Minimal theory of quantum electrodynamics

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Within the general framework of the Lehmann-Symanzik-Zimmermann axiomatic field theory, we obtain a simple and coherent formulation of quantum electrodynamics. The definitions of the current densities fulfill the one-particle stability condition, and the commutation relations for the interacting fields are obtained rather than being postulated *a priori*, thus avoiding the inconsistencies which appear in the canonical formalism. This is possible due to the fact that we use the integral form of the equations of motion in order to compute the propagators and the S matrix. The resulting spectral representations automatically fulfill the correct boundary conditions thus fixing the ubiquitous quasilocal operators in a unique fashion.

I. INTRODUCTION

Quantum electrodynamics (QED) persists as the most accurate theory in physics, and it is used as the ideal pattern to build other gauge—as well as more general—field theories. In spite of this, a coherent quantum-field-theory description of QED has failed to exist hitherto. It is well known that the original theory based on the canonical formalism is plagued with infinities and internal inconsistencies, which have to be remedied *a posteriori*, both in the interaction picture[1] and in the Heisenberg picture.[2] The original renormalization program[1-3] is at variance with the basic idea of the canonical formalism, which implicitly assumes that all the results in the theory should follow from the postulated Lagrangian and the field (anti-) commutation relations. Thus the parameters appearing in the original Lagrangian should be the physical parameters from the very beginning, as they are indeed for few-body theories, or for free fields, where the canonical formalism does give a satisfactory description. For the interacting-fields case, instead, we find that the product of operators at equal-spacetime events are ill defined;[4] knowledge of the commutation relations for a pair of arbitrary events requires the exact solution of the problem we are seeking to solve, while equal-time commutation relations for the interacting fields cannot be assumed in advance in a consistent fashion, thus questioning the very existence of a well-defined interaction Lagrangian in the Heisenberg picture.

The alternative presented by the axiomatic field theories[5-9] gives a framework to define relativistic quantum field theories, making some very general assumptions which seem to be compatible with one another. These include basic properties of the physical state space taken as a Hilbert space and transformations therein, causality, locality, uniqueness of the vacuum state, and asymptotic completeness, as is well known. In the Lehmann-Symanzik-Zimmermann (LSZ) version,[5,8,9] there is a relationship between the interacting field (or interpolating field) and a corresponding free field (*in* or *out*) through the asymptotic boundary conditions. These schemes usually leave open the question of how to build a specific theory, e.g., quantum electrodynamics, since their interest lies in defining a general framework.

This work is devoted to the construction of a field theory of QED with a minimum of assumptions—which can be taken essentially as those of the LSZ formulation—and avoids the incompatibilities which appear in the canonical approach.

Along with these general assumptions, we take the usual commutation relations for the photon asymptotic free field a_μ:

$$[a_\mu(x),a_\nu(x')]=-iD_{\mu\nu}(x-x')\ , \tag{1.1}$$

and anticommutation relations for the electron-positron free spinor field ϕ:

$$\{\bar{\phi}_\alpha(x),\phi_\beta(x')\}=-iS_{\alpha\beta}(x-x')\ , \tag{1.2}$$

where D and S are the Jordan-Pauli functions.

The stability of the one-particle states[3-9] is a consequence of the general assumptions and the asymptotic condition, both for the photon and the electron fields. This result has far-reaching consequences: in particular, it constrains the possible definitions of the corresponding currents, which appear as the source terms in the equations of motion. Let us recall that the usual canonical electron-positron current density $j_\mu(x)$, inferred from the classical theory, and involving a product of two Heisenberg fields with the same argument x, does *not* fulfill this stability condition. This is apparently the reason why a "renormalized" current[2,4] has to be redefined *a posteriori*. The same applies indeed to the interaction term $f(x)$ in the Dirac equation.

The fact that equal-time commutation relations for interacting fields are ill defined is well known.[6,9] In particular, Haag's theorem[9,10] shows that they cannot be assumed to be the canonical ones in a relativistic theory. It is one of the aims of the present work to show that, in fact, they are *determined* by the theory, once the asymptotic free-fields commutation relations (1.1), (1.2), and the elementary interaction are assumed. This fact can be il-

lustrated in a simple example proposed by Sokolov and Tumanov:[11] let us take a classical harmonic oscillator interacting with a quantized electromagnetic field obeying the canonical commutation relations (CCR). The equations of motion for the position q and momentum p are those of a forced harmonic oscillator with a classical self-interaction term, and where the forcing term is given by the interaction with the quantized field. Expanding in normal modes, the equations for the oscillator can be solved for q and p thus becoming functions of the fields. Using the latter's CCR, the commutator between q and p can be worked out. It is no surprise to find out[11] that the resulting commutator is *not* the canonical one, but instead depends on the interaction, and includes radiative effects. Only to the lowest order in the interaction do we get the CCR for the oscillator variables.

In this sense, the question of what is *not* assumed in this work becomes very relevant. We do not take any *a priori* form of the commutation relations for the interacting fields. Neither do we take the usual definition[2] for the current density and the interaction term in the equations of motion for the fields. They both follow as results, starting from the first-order interaction which defines QED.

