Transverse modes of a laser resonator with Gaussian mirrors

Dwight M. Walsh and Larry V. Knight

Analytical methods are presented for transverse mode analysis of a laser resonator having spherical mirrors with a Gaussian reflectivity profile. The modes of this type of resonator have a form similar to that of the conventional Gaussian modes, but it is necessary to define an additional beam parameter to meet the self-consistency requirement for resonator modes. Stability of both the conventional complex beam parameter and the additional parameter is discussed. It is predicted mathematically that small perturbations in the new beam parameter will cause the intensity profile of the higher-order modes to evolve into that of the fundamental mode. Mode losses and discrimination are also discussed. The results may be useful in the design of regenerative laser amplifiers.

I. Introduction

In the design of high-gain laser amplifiers, it is important to obtain oscillation of only the fundamental transverse mode of propagation. The presence of higher-order modes degrades the beam quality, since the intensity of such modes varies rapidly in any cross-sectional plane of the beam. Also, higher-order beam modes are difficult to focus, since much of the mode energy propagates off the resonator axis. For most applications, including laser fusion and the study of laser–plasma interactions, one generally desires a nearly uniform beam that may be focused onto a small target.

Discrimination against higher-order modes is especially important in regenerative amplifiers, since the beam is passed back and forth along exactly the same path. Therefore, perturbations in the beam due to small defects in the optical system tend to grow with each round trip of the beam through the system. Hence the higher-order modes resulting from these perturbations become more manifest in the beam.

In recent years, there has been much interest in developing laser resonators with high diffraction or transmission losses at the resonator mirrors. Such resonators can provide the required mode discrimination, since the losses of the higher-order modes are much greater than the losses of the fundamental mode. Of particular interest are optical resonators with mirrors having a Gaussian reflectivity profile (Gaussian mirrors), since such resonators have a high-quality fundamental mode.[1,2]

Resonators employing Gaussian mirrors were treated by Vakhimov[3] and Zucker[4] by solving the resonator integral equation for the fields at the resonator mirrors. Arnaud[5,6] formulated the resonator problem using the concept of the complex point eikonal. Casperson and Lunman[1] and Yariv and Yeh[2] derived the ABCD matrix for Gaussian apertures or mirrors and treated the problem of optical resonators employing such mirrors with the matrix techniques that describe the propagation of Gaussian beams. Although this method yields the correct fundamental mode, the higher-order modes as stated by Casperson and Lunman fail to satisfy the round-trip condition for resonator modes. These modes do, however, form a complete orthogonal set of confined solutions of the paraxial wave equation. Ganiel and Hardy[7] expanded the actual resonator modes in a series of the modes drived by Casperson and Lunman and solved for the coefficients of the expansion. While this approach is mathematically valid, it is difficult to apply it to obtain reasonably accurate solutions with the precision available on most computers.

In this paper, we preswent another analytical method for obtaining the transverse electromagnetic (TEM) modes of an optical resonator with Gaussian apertures or mirrors. This method involves defining a complex beam parameter in addition to the beam spot size and wave front radius of curvature to satisfy the round-trip condition as well as the paraxial wave equation. Also, we analyze stability and losses of these modes at the resonator mirrors. The methods pre-

The authors are with Brigham Young University, Department of Physics & Astronomy, Provo, Utah 84602.

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sented in each section are illustrated with the example of a simple symmetrical resonator.

II. Resonator Modes

Assuming a homogeneous medium, the paraxial wave equation is

$$\nabla_t^2 u - 2ik\frac{\partial u}{\partial z} = 0 \qquad k = \frac{2\pi n}{\lambda}, \tag{1}$$

where ∇_t^2 is the transverse Laplacian operator. To solve Eq. (1) for the confined modes of propagation, we start with the trial solution

$$u(x,y,z) = f\left[\frac{x}{w(z)}\right] g\left[\frac{y}{w(z)}\right]$$

$$\times \exp\left\{-1\left[P(z) + \frac{k(x^2+y^2)}{2q(z)} + \phi(z)\right]\right\}, \tag{2}$$

where

$$P(z) = -i\ln\left(1+\frac{z}{q_0}\right), \tag{3}$$

$$q(z) = q_0 + z. \tag{4}$$

If we define the real functions $R(z)$ and $W(z)$ so that

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{2i}{kW^2(z)}, \tag{5}$$

$R(z)$ and $W(z)$ may be identified as the wave front radius of curvature and beam spot size, respectively, of the fundamental mode.[1,8] Equation (2) is similar to the trial solution used in Marcuse's alternate Gaussian beam derivation.[9] However, it is assumed in this derivation that the parameter $w(z)$ is equal to the beam spot size $W(z)$. While Marcuse's trial solution leads to confined modes of propagation, these solutions are not sufficiently general to satisfy the round-trip condition for resonators with Gaussian apertures or mirrors. Therefore, we assume at this point that the function $w(z)$ remains to be determined. Henceforth, the function $q(z)$ and $w(z)$ will be referred to as the primary and secondary beam parameters, respectively.

