Resonant vibrations of free cylinders and disks*

S. N. Rasband

Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602
(Received 29 March 1974; revised 10 January 1975)

A complete solution is obtained for nonaxisymmetric resonant vibrations of a free cylinder or disk involving infinite sums. For axisymmetric longitudinal vibrations an alternative to previous solutions is included. In principle, the solutions satisfy exactly the stress-free boundary conditions, in contrast to the approximate bending-mode solutions due to Pickett or approximate solutions based on a small diameter/length ratio and small shearing stresses at the ends.

Subject Classification: 40.26: 40.24, 20.40.

INTRODUCTION

A recent application of resonating cylinders has been their use as gravitational wave detectors.[1] To date only the lowest axisymmetric longitudinal mode has been of interest. However, with the anticipated advent of cryogenically cooled aluminum cylinders weighing several tons a large number of cylinder modes may prove useful. Consequently, the need for a detailed understanding of these modes provides the primary motivation for this study.

Vibrations of rods and cylinders have been studied ever since the early work of Pochhammer in 1876 and Chree in 1889 which focused on the infinite-length case.[2] The extension of their work to actual cylinders of finite length has been difficult, however, because it is not possible to completely satisfy the zero stress boundary conditions in a simple way. Zemanek[3] has given a recent account of elastic waves in cylinders in which the semi-infinite case is solved and vibrations other than just the axisymmetric longitudinal ones are considered.

For flexural vibrations Pickett[4] has given an analysis leading to the eigenfrequencies for finite cylinders. This solution is also not shear free everywhere on the plane end faces, but it does have vanishing total shear and total moment on the ends. The numerical solutions of Pickett's equations have been studied by Tefft[5] and by Gram, Douglass, and Tyson[6] and appear in good agreement with experiment.

Hutchinson[7] has given a completely shear free solution for axisymmetric longitudinal vibrations of finite cylinders. Many important aspects of the present work have their root in Hutchinson's solution.

Section I presents the solutions to the differential equations of elasticity in cylindrical geometry with the attendant notational conventions. Section II considers the boundary problem and its solution. In Sec. III the special case of axisymmetric modes is considered with discussion in Sec. IV.

I. SOLUTIONS TO THE DIFFERENTIAL EQUATIONS

The notation for the geometrical and physical parameters of the cylinder follow. L and R denote the length and radius of the cylinder, respectively. We denote σ as Poisson's ratio, E as Young's modulus, $\mu = E/2(1+\sigma)$ as shear modulus, $\hat{\rho}$ as density, $C_t = (\mu/\hat{\rho})^{1/2}$ as transverse (shear) wave velocity, C_l as longitudinal wave velocity, and $\lambda = C_t/C_l = [(1-2\sigma)/2(1-\sigma)]^{1/2}$ as the ratio of the transverse to longitudinal wave velocity.

We use as cylindrical coordinates ρ, ϕ, z with the dimensionless forms being given by $r=\rho/R$, ϕ, $\zeta = z/R$. The angular frequency is given by ω and the dimensionless form by $\overline{\omega} = \omega R/C_t$. U_r, U_ϕ, U_z denote the components of the displacement vector in the radial, azimuthal, and axial directions, respectively.

In standard fashion the displacement vector **U** is expressed as the sum of the gradient of a scalar potential Φ and the curl of a vector potential ψ (components ψ^i, $i = 1, 2, 3$ corresponding to ρ, ϕ, z). The equations of elasticity then give, as differential equations for the potentials, the following:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial\zeta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \overline{\omega}^2\lambda^2\Phi = 0, \tag{1}$$

$$\frac{\partial^2\psi^1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi^1}{\partial r} - \frac{1}{r^2}\psi^1 + \frac{1}{r^2}\frac{\partial^2\psi^1}{\partial\phi^2} - \frac{2\partial R\psi^2}{r\partial\phi^2} + \frac{\partial^2\psi^1}{\partial\zeta^2} + \overline{\omega}^2\psi^2 = 0, \tag{2a}$$

$$\frac{\partial^2\psi^2}{\partial r^2} + \frac{3}{r}\frac{\partial\psi^2}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi^2}{\partial\phi^2} + \frac{\partial^2\psi^2}{\partial\zeta^2} + \frac{2}{Rr^3}\frac{\partial\psi^1}{\partial\phi} + \overline{\omega}^2\psi^2 = 0, \tag{2b}$$

$$\frac{\partial^2\psi^3}{\partial r^2} + \frac{1}{r}\frac{\partial\psi^3}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi^3}{\partial\phi^2} + \frac{\partial^2\psi^3}{\partial\zeta^2} + \overline{\omega}^2\psi^3 = 0\ . \tag{3}$$

We choose the potential solutions in the following form:

$$\Phi(r, \phi, \zeta) = 2AR^2 J_n(\alpha r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}, \tag{4a}$$

where

$$\alpha^2 + \delta^2 = \overline{\omega}^2\lambda^2 ; \tag{4b}$$

$$\psi^1(r, \phi, \zeta) = BR^2(n/r)J_n(\hat{\alpha} r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\sin\beta\zeta\\ -\cos\beta\zeta\end{Bmatrix}, \tag{5a}$$

and

$$R\psi^2(r, \phi, \zeta) = \frac{-BR^2}{r}\frac{\partial J_n}{\partial r}(\hat{\alpha} r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\sin\beta\zeta\\ -\cos\beta\zeta\end{Bmatrix}, \tag{5b}$$

where

$$\hat{\alpha}^2 + \beta^2 = \overline{\omega}^2 ; \tag{5c}$$

$$\psi^3(r, \phi, \zeta) = -CR^2 J_n(\overline{\alpha} r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\gamma\zeta\\ \sin\gamma\zeta\end{Bmatrix}, \tag{6a}$$

where

$$\overline{\alpha}^2 + \gamma^2 = \overline{\omega}^2\ . \tag{6b}$$

A, B, C are arbitrary constant coefficients and the J_n denote Bessel functions of the first kind of order n.