The way this is achieved is as follows. The asymptotic completeness of the free fields allows us to expand all physical quantities (interacting fields, currents, and S matrix) in terms of them.[8,9] Once the general framework is set, we can define our particular theory, namely, QED, through the free-field equations of motion (sourceless Maxwell and Dirac equations for the photon and electron *in* fields a_μ,ϕ, respectively), the commutation relations for them, Eqs. (1.1) and (1.2), and the interaction, given in terms of these asymptotic *in* fields as the first-order term in the S matrix. This is, in fact, equivalent to giving the interaction Lagrangian in the interaction picture,[3] and defines a local theory. The solution of the problem consists in finding the complete S matrix given the first-order term, much in the same way as we solve for the evolution operator in quantum mechanics, once we are given the Hamiltonian of the problem. From the S matrix we can find the current densities and then the interacting fields. The resulting equations are obviously coupled. Since all physical quantities are expanded in terms of the asymptotic fields, the direct iterative procedure leads to a perturbative expansion: assume S up to first order, obtain the currents and fields up to second order, the n-point propagators and hence the S matrix to higher orders, and then iterate the procedure.

The way to sidestep the use of preassumed commutation relations for the interacting fields is by means of the Yang-Feldman equations.[12] Since these constitute the integral form of the equations of motion, they allow us to derive general expressions for the two-point photon and electron propagators in terms of the current-current correlation functions, which have well-defined spectral properties.[3] The resulting expressions are nothing else but the Källen-Lehman spectral representations[2] with the correct ultraviolet behavior (i.e., "already renormalized"). The n-point propagators ($n>2$) follow from computing the corresponding functional derivatives of the two-point functions.

In Sec. II we present the general framework, starting from the Yang-Feldman integral equations and the definition of the interaction in terms of the asymptotic *in* fields. Section III deals with the derivation of the spectral representation for the two-point functions, while Sec. IV includes the second-order results. The equivalence with the renormalized Feynman diagrams is surmised in Sec. V. Finally, Sec. VI contains a general discussion and a comparison with other approaches.

II. GENERAL FRAMEWORK

We assume the existence of Heisenberg fields for the photon and electron $A_\mu(x)$, $\Psi(x)$, as well as the n-point propagators, formed as the vacuum expectation values of time-ordered products of them. These fields tend asymptotically[5] to free incoming fields $a_\mu(x)$, $\phi(x)$, when $t\to-\infty$, and free outgoing fields $a_\mu^{\text{out}}(x)$, $\phi^{\text{out}}(x)$ when $t\to\infty$. They both satisfy the sourceless equations

$$K_x a_\mu(x)\equiv\partial_\lambda\partial^\lambda a_\mu(x)=0\,, \tag{2.1a}$$

$$\mathscr{D}_x\phi(x)\equiv(i\gamma_\lambda\partial^\lambda-m)\phi(x)=0\,, \tag{2.1b}$$

in the Lorentz gauge.

We also assume asymptotic completeness in the usual sense,[5,9] so that the unitary S matrix relating *in* fields to *out* fields,

$$S^{-1}a_\mu(x)S=a_\mu^{\text{out}}(x)\,, \tag{2.2a}$$

$$S^{-1}\phi(x)S=\phi^{\text{out}}(x)\,, \tag{2.2b}$$

can be expanded in terms of normal-ordered charge-conserving products of the *in* fields:

$$S=1+\sum_{m,n=1}^{\infty}\int\frac{i^n(-)^m}{n!(m!)^2}C_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_m,z_1,\ldots,z_n)$$
$$\times:\bar\phi(x_1)\cdots\bar\phi(x_m)\phi(y_1)\cdots\phi(y_m)a_{\mu_1}(z_1)\cdots a_{\mu_n}(z_n):dx_1\cdots dz_n\,. \tag{2.3a}$$

The expansion coefficients are given by the reduction formulas[3,5,13] in terms of the $(2m+n)$-point propagator

$$C_{m,n}(x,y,z)=\mathscr{D}_1\cdots\mathscr{D}_m K_1\cdots K_n\langle T\bar\psi(x_1)\cdots\psi(y_1)\cdots A_{\mu_1}(z_1)\cdots\rangle\overleftarrow{\mathscr{D}}_1\cdots\overleftarrow{\mathscr{D}}_m\,. \tag{2.3b}$$

The interaction is now defined by the elementary vertex, as the first-order S-matrix term, and it contributes only to the three-point function:

$$S^{(1)}=-ie\int :\bar{\phi}(x)\gamma_\mu\phi(x)a^\mu(x):dx\ . \tag{2.4}$$

This is, of course, the "minimal coupling" interaction. It is local, conserves charge, and yields a gauge-invariant theory. As is well known, it is also C, P, and T invariant.

