best-fit Gaussian determined using Matlab's cftool is overlaid

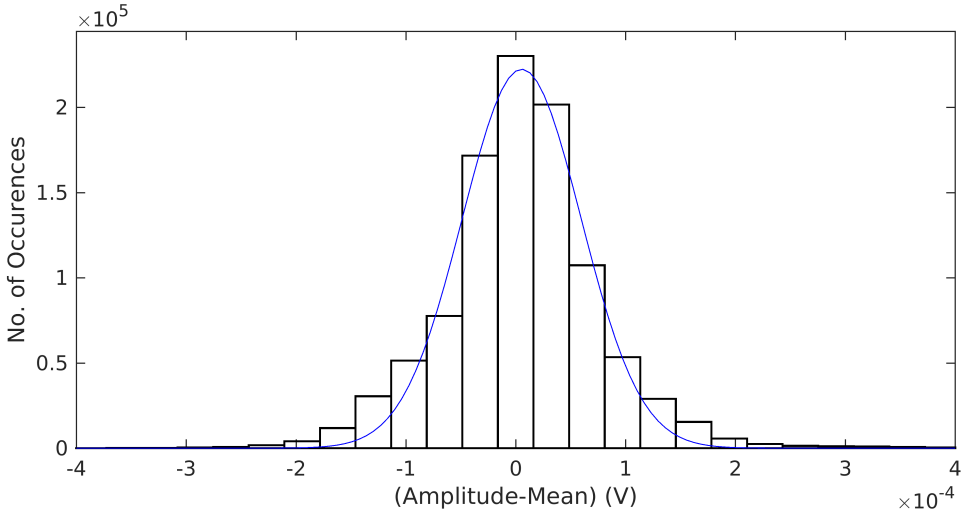

Figure 2: A histogram of one million points acquired from a measurement of the voltage across a 2.2 kΩ resistor with a current of 100 μA passing through it. The average value is 0.214901 V with a standard deviation of 71.2 μV. For plotting, the data have been shifted so they are centered on the average value. A best-fit Gaussian is overlaid on the data (the solid line). The Gaussian has an amplitude of 2.224×10^5 and $\sigma = 54.1\,\mu$V. As can be seen, the distribution is close to a Gaussian but with slightly broader wings.

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on the data. The equation for the Gaussian was

$$n(V) \;=\; 2.224 \times 10^{-5}\, \mathrm{e}^{-\left[\frac{(V-5.44\times10^{-6})^2}{2\,(5.41\times10^{-5})^2}\right]}$$

(amplitude = 2.224×10^{-5}, $\sigma = 54.1\,\mu$V, and the peak is shifted by 5.44×10^{-6} V). As can be seen, the Gaussian is narrower than the original data in the wings (the lower 1/3 of the peak).

The data were then divided into 1000 sets of 1000 points each, and a similar analysis was performed on the means of each of these 1000 sets. Figure 3 shows a histogram of these 1000 averages. They are plotted on the same horizontal scale as those in Figure 2 to illustrate the improvement from averaging. Figure 4 is the same histogram on an expanded scale so the details can be better seen. The best-fit Gaussian curve for the collection of means is overlaid on the data. This curve has the equation

$$n(V) \;=\; 102.4\, \mathrm{e}^{-\left[\frac{(V-1.14\times10^{-7})^2}{2\,(9.63\times10^{-6})^2}\right]}$$