By the usual route from potentials to displacement

TABLE I. Solutions of elasticity equations with arbitrary azimuthal symmetry. Primes denote differentiation with respect to the argument and $\psi_n(x) = xJ_{n-1}(x)/J_n(x) - (n+1)$.

	(A) $\alpha^2+\delta^2=\bar{\omega}^2\lambda^2$	(B) $\hat{\alpha}^2+\beta^2=\bar{\omega}^2$	(C) $\bar{\alpha}^2+\gamma^2=\bar{\omega}^2$
$\frac{U_r}{R}$	$2\alpha J_n'(\alpha r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}$	$\hat{\alpha}\beta J_n'(\hat{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\beta\zeta\\ \sin\beta\zeta\end{Bmatrix}$	$\frac{n}{r}J_n(\bar{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\gamma\zeta\\ \sin\gamma\zeta\end{Bmatrix}$
$\frac{U_\phi}{R}$	$\frac{2n}{r}J_n(\alpha r)\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}$	$\frac{\beta n}{r}J_n(\hat{\alpha}r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\beta\zeta\\ \sin\beta\zeta\end{Bmatrix}$	$\bar{\alpha}J_n'(\bar{\alpha}r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\gamma\zeta\\ \sin\gamma\zeta\end{Bmatrix}$
$\frac{U_z}{R}$	$2\delta J_n(\alpha r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}-\sin\delta\zeta\\ \cos\delta\zeta\end{Bmatrix}$	$\hat{\alpha}^2J_n(\hat{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\sin\beta\zeta\\ -\cos\beta\zeta\end{Bmatrix}$	0
$\frac{\sigma^{rr}}{2\mu}$	$\frac{2J_n(\alpha r)}{r^2}\left\{(n^2-1)+(r^2/2)(2\delta^2-\bar{\omega}^2)-\psi_n(\alpha r)\right\}\times\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}$	$\frac{\beta}{r^2}J_n(\hat{\alpha}r)\left\{(n^2-1)-(\hat{\alpha}r)^2-\psi_n(\hat{\alpha}r)\right\}\times\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\beta\zeta\\ \sin\beta\zeta\end{Bmatrix}$	$\frac{n}{r^2}J_n(\bar{\alpha}r)\psi_n(\bar{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\gamma\zeta\\ \sin\gamma\zeta\end{Bmatrix}$
$\frac{\sigma^{zz}}{2\mu}$	$(2\alpha^2-\bar{\omega}^2)J_n(\alpha r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}$	$\hat{\alpha}^2\beta J_n(\hat{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\cos\beta\zeta\\ \sin\beta\zeta\end{Bmatrix}$	0
$\frac{\sigma^{r\phi}}{\mu}$	$\frac{4n}{r^2}J_n(\alpha r)\psi_n(\alpha r)\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\delta\zeta\\ \sin\delta\zeta\end{Bmatrix}$	$\frac{2n\beta}{r^2}J_n(\hat{\alpha}r)\psi_n(\hat{\alpha}r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\beta\zeta\\ \sin\beta\zeta\end{Bmatrix}$	$\frac{2}{r^2}J_n(\bar{\alpha}r)\left\{(n^2-1)-\frac{(\bar{\alpha}r)^2}{2}-\psi_n(\bar{\alpha}r)\right\}\times\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\cos\gamma\zeta\\ \sin\gamma\zeta\end{Bmatrix}$
$\frac{\sigma^{rz}}{\mu}$	$4\alpha\delta J_n'(\alpha r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\begin{Bmatrix}-\sin\delta\zeta\\ \cos\delta\zeta\end{Bmatrix}$	$\hat{\alpha}(\hat{\alpha}^2-\beta^2)J_n'(\hat{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}\sin\beta\zeta\\ -\cos\beta\zeta\end{Bmatrix}$	$\frac{n\gamma}{r}J_n(\bar{\alpha}r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\begin{Bmatrix}-\sin\gamma\zeta\\ \cos\gamma\zeta\end{Bmatrix}$
$\frac{\sigma^{\phi z}}{\mu}$	$\frac{4\delta n}{r}J_n(\alpha r)\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}\begin{Bmatrix}-\sin\delta\zeta\\ \cos\delta\zeta\end{Bmatrix}$	$(\hat{\alpha}^2-\beta^2)\frac{n}{r}J_n(\hat{\alpha}r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}\sin\beta\zeta\\ -\cos\beta\zeta\end{Bmatrix}$	$\gamma\bar{\alpha}J_n'(\bar{\alpha}r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\begin{Bmatrix}-\sin\gamma\zeta\\ \cos\gamma\zeta\end{Bmatrix}$

vector, from displacement vector to strain tensor, from strain tensor to stress tensor via Hooke's law we obtain the solutions as given in Table I. The functions in brackets, for example, in Eq. 4a, are alternative independent solutions. Thus Eq. 4a summarizes four independent solutions for the potential Φ.