The next step is to define the current densities $j_\mu(x)$ and $f(x)$ from the S matrix:

$$ej_\mu(x)=S^{-1}K_x[Ta_\mu(x)S]\ , \tag{2.5a}$$

$$ef(x)=S^{-1}\mathscr{D}_x[T\phi(x)S]\ . \tag{2.5b}$$

(This is, of course, equivalent to the definition of the currents in terms of functional derivatives with respect to the fields.[3,7,9,15]) Here T is Wick's time-ordering operator.[3,14] Since it contains step functions in the time t, it does not commute with K, $\mathscr{D}$, or with the integrations appearing in Eq. (2.3) in general. We can easily show, however, that Wick's theorem[14] is valid in this case, so the combined effect of K and T in Eq. (25) is to "drop" one of the a's in Eq. (2.3), and substitute its argument by x in the integral. By the same token, we can prove[8,9] that

$$a_\mu^{\text{out}}(x)-a_\mu(x)=S^{-1}[a_\mu(x),S]$$
$$=-e\int D_{\mu\nu}(x-x')j^\nu(x')dx'\ , \tag{2.6a}$$

$$\phi^{\text{out}}(x)-\phi(x)=S^{-1}[\phi(x),S]$$
$$=e\int S(x-x')f(x')dx'\ , \tag{2.6b}$$

thus showing that j_μ and f are the correct current densities.

The Heisenberg fields $A_\mu(x),\Psi(x)$ obey now the equations of motion

$$K_xA_\mu(x)=ej_\mu(x)\ , \tag{2.7a}$$

$$\mathscr{D}_x\Psi(x)=ef(x) \tag{2.7b}$$

in the Lorentz gauge. The integral form for these equations in terms of the *in* fields is[12]

$$A_\mu(x)=a_\mu(x)-e\int D_{\mu\nu}^{\text{ret}}(x-x')j^\nu(x')dx'\ , \tag{2.8a}$$

$$\Psi(x)=\phi(x)+e\int S^{\text{ret}}(x-x')f(x')dx'\ . \tag{2.8b}$$

From the asymptotic conditions and Eqs. (2.7), it follows that these current densities fulfill the stability conditions[9]

$$\langle 0\,|\,j_\mu(x)\,|\,1\text{ ph}\rangle=0\ , \tag{2.9a}$$

$$\langle 0\,|\,f(x)\,|\,1\text{ el}\rangle=0\ , \tag{2.9b}$$

in contradistinction with the canonical ones.

From the definition of the currents Eq. (2.5), and the interaction given by Eq. (2.4), we find the usual lowest-order terms:

$$ej_\mu^{(1)}(x)=e:\bar{\phi}(x)\gamma_\mu\phi(x):\ , \tag{2.10a}$$

$$ef^{(1)}(x)=e\mathscr{A}(x)\phi(x)\ . \tag{2.10b}$$

III. TWO-POINT PROPAGATORS

We can now find the spectral representation for the photon and electron propagators,[16,17] with the aid of the integral equations (2.8) and the stability condition (2.9).

Let us start with the photon propagator, defined as

$$\mathscr{D}_{\mu\nu}^c(x_1-x_2)=i\langle TA_\mu(x_1)A_\nu(x_2)\rangle\ . \tag{3.1}$$

In order to have a correct relativistic definition of the T product, we introduce the scalar function

$$\mathscr{D}^c(x_1-x_2)=\frac{i}{3}\langle TA_\mu(x_1)A^\mu(x_2)\rangle$$
$$=\frac{i}{3}[\theta(\tau)\langle A_\mu(x_1)A^\mu(x_2)\rangle$$
$$+\theta(-\tau)\langle A^\mu(x_2)A_\mu(x_1)\rangle]\ ,$$
$$\tau=t_1-t_2\ , \tag{3.2}$$

and use the transversality property[3,18]

$$\mathscr{D}_{\mu\nu}^c(k)-D_{\mu\nu}^c(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)[\mathscr{D}^c(k)-D^c(k)] \tag{3.3}$$

in momentum space. This is a consequence[18] of the fact that the longitudinal part of $\mathscr{D}$ cancels out in Eq. (3.3), thus yielding a gauge-invariant expression, as is well known. Substituting now Eq. (2.8a) into Eq. (3.2)

$$\mathscr{D}^c(x_1-x_2)=D^c(x_1-x_2)+\frac{ie^2}{3}\left[\theta(\tau)\int D^{\text{ret}}(x_1-y_1)D^{\text{ret}}(x_2-y_2)+\langle j_\mu(y_1)j^\mu(y_2)\rangle dy_1dy_2+(1\leftrightarrow 2)\right]\ . \tag{3.4}$$

There are several points worth remarking about in this expression. The first one is that the cross terms vanish, in view of the one-photon stability property (2.2a). The second one is that the causal character of the propagator follows directly from its definition in terms of the T product, so it becomes irrelevant whether we use retarded functions in the integrand, or any other photon free propagator. Third, we notice that the argument in the step functions refers to the *original* variables, and *not* to the integration variables, appearing as arguments of the current densities. Finally, the current-current correlation function is a well-defined function (more precisely a tempered distribution[3,7]) with well-known spectral properties.[3] Its Fourier transform has the form

$$-\frac{i}{3}\int\langle j_\mu(y_1)j^\mu(y_2)\rangle e^{iky}dy=J(k^2)\theta(k^0)\ ,$$
$$y=y_1-y_2\ , \tag{3.5}$$

where J depends on k^2 only. It can be obviously calculated knowing the current density $j_\mu(y)$.