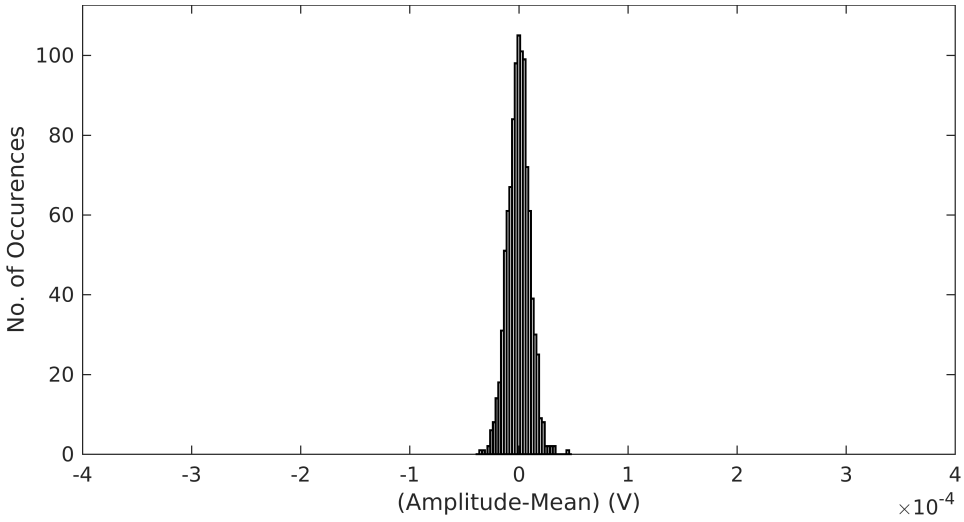

Figure 3: A histogram of 10^3 averages of 10^3 points each using successive subsets of the 10^6 values represented in Figure 2. The data have been shifted, so they are centered on the average value. The mean of the collection of average values is 0.214901 V and the standard deviation of the average values is 9.96 μV. For illustration purposes, this histogram was plotted on the same horizontal scale as Figure 2. From the theoretical effect of signal averaging discussed here we would expect $\sigma_{\bar{v}} = \sigma_v/\sqrt{N}$ where $\sigma_{\bar{v}}$ is the standard deviation of the group of averages, σ_v is the standard deviation of the original data (71.2 μV), and N is the number of points used to determine each individual average (1000). This gives an ideal value of $\sigma_{\bar{v}} = 2.25\,\mu$V. The actual standard deviation is a factor of 4 larger than the predicted value but still a factor of 7 better than without averaging.

(amplitude = 102.4, $\sigma = 9.63\,\mu$V, and the peak is shifted by 0.1143 μV). The Gaussian fit to the collection of means is considerably better than to the original data. The average of the means is the same as that for the original data (0.214901 V) as would be expected. The standard deviation is 9.96 μV, quite close to that of the best-fit Gaussian and a factor of 7.1 smaller than that of the original data. The theory says the improvement should be a factor of $\sqrt{1000} = 31.6$. The discrepancy is likely due to the deviation of the original data from a Gaussian distribution.

Further illustrations of the effects of number of samples, sampling rate, and sampling duration on signal averaging can be found in the Noise Plots found on Learning Suite in `Content` ⇒ `LabVIEW Basics Course` ⇒ `Measurements, Uncertainties, and Noise` or on the course web page.

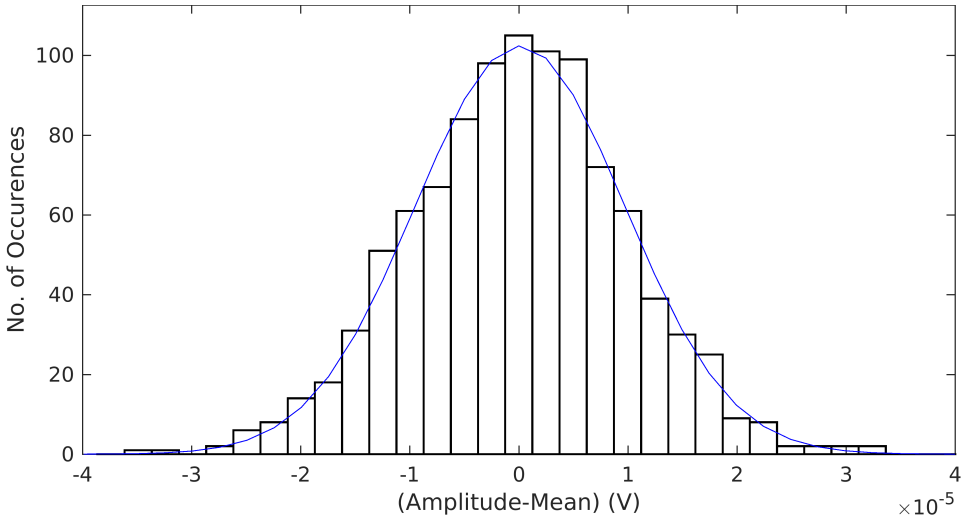

Figure 4: The same histogram as shown in Figure 3 but on an expanded scale to show the detail in the distribution. A best-fit Gaussian is overlaid on this data (the solid line). It has an amplitude of 102.4 and $\sigma = 9.63 \times 10^{-6}$. The collection of averages is fairly close to a Gaussian distribution.

5.2 Filtering to reduce noise

If you have noise that is not random, such as 60-Hz AC noise or RF noise from radios or computers, you may be able to use a high-pass filter if your signal of interest is higher frequency or a low-pass filter if your signal is lower frequency. It is also possible to use a bandpass filter to reduce the offending noise signal if the signal of interest is in a small frequency range or a notch filter if the noise is only present in a small frequency range.

Filtering the signal will reduce the bandwidth (smaller Δf) which will significantly reduce the Johnson-Nyquist, shot, or $1/f$ noise and improve the sensitivity to small signals.

6.0 Systematic errors

It is possible that you will have an uncertainty in your measurements due to what are known as systematic errors. These are usually design errors or electronic features that cause a systematic shift in the measurement. If these effects are relatively constant over the duration of the measurement, you may be able to determine them in some

fashion and properly compensate for the error in your measurements (an offset voltage in an amplifier is an example). It is more complex if these errors vary over the course of the experiment and compensation becomes a difficult problem you will have to solve.

Unfortunately, systematic errors are usually unique to a particular measurement and there are no standard methods for dealing with these errors. The better you understand the details of your experiment and the measurements, the more likely it will be that you can find and overcome systematic errors.

[Modified: January 16, 2019]