II. BOUNDARY CONDITIONS

In the past the major obstacle to obtaining a complete solution for the resonant vibrations of a cylinder has been the boundary conditions. In terms of stress tensor components, these boundary conditions are as follows:

$$\sigma^{rr}(r=1)=0, \tag{7a}$$

$$\sigma^{r\phi}(r=1)=0, \tag{7b}$$

$$\sigma^{rz}(r=1)=0, \tag{7c}$$

$$\sigma^{zz}(\zeta=\pm h)=0, \tag{7d}$$

$$\sigma^{rz}(\zeta=\pm h)=0, \tag{7e}$$

$$\sigma^{\phi z}(\zeta=\pm h)=0, \tag{7f}$$

where $h=L/2R$.

In order to satisfy Eqs. 7, it is necessary to take an infinite sum of solutions, each in turn partially satisfying the boundary conditions. The infinite sum is then required to be orthogonal to a complete set of functions over the appropriate interval. For $0\leq r\leq 1$ we choose Bessel functions and for $-h\leq\zeta\leq h$ we choose the Fourier functions, sines and cosines. To obtain the solutions partially satisfying the boundary conditions we proceed as follows.

Let $\delta=\beta=\gamma$ and then make the notational change $\delta=\beta=\gamma\to\alpha$, $\alpha\to\delta$, $\hat{\alpha}=\bar{\alpha}\to\beta$. The solutions of Table I take on the forms given in Table II. We then take a superposition of the (A)-, (B)-, (C)-type solutions and substitute into Eqs. 7. The resulting boundary condition equations for the upper ϕ function are the following:

$$\left\{2AJ_n(\delta)\left[(n^2-1)+\frac{(\alpha^2-\beta^2)}{2}-\psi_n(\delta)\right]+B\alpha J_n(\beta)\times\left[(n^2-1)-\beta^2-\psi_n(\beta)\right]+CnJ_n(\beta)\psi_n(\beta)\right\}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}=0; \tag{8a}$$

$$\left\{2AnJ_n(\delta)\psi_n(\delta)+Bn\alpha J_n(\beta)\psi_n(\beta)+CJ_n(\beta)\left[(n^2-1)-(\beta^2/2)-\psi_n(\beta)\right]\right\}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}=0; \tag{8b}$$

$$\left\{4A\alpha\delta J_n'(\delta)+B\beta(\alpha^2-\beta^2)J_n'(\beta)+Cn\alpha J_n(\beta)\right\}\times\begin{Bmatrix}\sin\alpha\zeta\\ \cos\alpha\zeta\end{Bmatrix}=0; \tag{8c}$$

$$\left\{A(2\delta^2-\omega^2)J_n(\delta r)+B\beta^2\alpha J_n(\beta r)\right\}\begin{Bmatrix}\cos\alpha h\\ \sin\alpha h\end{Bmatrix}=0; \tag{8d}$$

$$\left\{4A\alpha\delta J_n'(\delta r)+B\beta(\alpha^2-\beta^2)J_n'(\beta r)+\frac{Cn\alpha}{r}J_n(\beta r)\right\}\begin{Bmatrix}\sin\alpha h\\ \cos\alpha h\end{Bmatrix}=0; \tag{8e}$$

TABLE II. Specialized solutions of elasticity equations with arbitrary azimuthal symmetry. The solutions listed in this table are obtained from those of Table I by letting $\delta=\beta=\gamma\rightarrow\alpha$, $\hat{\alpha}=\overline{\alpha}\rightarrow\beta$, and $\alpha\rightarrow\delta$.

	(A)	(B)	(C)	$\alpha^2+\delta^2=\overline{\omega}^2\lambda^2$ $\alpha^2+\beta^2=\overline{\omega}^2$
$\frac{U_r}{R}$	$2\delta J_n'(\delta r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\alpha\beta J_n'(\beta r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$n\frac{J_n(\beta r)}{r}\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}$
$\frac{U_\phi}{R}$	$2n\frac{J_n(\delta r)}{r}\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}$	$\alpha n\frac{J_n(\beta r)}{r}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\beta J_n'(\beta r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}$
$\frac{U_z}{R}$	$2\alpha J_n(\delta r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\beta^2 J_n(\beta r)\begin{Bmatrix}-\sin n\phi\\ \cos n\phi\end{Bmatrix}$	0	$\begin{Bmatrix}-\sin\alpha\zeta\\ \cos\alpha\zeta\end{Bmatrix}$
$\frac{\sigma^{rr}}{2\mu}$	$\frac{2J_n(\delta r)}{r^2}[(n^2-1)+\frac{r^2}{2}(\alpha^2-\beta^2)-\psi_n(\delta r)]\times\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\frac{\alpha J_n(\beta r)}{r^2}[(n^2-1)-(\beta r)^2-\psi_n(\beta r)]\times\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$\frac{nJ_n(\beta r)}{r^2}\psi_n(\beta r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}$
$\frac{\sigma^{zz}}{2\mu}$	$(2\delta^2-\overline{\omega}^2)J_n(\delta r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\alpha\beta^2 J_n(\beta r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	0	$\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}$
$\frac{\sigma^{r\phi}}{\mu}$	$\frac{4n}{r^2}J_n(\delta r)\psi_n(\delta r)\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}$	$\frac{2n\alpha}{r^2}J_n(\beta r)\psi_n(\beta r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\frac{2}{r^2}J_n(\beta r)[(n^2-1)-\frac{(\beta r)^2}{2}-\psi_n(\beta r)]\times\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}$
$\frac{\sigma^{rz}}{\mu}$	$4\alpha\delta J_n'(\delta r)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\beta(\alpha^2-\beta^2)J_n'(\beta r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$\frac{n\alpha}{r}J_n(\beta r)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}$	$\begin{Bmatrix}-\sin\alpha\zeta\\ \cos\alpha\zeta\end{Bmatrix}$
$\frac{\sigma^{\phi z}}{\mu}$	$\frac{4\alpha n}{r}J_n(\delta r)\begin{Bmatrix}\cos n\phi\\ -\sin n\phi\end{Bmatrix}$	$(\alpha^2-\beta^2)\frac{n}{r}J_n(\beta r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\alpha\beta J_n'(\beta r)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$\begin{Bmatrix}-\sin\alpha\zeta\\ \cos\alpha\zeta\end{Bmatrix}$