The last step to obtain the spectral representation of $\mathscr{D}^c(k)$ is to take the Fourier transform of Eq. (3.4), yielding a convolution in ω, the conjugate variable of the time difference τ, using the representation of θ:

$$\theta(\pm\tau)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}}{\pm\omega+i\epsilon}d\omega\ . \tag{3.6}$$

So we finally have

$$\mathscr{D}^c(k)=D^c(k) -\frac{e^2}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{J(q^2)}{q^4}\left[\frac{\theta(\omega)}{k^0-\omega+i\epsilon}+\frac{\theta(-\omega)}{-k^0+\omega+i\epsilon}\right], \tag{3.7}$$

where

$$q^\mu=(\omega,\mathbf{k})\ . \tag{3.8}$$

As mentioned above, we do not need to specify the character of the Green's function $1/q^2$.

If we now change the integration variable in Eq. (3.7) to the scalar

$$\lambda\equiv q^2=k^2-(k^0)^2+\omega^2 \tag{3.9}$$

we get

$$\mathscr{D}^c(k)=D^c(k)+\frac{e^2}{2\pi}\int_0^{\infty}\frac{J(\lambda)}{\lambda^2}\frac{1}{\lambda-k^2-i\epsilon}d\lambda\ . \tag{3.10}$$

This spectral representation shows that the correct spectral density for the interacting part of the photon propagator is $J(\lambda)/\lambda^2$.

The photon proper energy function is defined through the relation[18]

$$\mathscr{D}^c=D^c+D^c\Pi D^c\ , \tag{3.11a}$$

$$\Pi(k^2)=k^2[\mathscr{D}^c(k)-D^c(k)]k^2\ , \tag{3.11b}$$

and the transverse projection gives the gauge-invariant function

$$\Pi_{\mu\nu}(k)=\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\Pi(k^2)\ . \tag{3.12}$$

It then follows immediately, from Eqs. (3.10) and (3.11), that

$$\Pi(k^2)=\frac{e^2}{2\pi}k^4\int_0^{\infty}\frac{J(\lambda)}{\lambda^2}\frac{1}{\lambda-k^2-i\epsilon}d\lambda\ , \tag{3.13}$$

satisfying

$$\Pi(\lambda)\,|_0=\Pi'(\lambda)\,|_0=0\ . \tag{3.14}$$

These are usually imposed as renormalization conditions for the photon propagator.[1-3,18] It is also evident that a dispersion relation appropriately subtracted obtains directly from Eq. (3.13)

$$\Pi(k^2)=\frac{k^4}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\Pi(\lambda)}{\lambda^2(\lambda-k^2-i\epsilon)}d\lambda\ . \tag{3.15}$$

We can use a similar procedure[16] for the electron propagator *mutatis mutandis*, namely, using Eqs. (2.8b) and (2.9b) in

$$\mathscr{G}^c_{\alpha\beta}(x_1-x_2)=-i\langle T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle =-i[\theta(\tau)\langle\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle -\theta(-\tau)\langle\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle]\ , \tag{3.16}$$

so that

$$\mathscr{G}^c_{\alpha\beta}(x_1-x_2)=S^c_{\alpha\beta}(x_1-x_2)-ie^2\left[\theta(\tau)\int S^{\mathrm{ret}}_{\alpha\gamma}(x_1-y_1)\langle f_\gamma(y_1)\bar{f}_\delta(y_2)\rangle\bar{S}^{\mathrm{ret}}_{\delta\beta}(x_2-y_2)dy_1dy_2 -\theta(-\tau)\int\langle\bar{f}_\delta(y_2)\bar{S}^{\mathrm{ret}}_{\delta\beta}(x_2-y_2)S^{\mathrm{ret}}_{\alpha\gamma}(x_1-y_1)f_\gamma(y_2)\rangle dy_1dy_2\right], \tag{3.17}$$

where $\bar{S}=\gamma^0S^+\gamma^0$. We now introduce the well-defined spectral representation of the f-f correlation function:[2,3]

$$\mathscr{F}\langle f(y_1)\bar{f}(y_2)\rangle=\theta(p^0)[\not{p}R_1(p^2)+R_2(p^2)]\ , \tag{3.18}$$

where R_1 and R_2 are functions of the scalar p^2, in view of the Lorentz and parity invariance. Hence Eq. (3.17) gives the matrix

$$\mathscr{G}^c(p)=S^c(p)+\frac{e^2}{2\pi}\int_{-\infty}^{\infty}d\omega\left[\frac{\theta(\omega)}{p^0-\omega+i\epsilon}+\frac{\theta(-\omega)}{-p^0+\omega+i\epsilon}\right]S^0(q)[\not{q}R_1(q^2)+R_2(q^2)]S^0(q)\ , \tag{3.19}$$

where

$$S^0(q)=\frac{\not{q}+m}{q^2-m^2} \tag{3.20}$$

and ω substitutes p^0 in

$$q^\mu=(\omega,\mathbf{p}) \tag{3.21}$$

so now

$$\lambda\equiv q^2=p^2-(p^0)^2+\omega^2 \tag{3.22}$$

is the new integration variable. The spectral representation of the electron propagator reads

$$\mathscr{S}^c(p)=S^c(p)-\frac{e^2}{2\pi}\int_0^\infty \frac{\Sigma_1(\lambda)+\not{p}\Sigma_2(\lambda)}{(\lambda-m^2)^2(\lambda-p^2-i\epsilon)}d\lambda\ , \quad (3.23)$$

in terms of the functions

$$\Sigma_1(\lambda)=2\lambda m R_1(\lambda)+(\lambda+m^2)R_2(\lambda)\ , \quad (3.24a)$$

$$\Sigma_2(\lambda)=(\lambda+m^2)R_1(\lambda)+2mR_2(\lambda)\ . \quad (3.24b)$$

The electron proper energy

$$\Sigma(\not{p})=(\not{p}-m)[\mathscr{S}^c(p)-S^c(p)](\not{p}-m) \quad (3.25)$$

is hence given by

$$\Sigma(\not{p})=-\frac{e^2}{2\pi}(\not{p}-m)^2\int_0^\infty \frac{\Sigma_1(\lambda)+\not{p}\Sigma_2(\lambda)}{(\lambda-m^2)^2(\lambda-p^2-i\epsilon)}d\lambda \quad (3.26)$$

automatically fulfilling the "renormalization conditions"

$$\Sigma(\not{p})\,|_{\not{p}=m}=\Sigma'(\not{p})\,|_{\not{p}=m}=0\ . \quad (3.27)$$

The origin of the subtraction terms in Eqs. (3.13) and (3.26) is the appearance of the inverse of K and $\mathscr{D}$, respectively, when solving the equations of motion in Eq. (2.8). This is in agreement with the fact that the interaction implies necessarily subtracted dispersion relations.[19]

The contribution of the two-point propagators to the S matrix obtains from the reduction formulas, Eq. (2.3b),

$$C_{0,2}(z_1,z_2)=-i\Pi(z_1-z_2) \quad (3.28a)$$

and

$$C_{1,0}(x_1,y_1)=i\Sigma(x_1-y_1)\ . \quad (3.28b)$$

Finally, let us obtain the (anti-) commutation relations. Starting from Eq. (3.10) in configuration space

$$\mathscr{D}^c(x_1-x_2)=D^c(x_1-x_2)-\frac{e^2}{2\pi}\int_0^\infty \frac{J(\lambda)}{\lambda^2}D^c(x_1-x_2;\lambda)d\lambda\ , \quad (3.29)$$

with

$$(\partial^2+\lambda)D^c(x;\lambda)=-\delta(x)\ . \quad (3.30)$$

Taking the even part of Eq. (3.1), and using Eqs. (3.3) and (3.29), we get

$$\langle[A_\mu(x_1),A_\nu(x_2)]\rangle=-i\left[D_{\mu\nu}(x_1-x_2)-\frac{e^2}{2\pi}\int_0^\infty \frac{J(\lambda)}{\lambda^2}D_{\mu\nu}(x_1-x_2;\lambda)d\lambda\right]. \quad (3.31)$$

Taking the time derivative of Eq. (3.31) and evaluating it at equal times $t_1=t_2$, we get

$$\langle[\partial_t A_\mu(x),A_\nu(x')]\rangle\,|_{t=t'}=-i\delta(\mathbf{x}-\mathbf{x}')g_{\mu\nu}\left[1-\frac{e^2}{2\pi}\int_0^\infty \frac{J(\lambda)}{\lambda^2}d\lambda\right] \quad (3.32)$$

in the Lorentz gauge.

Similarly, for the electron anticommutation relation, we get

$$\langle\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}\rangle\,|_{t=t'}=\gamma^0_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')\left[1+\frac{e^2}{2\pi}\int_0^\infty \frac{\sqrt{\lambda}\Sigma_1(\lambda)+\Sigma_2(\lambda)}{(\lambda-m^2)^2}d\lambda\right]. \quad (3.33)$$

Just as in the case[11] mentioned in the Introduction, the commutation relations depend on the interaction. Furthermore, since the δ functions factor out, we are left with a divergent constant multiplying the δ function, in both cases.

IV. SECOND-ORDER TERMS

All the tree diagrams are calculated directly from functional derivatives of the free electron propagator S^c (Refs. 13, 20, and 21). The derivative with respect to a_μ is obtained, by introducing an infinitesimal change $a_\mu\to a_\mu+\delta a_\mu$, taking δa_μ as an external field.

To lowest order, we can use the equation of motion Eq. (2.8b), together with (2.10b), to get

$$\mathscr{D}_x\delta S^c(x,y)=e\delta a(x)S^c(x,y)\ . \quad (4.1)$$

(We use the shorthand notation $S^c(x,y)$ to denote $S^c[x,y;\delta a_\mu]$ which has a functional dependence on the (infinitesimal) external field.) Hence the first functional derivative yields[9,13,20]

$$\frac{\delta S^c(x,y)}{\delta a_\mu(\xi)}=eS^c(x,\xi)\gamma^\mu S^c(\xi,y)\ . \quad (4.2)$$

This gives an extra elementary vertex, with a factor of e. Hence the second derivative will give a second-order term. It represents the lowest-order Compton scattering term:

$$\frac{\delta^2 S^c(x,y)}{\delta a_\mu(\xi)\delta a_\nu(\eta)}=e^2[S^c(x,\xi)\gamma^\nu S^c(\xi,\eta)\gamma^\mu S^c(\eta,y)+S^c(x,\eta)\gamma^\mu S^c(\eta,\xi)\gamma^\nu S^c(\xi,y)]\ . \quad (4.3)$$