$$\left\{\frac{4A\alpha n}{r}J_n(\delta r)+B(\alpha^2-\beta^2)(n/r)J_n(\beta r)+C\alpha\beta J_n'(\beta r)\right\}\times\begin{Bmatrix}\sin\alpha h\\ \cos\alpha h\end{Bmatrix}=0\,. \tag{8f}$$

Hereafter we speak of even and odd solutions according to whether or not U_z is even or odd. Even (odd) solutions correspond to the lower (upper) ζ function.

Choosing two from among Eqs. 8a–8c, the constants A, B, C are determined to within a common factor. Choosing Eqs. 8a and 8b, then $(A, B, C)=(\overline{A}, \overline{B}, \overline{C})$, where

$$\overline{A}=\alpha J_n(\beta)\{[(n^2-1)-\beta^2/2-\psi_n(\beta)][(n^2-1)-\beta^2-\psi_n(\beta)]-n^2\psi_n^2(\beta)\}\,, \tag{9a}$$

$$\overline{B}=2J_n(\delta)\{[\psi_n(\beta)+\beta^2/2-(n^2-1)]\times[(n^2-1)+(\alpha^2-\beta^2)/2-\psi_n(\beta)]+n^2\psi_n(\beta)\psi_n(\delta)\}\,, \tag{9b}$$

$$\overline{C}=2\alpha nJ_n(\delta)\{[(n^2-1)+(\alpha^2-\beta^2)/2\,[\psi_n(\beta)-\psi_n(\delta)]+(\overline{\omega}^2/2)\psi_n(\delta)\}\,. \tag{9c}$$

Choosing Eqs. 8a and 8c, then $(A, B, C)=(\tilde{A}, \tilde{B}, \tilde{C})$, where

$$\tilde{A}=nJ_n(\beta)\{-\alpha^2[(n^2-1)-\beta^2-\psi_n(\beta)]+(\alpha^2-\beta^2)\psi_n(\beta)[\psi_n(\beta)+1]\}\,, \tag{10a}$$

$$\tilde{B}=2\alpha nJ_n(\delta)\{[(n^2-1)+(\alpha^2-\beta^2)/2-\psi_n(\delta)]-2\psi_n(\beta)[\psi_n(\beta)+1]\}\,, \tag{10b}$$

$$\tilde{C}=2J_n(\delta)(\psi_n(\delta)\{2\alpha^2[(n^2-1)-\beta^2]+(\alpha^2-\beta^2)\}-\psi_n(\beta)\{(\alpha^2-\beta^2)[(n^2-1)+(\alpha^2-\beta^2)/2]+2\alpha^2\}-\overline{\omega}^2[\psi_n(\delta)\psi_n(\beta)-(n^2-1)+\overline{\omega}^2/2])\,. \tag{10c}$$

Equations 8b and 8c give $(A, B, C)=(\hat{A}, \hat{B}, \hat{C})$, where

$$\hat{A}=J_n(\beta)\{(\alpha^2-\beta^2)[\psi_n(\beta)+1][(n^2-1)-\beta^2/2-\psi_n(\beta)]-n^2\alpha^2\psi_n(\beta)\}\,, \tag{11a}$$

$$\hat{B}=2\alpha J_n(\delta)\{2[\psi_n(\delta)+1][\psi_n(\beta)+\beta^2/2-(n^2-1)]+n^2\psi_n(\delta)\}\,, \tag{11b}$$

$$\hat{C}=2nJ_n(\delta)\{2\alpha^2\psi_n(\beta)-(\alpha^2-\beta^2)\psi_n(\delta)+\overline{\omega}^2\psi_n(\beta)\psi_n(\delta)\}\,. \tag{11c}$$

Partial satisfaction of boundary conditions of Eqs. 8d–8f are obtained by letting $\alpha=\mu_m\equiv m\pi/h$ or $\alpha=\nu_m\equiv(2m+1)\pi/2h$, with $m=0, 1, 2, \ldots$.

Table III enumerates the different solutions obtained, each of which satisfies the boundary conditions of Eqs. 7a–7f in part. Repeating the previous steps of this section for the lower ϕ function leads to the same results. The solution to which we apply the condition of orthogonality is obtained by adding together all of the solutions of Table III.