Setting $\delta a_\mu=0$, and using the reduction formula Eq. (2.3b) together with Eq. (4.1) we get

$$S^{(2)}_{\text{Compton}}=-\frac{ie^2}{2}\left[\int :\bar{\phi}(\xi)a(\xi)S^c(\xi-\eta)a(\eta)\phi(\eta):d\xi d\eta-\int :a(\xi)\phi(\xi)S^c(\eta-\xi)\bar{\phi}(\eta)a(\eta):d\xi d\eta\right]. \quad (4.4)$$

(See Fig. 1.) The rest of the tree diagrams are obtained in a similar way. Additional functional derivatives give rise to higher-order terms with additional vertices. Extra electron terms are obtained by taking functional derivatives with respect to ϕ and $\bar{\phi}$, in a well-known procedure.[1,3,20,21]

Since functional differentiation does *not* create any closed loop, all the results for tree diagrams are obviously the same as in the usual unrenormalized Feynman-Dyson-Wick procedure.

Let us now look at the most interesting case of the lowest-order radiative corrections for the two-point propagators, starting from the spectral representations Eqs. (3.13) and (3.26). In the case of vacuum polarization we need to compute the j-j correlation function. To lowest order the current is

$$ej_\mu^{(1)}(x)=e:\bar{\phi}(x)\gamma_\mu\phi(x): \tag{2.10a}$$

so that

$$\begin{aligned} e^2\langle j_\mu(x_1)j^\mu(x_2)\rangle^{(2)} &= e^2\langle :\bar{\phi}(x_1)\gamma_\mu\phi(x_1)::\bar{\phi}(x_2)\gamma^\mu\phi(x_2):\rangle \\ &= -e^2\,\mathrm{tr}[S^-(x_1-x_2)\gamma_\mu S^+(x_2-x_1)\gamma^\mu]\,, \end{aligned} \tag{4.5}$$

and the function J in Eq. (3.5) is approximated as

$$e^2J^{(2)}(\lambda)=-\frac{e^2}{3(2\pi)}\theta(\lambda-4m^2)\left[\frac{\lambda-4m^2}{\lambda}\right]^{1/2}(\lambda+2m^2) \tag{4.6}$$

and hence

$$\Pi^{(2)}(k^2)=-\frac{e^2}{12\pi^2}k^4\int_{4m^2}^{\infty}\frac{d\lambda}{\lambda^2}\left[\frac{\lambda-4m^2}{\lambda}\right]^{1/2}\frac{\lambda+2m^2}{\lambda-k^2-i\epsilon}\,. \tag{4.7}$$

For small k

$$\Pi^{(2)}(k^2)\underset{k^2\to 0}{\sim}-\frac{\alpha}{15\pi m^2}k^4\,, \tag{4.8}$$

where $\alpha=e^2/4\pi$ is the fine-structure constant.

It is evident from Eq. (4.5) that a loop is formed in the correlation function. However, since there is no multiplication by step functions, the product in Eq. (4.5) is well defined as a product of distributions.[7] The diagram in Fig. 2 is then similar to the one used in S-matrix theory.[22,23] However, in the present theory there are no assumptions about analytic properties, and the subtractions in the dispersion relations Eq. (3.15) are derived,

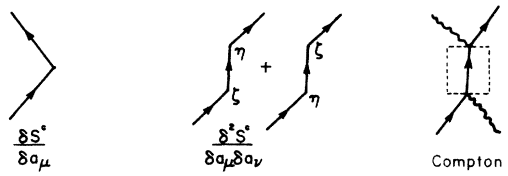

FIG. 1. The Compton scattering term can be easily computed by taking the second functional derivative of the free electron propagator.

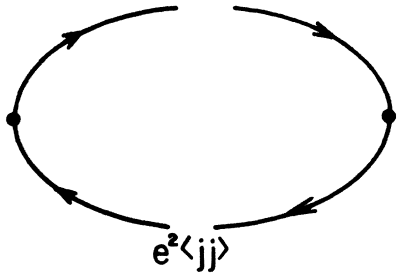

FIG. 2. The j-j correlation function gives rise to a loop associated with well-defined product of distributions.

rather than being imposed.[18]

A similar calculation gives the radiative correction to the electron propagator. To order e^2, it is enough to take

$$ef^{(1)}(x)=e\alpha(x)\phi(x) \tag{2.10b}$$

so the f-f correlation function yields

$$\begin{aligned} e^2\langle f(x_1)\bar{f}(x_2)\rangle^{(2)} &= e^2\langle\gamma^\mu a_\mu(x_1)\phi(x_1)\bar{\phi}(x_2)a_\nu(x_2)\gamma^\nu\rangle \\ &= e^2D^+(x_1-x_2)\gamma^\mu S^+(x_1-x_2)\gamma_\mu\,. \end{aligned} \tag{4.9}$$

Hence, from Eqs. (3.18) and (3.24), we get

$$e^2\Sigma_1^{(2)}(\lambda)=\frac{e^2}{4(2\pi)}\theta(\lambda-m^2)\left[\frac{2(m^4-\lambda^2)m}{\lambda}\right], \tag{4.10a}$$

$$e^2\Sigma_2^{(2)}(\lambda)=\frac{e^2}{4(2\pi)}\theta(\lambda-m^2)\left[\frac{4m^2(\lambda-m^2)}{\lambda}-\frac{(\lambda-m^2)^3}{\lambda^2}\right], \tag{4.10b}$$

so we finally get

$$\begin{aligned} \Sigma^{(2)}(\not p)=&-\frac{\alpha}{4\pi}(\not p-m)^2 \\ &\times\int_{m^2}^{\infty}\frac{d\lambda}{\lambda-p^2-i\epsilon} \\ &\times\left[\frac{2m(\lambda+m^2)}{\lambda(\lambda-m^2)}\right. \\ &\left.+\not p\left[\frac{\lambda-m^2}{\lambda^2}-\frac{4m^2}{\lambda(\lambda-m^2)}\right]\right] \end{aligned} \tag{4.11}$$

disregarding the harmless "infrared catastrophe."

We notice again from Fig. 3 that a loop is formed in the computation of the f-f correlation function. In gen-

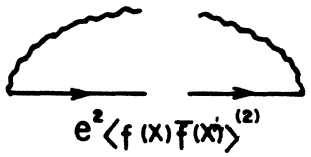

FIG. 3. The f-f correlation function loop also corresponds to well-defined product of distributions.

eral, *all* the new loops result from the contractions appearing in the spectral densities $J(\lambda)$, $\Sigma_1(\lambda)$, and $\Sigma_2(\lambda)$. These are not time-ordered functions, and are well behaved. Furthermore, since functional differentiation does not produce any new loops, higher-order multilegged diagrams can be safely computed without introducing any ultraviolet divergences.

V. FEYNMAN DIAGRAMS

The higher-order terms can be computed by iteration: from the two- and four-point propagators to second order, we construct the S matrix, and hence the currents to the same order, using Eqs. (2.3) and (2.5). At the same time, we can compute the third-order contributions to the vertex by taking the functional derivative $\delta\Sigma/\delta a_\mu$ which yields the S matrix and the currents to third order. Substituting these expressions for the currents in the spectral representations for the photon and electron proper energies, Eqs. (3.13) and (3.26), gives their correct expression to fourth order, and the process can be iterated.

The results obtained in this way coincide with the usual perturbation expansion, once the latter has been properly renormalized (3). It seems then more illustrative to reproduce the Feynman diagram expressions directly instead of following in detail the iterative procedure discussed above. It is evident from Eqs. (2.4) and (3.17), however, that Wick's theorem cannot be applied in the expressions as they stand, since the arguments of the step functions do not refer to the same variables as the currents. We can rewrite both proper energies in terms of T products, if we are willing to tolerate the presence of divergent integrals (compensated by divergent contact terms), namely, express the propagators in terms of "unrenormalized" functions.

In the case of the photon, we define formally $\tilde{\Pi}$ as

$$\tilde{\Pi}(k^2)=\frac{e^2}{2\pi}\int_0^\infty \frac{J(\lambda)d\lambda}{\lambda-k^2-i\epsilon} \tag{5.1}$$

[Eq. (5.1) should be understood as a regularized quantity, since it diverges, written as it is[1,3,13]] and write Eq. (3.13) as

$$\Pi(k^2)=\tilde{\Pi}(k^2)-\tilde{\Pi}(0)-k^2\tilde{\Pi}'(0)\ . \tag{5.2}$$

Although each term in Eq. (5.2) would diverge if calculated directly, the definition (5.1) has the advantage that in x space it has the simple form

$$\tilde{\Pi}(x-x')=\frac{ie^2}{3}\langle Tj_\mu(x)j^\mu(x')\rangle \tag{5.3}$$

to which Wick's theorem does apply.

Similarly, for the electron proper energy, we define

$$\tilde{\Sigma}(\not p)=-\frac{e^2}{2\pi}\int_0^\infty \frac{\not p R_1(\lambda)+R_2(\lambda)}{\lambda-p^2-i\epsilon}d\lambda \tag{5.4}$$

and rewrite Eq. (3.26) as

$$\Sigma(\not p)=\tilde{\Sigma}(p)-\tilde{\Sigma}(\not p=m)-(\not p-m)\tilde{\Sigma}'(\not p=m) \tag{5.5}$$

and again,

$$\tilde{\Sigma}(x-x')=-ie^2\langle Tf(x)\bar{f}(x')\rangle \tag{5.6}$$

FIG. 4. The unrenormalized Feynman diagram (a) gives rise to an unrenormalized vertex diagram.

is a T product.

Let us now compute Eq. (5.6) to second order,

$$\begin{aligned}\tilde{\Sigma}^{(2)}(x-x')&=-ie^2\langle T:a(x)\phi(x)::\bar{\phi}(x')a(x'):\rangle\\&=-ie^2\gamma^\mu S^c(x-x')\gamma_\mu D^c(x-x')\ ,\end{aligned} \tag{5.7}$$

or, in p space

$$\tilde{\Sigma}^{(2)}(p)=-\frac{ie^2}{(2\pi)^4}\int \gamma^\mu S^c(\not p-\not q)\gamma_\mu D^c(q)d^4q \tag{5.8}$$

which corresponds to the (unrenormalized) Feynman diagram in Fig. 4(a). Substituting this in expression (5.5), we obviously get the correct renormalized proper energy, Eq. (4.11).