We let $\overline{f}_n(r;\alpha)=2\beta\overline{A}J_n'(\beta r)+\overline{B}\alpha\beta J_n'(\beta r)+(\overline{C}n/r)J_n(\beta r)$. $\tilde{f}_n(r;\alpha)$ and $\hat{f}_n(r;\alpha)$ have similar definitions but with a tilde or caret over the A, B, C functions (cf. Eqs. 9–11). Using this definition we write out as an example the complete solution for U_r, omitting the sinusoidal function of time.

TABLE III. Coefficients for solutions partially satisfying boundary conditions.

Odd (upper)					Even (lower)				
Coefficient	α	(A)	(B)	(C)	Coefficient	α	(A)	(B)	(C)
A	μ_m	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\mathfrak{A}$	μ_m	$\bar{A}$	$\bar{B}$	$\bar{C}$
B	μ_m	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\mathfrak{B}$	μ_m	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
C	μ_m	$\hat{A}$	$\hat{B}$	$\hat{C}$	$\mathfrak{C}$	μ_m	$\hat{A}$	$\hat{B}$	$\hat{C}$
D	ν_m	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\mathfrak{D}$	ν_m	$\bar{A}$	$\bar{B}$	$\bar{C}$
E	ν_m	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\mathfrak{E}$	ν_m	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
F	ν_m	$\hat{A}$	$\hat{B}$	$\hat{C}$	$\mathfrak{F}$	ν_m	$\hat{A}$	$\hat{B}$	$\hat{C}$

$$\frac{U_r}{R}=\sum_{n=0}^{\infty}\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}\sum_{m=0}^{\infty}\{A_{nm}\bar{f}_n(r;\mu_{m+1})\cos\mu_{m+1}\zeta$$
$$+[B_{nm}\tilde{f}_n(r;\mu_m)+C_{nm}\hat{f}_n(r;\mu_m)]\cos\mu_m\zeta$$
$$+[D_{nm}\bar{f}_n(r;\nu_m)+E_{nm}\tilde{f}_n(r;\nu_m)+F_{nm}\hat{f}_n(r;\nu_m)]\cos\nu_m\zeta$$
$$+\mathfrak{A}_{nm}\bar{f}_n(r;\mu_m)\sin\mu_m\zeta+[\mathfrak{B}_{nm}\tilde{f}_n(r;\mu_{m+1})$$
$$+\mathfrak{C}_{nm}\hat{f}_n(r;\mu_{m+1})]\sin\mu_{m+1}\zeta+[\mathfrak{D}_{nm}\bar{f}_n(r;\nu_m)$$
$$+\mathfrak{E}_{nm}\tilde{f}_n(r;\nu_m)+\mathfrak{F}_{nm}\hat{f}_n(r;\nu_m)]\sin\nu_m\zeta. \tag{12}$$

The stress tensor components are formed in the same fashion as Eq. 12 and then are required to satisfy the boundary conditions of Eqs. 7 by being orthogonal to the appropriate complete set. The equations

$$\int_{-h}^{h}\sigma^{rr}(r=1,\zeta)\sin\nu_k\zeta\,d\zeta=0 \tag{13a}$$

and

$$\int_{-h}^{h}\sigma^{rr}(r=1,\zeta)\cos\mu_k\zeta\,d\zeta=0 \tag{13b}$$

lead, respectively, to the following equations for the coefficients, where the subscript n is hence forth suppressed.

$$\sum_m\hat{C}_{km}\mathfrak{C}_m+\mathfrak{F}_k\hat{F}_k=0 \tag{14a}$$

and

$$C_k\hat{C}_k+\sum_m\hat{F}_{km}F_m=0, \tag{14b}$$

where

$$C\binom{\mu}{\nu}=2\hat{A}J_n(\delta)\left[(n^2-1)+\frac{(\alpha^2-\beta^2)}{2}-\psi_n(\delta)\right]$$
$$+\hat{B}\alpha J_n(\beta)[(n^2-1)-\beta^2-\psi_n(\beta)]+\hat{C}nJ_n(\beta)\psi_n(\beta). \tag{15}$$

The choice of μ or ν is made depending on whether α equals μ or ν. We have defined

$$\hat{C}_k=2^jC(\mu_k)h(-1)^k \tag{16a}$$

and

$$\hat{F}_k=C(\nu_k)h(-1)^k, \tag{16b}$$

where

$$j=\begin{cases}1, & k=0\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

Also we use

$$\hat{C}_{km}=C(\mu_{m+1})\mu_{m+1}\tau_{k,m+1} \tag{18a}$$

and

$$\hat{F}_{km}=C(\nu_m)\tau_{mk}\nu_m, \tag{18b}$$

where

$$\tau_{km}=2(-1)^k/(\nu_k^2-\mu_m^2). \tag{19}$$

The equations

$$\int_{-h}^{h}\sigma^{r\phi}(r=1,\zeta)\sin\nu_k\zeta=0 \tag{20a}$$

and

$$\int_{-h}^{h}\sigma^{r\phi}(r=1,\zeta)\cos\mu_k\zeta=0 \tag{20b}$$

lead, respectively, to the following equations for the coefficients:

$$\sum_m\hat{B}_{km}\mathfrak{B}_m+\mathfrak{E}_k\hat{E}_k=0 \tag{21a}$$

and

$$B_k\hat{B}_k+\sum_m\hat{E}_{km}E_m=0, \tag{21b}$$

where

$$\hat{B}_k=2^jB(\mu_k)h(-1)^k, \tag{22a}$$
$$\hat{E}_k=B(\nu_k)h(-1)^k, \tag{22b}$$
$$\hat{B}_{km}=B(\mu_{m+1})\mu_{m+1}\tau_{k,m+1}, \tag{23a}$$
$$\hat{E}_{km}=B(\nu_m)\tau_{mk}\nu_m, \tag{23b}$$

with

$$B\binom{\mu}{\nu}=2\tilde{A}nJ_n(\delta)\psi_n(\delta)+\tilde{B}\alpha nJ_n(\beta)\psi_n(\beta)$$
$$+\tilde{C}J_n(\beta)[(n^2-1)-(\beta^2/2)-\psi_n(\beta)]. \tag{24}$$

In Eq. 24 μ or ν is chosen depending on whether α equals μ or ν. In similar fashion, the equations

$$\int_{-h}^{h}\sigma^{rz}(r=1,\zeta)\sin\nu_k\zeta\,d\zeta=0 \tag{25a}$$

and

$$\int_{-h}^{h}\sigma^{rz}(r=1,\zeta)\cos\mu_k\zeta\,d\zeta=0 \tag{25b}$$

lead to

$$\sum_m\hat{A}_{km}A_m+\hat{D}_kD_k=0 \tag{26a}$$

and

$$\mathfrak{A}_k\hat{A}_k+\sum_m\hat{D}_{km}\mathfrak{D}_m=0. \tag{26b}$$

We have defined

$$\hat{A}_k=2^jA(\mu_k)h(-1)^k, \tag{27a}$$
$$\hat{D}_k=A(\nu_k)h(-1)^k, \tag{27b}$$
$$\hat{A}_{km}=A(\mu_{m+1})\mu_{m+1}\tau_{k,m+1}, \tag{28a}$$

$$\hat{D}_{km} = A(\nu_m)\nu_m\tau_{mk}\,, \tag{28b}$$

where

$$A\binom{\mu}{\nu} = 4\bar{A}\alpha\delta J_n'(\delta) + \bar{B}\beta(\alpha^2-\beta^2)J_n'(\beta) + \bar{C}n\alpha J_n(\beta)\,, \tag{29}$$

with again the choice of μ or ν determined by α.

For the remaining boundary conditions, we make use of the functions $r^{1/2}J_n(y_k^n r)$ which are orthogonal over the interval $0 \leq r \leq 1$, where y_k^n denotes the kth root of $J_n'(x)=0$. Rather than using Eq. 7d in the form given we use the two equivalent equations.

$$\sigma^{zz}(\zeta = h) + \sigma^{zz}(\zeta = -h) = \sigma^{zz}_{\text{even}}(\zeta = h) = 0 \tag{30a}$$

and

$$\sigma^{zz}(\zeta = h) - \sigma^{zz}(\zeta = -h) = \sigma^{zz}_{\text{odd}}(\zeta = h) = 0 \;. \tag{30b}$$

Thus we require

$$\int_{-h}^{h} r\sigma^{zz}_{\text{even}}(r, \zeta = h)J_n(y_k^n r)\,dr = 0 \tag{31a}$$

and

$$\int_{-h}^{h} r\sigma^{zz}_{\text{odd}}(r, \zeta = h)J_n(y_k^n r)\,dr = 0\,. \tag{31b}$$

These yield the following equations for the coefficients:

$$\sum_m A_m\hat{A}_{km}(\mu_{m+1}) + B_m\hat{B}_{km}(\mu_m) + C_m\hat{C}_{km}(\mu_m) = 0 \tag{32a}$$

and

$$\sum_m \mathfrak{D}_m\hat{A}_{km}(\nu_m) + \mathfrak{E}_m\hat{B}_{km}(\nu_m) + \mathfrak{F}_m\hat{C}_{km}(\nu_m) = 0\,, \tag{32b}$$

where

$$\hat{A}_{km}\binom{\mu}{\nu} = J_n(y_k^n)\left\{\frac{\bar{A}(2\delta^2-\bar{\omega}^2)\delta J_n'(\delta)}{(y_k^n)^2-\delta^2} + \frac{\bar{B}\alpha\beta^3 J_n'(\beta)}{(y_k^n)^2-\beta^2}\right\}\;. \tag{33}$$

Once again the choice of μ or ν depends on whether α equals μ_m or ν_m.

$$\hat{B}_{km}\binom{\mu}{\nu} \text{ and } \hat{C}_{km}\binom{\mu}{\nu}$$

are given by Eq. 33 with $(\bar{A}, \bar{B}) \to (\tilde{A}, \tilde{B})$ and $(\hat{A}, \hat{B})$, respectively.

As was done with Eqs. 7d, we focus on the even and odd parts of Eqs. 7e and 7f. We find

$$\sigma^{rz}_{\text{even}}(r, \zeta = h) = 0\,, \tag{34a}$$

$$\sigma^{rz}_{\text{odd}}(r, \zeta = h) = 0\,, \tag{34b}$$

$$\sigma^{\phi r}_{\text{even}}(r, \zeta = h) = 0\,, \tag{34c}$$

$$\sigma^{\phi z}_{\text{odd}}(r, \zeta = h) = 0\,. \tag{34d}$$

Because of the nature of the Bessel functions occurring in these equations it is more convenient to consider the sum and difference of Eqs. 34a and 34c and likewise the sum and difference of Eqs. 34b and 34d. The use of recursion relations makes for considerable simplification. We find that

$$\int_{-0}^{1} r[\sigma^{rz}_{\text{even}} + \sigma^{\phi z}_{\text{even}}]J_{n-1}(y_k^{n-1}r)\,dr = 0 \tag{35a}$$

and

$$\int_{-0}^{1} r[\sigma^{rz}_{\text{even}} - \sigma^{\phi z}_{\text{even}}]J_{n+1}(y_k^{n+1}r)\,dr = 0 \tag{35b}$$

result in the following equations for the coefficients:

$$\sum_m \mathfrak{A}_m U_{km}(\mu_m) + \mathfrak{B}_m V_{km}(\mu_{m+1}) + \mathfrak{C}_m W_{km}(\mu_{m+1}) = 0 \tag{36a}$$

and

$$\sum_m \mathfrak{A}_m R_{km}(\mu_m) + \mathfrak{B}_m S_{km}(\mu_{m+1}) + \mathfrak{C}_m T_{km}(\mu_{m+1}) = 0\,, \tag{36b}$$

where

$$U_{km}\binom{\mu}{\nu} = J_{n-1}(y_k^{n-1})\left\{\frac{4\bar{A}\alpha\delta^2 J_{n-1}'(\delta)}{(y_k^{n-1})^2-\delta^2} + \frac{[\bar{B}(\alpha^2-\beta^2)+\bar{C}\alpha]\beta^2 J_{n-1}'(\beta)}{(y^{n-1})^2-\beta^2}\right\}\,. \tag{37}$$

$$V_{km}\binom{\mu}{\nu} \text{ and } W_{km}\binom{\mu}{\nu}$$

are obtained from Eq. 37 by letting $(\bar{A}, \bar{B}, \bar{C}) \to (\tilde{A}, \tilde{B}, \tilde{C})$ and $(\hat{A}, \hat{B}, \hat{C})$, respectively.

$$R_{km}\binom{\mu}{\nu},\quad S_{km}\binom{\mu}{\nu},\quad \text{and}\quad T_{km}\binom{\mu}{\nu}$$

are obtained from

$$U_{km}\binom{\mu}{\nu},\quad V_{km}\binom{\mu}{\nu},\quad \text{and}\quad W_{km}\binom{\mu}{\nu}\,,$$

respectively, by letting $(n-1) \to (n+1)$. Again the choice of μ or ν is determined by α.

The requirements that

$$\int_{0}^{1} r[\sigma^{rz}_{\text{odd}} + \sigma^{\phi z}_{\text{odd}}]J_{n-1}(y_k^{n-1}r)\,dr = 0 \tag{38a}$$

and

$$\int_{0}^{1} r[\sigma^{rz}_{\text{odd}} - \sigma^{\phi z}_{\text{odd}}]J_{n+1}(y_k^{n+1}r)\,dr = 0 \tag{38b}$$

yield for the coefficients

$$\sum_m D_m U_{km}(\nu_m) + E_m V_{km}(\nu_m) + F_m W_{km}(\nu_m) = 0 \tag{39a}$$

and

$$\sum_m D_m R_{km}(\nu_m) + E_m S_{km}(\nu_m) + F_m T_{km}(\nu_m) = 0\,. \tag{39b}$$

Equations 14b, 21b, 26a, 32a, 39a, and 39b are conveniently summarized in matrix form.

$$\begin{bmatrix} \hat{A}_{km} & 0 & 0 & \hat{D}_k & 0 & 0 \\ 0 & \hat{B}_k & 0 & 0 & \hat{E}_{km} & 0 \\ 0 & 0 & \hat{C}_k & 0 & 0 & \hat{F}_{km} \\ \hat{A}_{km}(\mu_{m+1}) & \hat{B}_{km}(\mu_m) & \hat{C}_{km}(\mu_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{km}(\nu_m) & V_{km}(\nu_m) & W_{km}(\nu_m) \\ 0 & 0 & 0 & R_{km}(\nu_m) & S_{km}(\nu_m) & T_{km}(\nu_m) \end{bmatrix} \begin{bmatrix} A \\ \vdots \\ B \\ \vdots \\ C \\ \vdots \\ D \\ \vdots \\ E \\ \vdots \\ F \\ \vdots \end{bmatrix} = 0 . \tag{40}$$

Equations 14a, 21a, 26b, 32b, 36a, and 36b are contained in the matrix equation

$$\begin{bmatrix} \hat{A}_k & 0 & 0 & \hat{D}_{km} & 0 & 0 \\ 0 & \hat{B}_{km} & 0 & 0 & \hat{E}_k & 0 \\ 0 & 0 & \hat{C}_{km} & 0 & 0 & \hat{F}_k \\ 0 & 0 & 0 & \hat{A}_{km}(\nu_m) & \hat{B}_{km}(\nu_m) & \hat{C}_{km}(\nu_m) \\ U_{km}(\mu_m) & V_{km}(\mu_{m+1}) & W_{km}(\mu_{m+1}) & 0 & 0 & 0 \\ R_{km}(\mu_m) & S_{km}(\mu_{m+1}) & T_{km}(\mu_{m+1}) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{A} \\ \vdots \\ \mathfrak{B} \\ \vdots \\ \mathfrak{C} \\ \vdots \\ \mathfrak{D} \\ \vdots \\ \mathfrak{E} \\ \vdots \\ \mathfrak{F} \\ \vdots \end{bmatrix} = 0 . \tag{41}$$

Submatrices with only one index are diagonal matrices with their diagonal elements given by the indicated quantity.