Let us now calculate the corresponding contribution of Eq. (5.7) to the vertex, taking the functional derivative

$$e\Lambda_\mu(x,y;\xi)=\frac{\delta\Sigma(x,y)}{\delta a^\mu(\xi)}\ . \tag{5.9}$$

Using Eq. (4.2), we obtain the third-order diagram in Fig. 4(b) as

$$\begin{aligned}e\tilde{\Lambda}_\mu^{(3)}(p';p)=&-\frac{ie^3}{(2\pi)^4}\\&\times\int \gamma^\nu S^c(\not p'-\not q)\gamma_\mu S^c(\not p-\not q)\gamma_\nu D^c(q)dq\ .\end{aligned} \tag{5.10}$$

From Eqs. (5.5) and (5.9) we get

$$e\Lambda_\mu^{(3)}(p',p)=e\tilde{\Lambda}_\mu^{(3)}(p',p)+e\gamma_\mu B^{(2)}\ , \tag{5.11}$$

where

$$B^{(2)}=\tilde{\Sigma}'^{(2)}(\not p=m)\ . \tag{5.12}$$

This is obviously the usual renormalized result for the lowest-order radiative vertex correction. (The vacuum polarization term in the vertex appears in this instance from the bubble diagram in $\tilde{\Sigma}[x_1,x_2;\delta a_\mu]$, which does not vanish for an external field.[13,18])

We notice, *en passant*, that, from Eq. (5.9) the Ward identity is fulfilled:

$$B=\tilde{\Sigma}'(\not p=m)\ . \tag{5.13}$$

VI. DISCUSSION

The LSZ formulation has been used as a starting point to obtain a consistent renormalized field theory in the past.[24–28] The usual starting point is the generalized unitarity condition[5,8,9] from which a parametric dispersion

relation can be obtained.[19] In order to define the theory uniquely, one has to *impose* the number of subtractions in the dispersion relations. A similar approach is used by Fried,[24] deriving a perturbation theory for retarded propagators, instead of subtracted dispersion relations. An alternative approach was followed by Pugh,[25] Chen,[26] and Wray,[27] who start off with the asymptotic conditions[5] for the free fields, together with the causality condition.[3,7] In this case, the nonuniqueness appears in terms of the solutions to the homogeneous equations for the (amputated) propagators, and an infinite number of boundary conditions have to be imposed in terms of the asymptotic values of the momenta,[25] or on the mass shell.[27] The work of Steinmann[28] gives a rigorous mathematical justification to these approaches, in the framework of perturbation theory, exploiting the generalized unitarity conditions for the retarded propagators, and relating the ambiguities to the renormalization-group approach,[3] corresponding to finite renormalizations of the mass and the coupling constant.

Our own approach shares with the above-mentioned ones the advantage of yielding a simple, logical, and coherent picture, without having to introduce (unphysical) cutoffs or renormalizations, dealing directly with the physical propagators and fields. In the present paper we have shown, however, that the physical condition of particle stability, derivable from the LSZ postulates, is enough to develop a coherent formulation of quantum electrodynamics, with no need to impose any further boundary or normalization conditions. It should also be evident that similar results are obtained for the retarded propagators, by simply using the corresponding definition. The causal propagators enjoy the advantage of being related to the functional derivatives in a straightforward fashion.

Our approach seems also to be related to Schwinger's source theory,[29] who uses first quantized fields in an iterative way. Schwinger shows (at least to second order) that the correct expressions for the electron and the photon propagators are obtained by expressing them in terms of external sources. The "source coupling" form[29] introduces subsequently a product of two free propagators in the respective spectral representations, much in the same way as they appear in Eqs. (3.13) and (3.26) of the present paper. The use of the Yang-Feldman equation for the field, in terms of the interaction (instead of external sources) allows us, however, to obtain a closed result, independent of perturbation theory.

A final point which we would like to emphasize is the fact that, in our case, the starting point for the chain of functional derivatives is not the vacuum-vacuum amplitude (or generating functional).[13,21] Instead, we have started with the *two-point propagators.* In this way, the interaction is defined in a unique way[19] and what we compute is the modification of the free propagators due to the interaction. It is not clear how to define a vacuum-vacuum amplitude free of ambiguities (even with a renormalized Lagrangian), since the renormalization conditions are imposed on the two-point propagators in the usual theory.

In conclusion, a coherent and systematic picture of QED has been obtained by defining current densities which obey the one-particle stability condition, and taking the integral form of the equations of motion as a starting point. The commutation relations for the interacting fields are computed *a posteriori,* instead of being assumed, as is done in canonical theory. In this way we work with "renormalized" fields from the very beginning. The resulting expressions for the two-point causal propagators turn out to be the correct ones. They are obtained as spectral representations of the source-source correlation functions with the appropriate subtractions. These expressions do not have any ambiguities and fulfill automatically the "renormalization conditions."[1-3] This result is a direct consequence of the use of the vacuum and the one-particle stability conditions.

In order to have a closed system of equations, the source terms are expressed in terms of the S-matrix, which in turn follows from the propagators through the usual reduction formulas.

The rest of the n-point functions are computed by taking the appropriate functional derivatives, thus inheriting the correct boundary conditions, and no quasilocal operators remain undetermined.[3,7,8,29]

We should also add that we have not made use of the generalized unitarity relations for the propagators in our procedure.

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