For a nontrivial solution, the determinant of the coefficient matrix must vanish, which determines the allowed frequencies. The matrix of Eq. 40 gives the frequencies for odd modes and that of Eq. 41 for the even modes.

III. THE AXISYMMETRIC MODE

The determinants associated with Eqs. 40 and 41 are different of course for each n. The axisymmetric mode $n=0$ requires special attention since some coefficients vanish for $n=0$ (cf., Eqs. 10a and 10b) and since it is in this mode that comparison with previous work[7] is possible.

Examination of the boundary conditions contained in Eqs. 8 shows that for $n=0$ there is a natural splitting. Equations 8b and 8f concern only the (C)-type solution of Table I and the remaining boundary conditions couple the (A) and (B) types together. The boundary conditions for the (C)-type solution are easily satisfied by choosing α equal to μ_m or ν_m and $\bar{\omega}$ satisfying $\beta^2 J_2(\beta)=0$. These are the well-known torsional oscillations and are not considered further here.

For the axisymmetric longitudinal mode the Eqs. 40 and 41 reduce to

$$\begin{bmatrix} \hat{A}_{km} & 0 & \hat{D}_k & 0 \\ 0 & -\hat{A}_k & 0 & -\hat{D}_{km} \\ \hat{A}_{km}(\mu_{m+1}) & \hat{C}_{km}(\mu_m) & 0 & 0 \\ 0 & 0 & R_{km}(\nu_m) & T_{km}(\nu_m) \end{bmatrix} \begin{bmatrix} A \\ \vdots \\ C \\ \vdots \\ D \\ \vdots \\ F \\ \vdots \end{bmatrix} = 0 \tag{42}$$

and

$$\begin{bmatrix} \hat{A}_k & 0 & \hat{D}_{km} & 0 \\ 0 & -\hat{A}_{km} & 0 & -\hat{D}_k \\ 0 & 0 & \hat{A}_{km}(\nu_m) & \hat{C}_{km}(\nu_m) \\ R_{km}(\mu_m) & T_{km}(\mu_{m+1}) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{U} \\ \vdots \\ \mathfrak{C} \\ \vdots \\ \mathfrak{D} \\ \vdots \\ \mathfrak{F} \\ \vdots \end{bmatrix} = 0 . \tag{43}$$

The matrix coefficients are given by the equations in the previous section specialized to the case $n=0$.

IV. DISCUSSION

In the previous sections a complete solution for the resonating modes of a free disk or cylinder are obtained in the form of an infinite series for each mode. A particular mode is selected by fixing n and then choosing a particular frequency solution to either the even or odd parity equation given by requiring the vanishing of the determinant of Eq. 41 or 40. Thus, a particular mode is labeled by $(n,\pm,l)$. The label n determines the geometric character of the mode. For example, $n=0$ selects the axisymmetric longitudinal or torsional modes (cf. Sec. III, Refs. 2, 3, and 7) and the exact solution of this paper is an alternative to that given by Hutchinson; $n=1$ selects the bending modes [Refs. 4, 5, and 6]; $n=2$ selects the longitudinal modes with quadrupole symmetry. The plus or minus sign denotes mode parity. In contrast to the Pochhammer–Chree solution (the infinite rod solutions as applied to finite cylinders by requiring the normal stress at the ends to vanish but ignoring the tangential stress), it is not possible to determine the radial node structure without a detailed examination to U_z as a function of r.

Exploratory numerical computations have been done to obtain the frequencies giving a vanishing determinant for the matrices in Eqs. 42 and 43. Excellent agreement with Hutchinson's results was obtained at selected points. It should be noted, however, that the number of terms included in the expansions, necessary to obtain equivalent accuracy, was somewhat larger. Also we remark that no false roots appear since terms such as $J_n'(\delta)/[(y_k^n)^2-\delta^2]$ which occur in Eqs. 33, 37, etc. have a finite limit as $\delta \to y_k^n$.

ACKNOWLEDGMENTS

The author wishes to express gratitude to R. G. Hier and B. K. Harrison for helpful discussions. Also appreciation is expressed to J. R. Hutchinson for enlightenment on various aspects of his work.

*This research was supported in part by a grant from the Research Corporation.

[1]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 37.

[2]A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944), pp. 287–292.

[3]J. Zemanek, J. Acoust. Soc. Am. **51**, 265 (1972). For an earlier excellent review, see T. R. Meeker and A. H. Meitzer, "Guided Wave Propagation in Elongated Cylinders and Plates," in *Physical Acoustics*, W. P. Mason, Ed. (Academic, New York, 1964), Vol. 1.

[4]G. Pickett, J. Appl. Phys. **16**, 820 (1945).

[5]W. E. Tefft, J. Res. NBS **64B**, No. 4, 237 (1960).

[6]R. Q. Gram, D. H. Douglass, and J. A. Tyson (preprint).

[7]J. R. Hutchinson, J. Acoust. Soc. Am. **51**, 233 (1972).