Substituting the trial solution (2) into the wave equation (1), we have

$$\frac{f''}{f} + 2ikx\left(\frac{dw}{dz} - \frac{w}{q}\right)\frac{f'}{f} + \frac{\ddot{g}}{g} + 2iky\left(\frac{dw}{dz} - \frac{w}{q}\right)\frac{\dot{g}}{g} - 2kw^2\frac{d\phi}{dz} = 0, \tag{6}$$

where primes and dots denote differentiation with respect to the arguments x/w and y/w, respectively. If we now require

$$\frac{dw}{dz} - \frac{w}{q} = \frac{2i}{kw} \quad \text{or} \quad \frac{dw^2}{dz} - \frac{2w^2}{q_0+z} = \frac{4i}{k} \tag{7}$$

and make the change of variables $t = (\sqrt{2}x)/w$ and $\tau = (\sqrt{2}y)/w$, Eq. (6) becomes

$$\frac{1}{f}\frac{d^2f}{dt^2} - 2t\frac{df}{dt} + \frac{1}{g}\frac{d^2g}{d\tau^2} - 2\tau\frac{dg}{d\tau} - 2kw^2\frac{d\phi}{dz} = 0. \tag{8}$$

In Eq. (8), the first two terms depend only on t, the next two terms depend only on τ, and the last term on the left-hand side depends only on z. Therefore, we may use the separation of variable technique to obtain

$$\frac{d^2f}{dt^2} - 2t\frac{df}{dt} + 2nf = 0, \tag{9}$$

$$\frac{d^2g}{d\tau^2} - 2\tau\frac{dg}{d\tau} + 2mg = 0, \tag{10}$$

$$\frac{d\phi}{dz} = -2\left(\frac{n+m}{kw^2}\right). \tag{11}$$

The solutions of Eqs. (9)–(11) must lead to confined beam solutions of Eq. (1). Therefore, n and m must both be positive integers, and we identify the solutions of Eqs. (9) and (10) as the Hermite polynomials. Therefore, we have

$$f\left(\frac{x}{w}\right) = H_n\left(\frac{\sqrt{2}x}{w}\right) \quad \text{and} \quad g\left(\frac{y}{w}\right) = H_m\left(\frac{\sqrt{2}y}{w}\right), \tag{12}$$

where H_n and H_m are Hermite polynomials.

Let us now solve Eq. (7) to determine the parameter w^2. Multiplying this equation by the integrating factor $(q_0+z)^{-2}$, we have

$$(q_0+z)^{-2}\frac{dw^2}{dz} - 2w^2(q_0+z)^{-3} = \frac{4i}{k}(q_0+z)^{-2}$$

or

$$\frac{d}{dz}[(q_0+z)^{-2}w^2] = \frac{4i}{k}(q_0+z)^{-2}.$$

Integrating this equation and solving for w^2 give us

$$w^2(z) = (q_0+z)\left[\frac{w_0^2}{q_0} + \left(\frac{w_0^2}{q_0^2} + \frac{4i}{kq_0}\right)z\right], \tag{13}$$

where w_0 is the secondary beam parameter at $z = 0$. At this point, it should be indicated that Eqs. (4) and (5) may be used to show that the spot size $W(z)$ is also a solution of Eq. (7). Therefore, the conventional Gaussian modes form a subset of the more general Gaussian modes derived in this section.

It remains now to determine the function $\phi(z)$. From Eq. (11), we have

$$\phi(z) = -2\int_0^z \frac{n+m}{kw^2(z')}dz' = -(n+m)\eta(z), \tag{14}$$

where

$$\eta(z) = 2q_0^2\int_0^z \frac{dz'}{(q_0+z')[kw_0^2q_0 + (kw_0^2+4iq_0)z']}, \tag{15}$$

and ϕ was arbitrarily chosen to be zero at $z = 0$. This integral may be evaluated by the method of partial fractions. The result is

$$\eta(z) = \frac{i}{2}\ln\left[\frac{kw_0^2(q_0+z)}{q_0kw_0^2 + (kw_0^2+4iq_0)z}\right]. \tag{16}$$

Hence we have determined the higher-order modes of a beam propagating in the positive z direction, assuming the primary and secondary beam parameters are both known at $z = 0$. Gathering our results, we have

$$u_{n,m}(x,y,z) = H_n\left(\frac{\sqrt{2}x}{w}\right) H_m\left(\frac{\sqrt{2}y}{w}\right)$$
$$\times \exp\left\{-i\left[P(z) + \frac{k(x^2+y^2)}{2(q_0+z)} - (n+m)\eta(z)\right]\right\}, \quad (17)$$

where $w(z)$ and $\eta(z)$ are given in Eqs. (13) and (16), respectively.

We may also solve Eq. (1) in cylindrical coordinates to obtain the Laguerre-Gaussian modes. These modes are given by

$$u_{n,m}(r,\theta,z) = \left(\frac{\sqrt{2}r}{w}\right)^m L_n^m\left(\frac{2r^2}{w^2}\right)\sin(m\theta)$$
$$\times \exp\left\{-1\left[P(z) + \frac{kr^2}{2(q_0+z)} - (2n+m)\eta(z)\right]\right\}, \quad (18)$$

where the L_n^m are the associated Laguerre polynomials, and the functions $P(z)$, $w(z)$, and $\eta(z)$ are the same as in Eq. (17).

Let us now consider a high-order Gaussian beam incident on a simple optical element. As indicated in Ref. 8, the optical element transforms the primary beam parameter according to

$$q_{\text{out}} = \frac{Aq_{\text{in}} + B}{Cq_{\text{in}} + D}, \quad (19)$$

where A, B, C, and D are the elements of the ray matrix for that optical element. Such matrices for a few simple optical elements are shown in Fig. 1. Casperson and Lunman[1] show that the ray (or ABCD) matrix for a Gaussian aperture is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{-2i}{kW_m^2} & 1 \end{pmatrix}, \quad (20)$$

where W_m is the Gaussian reflectivity spot size of the mirror. Let us establish a transformation law similar to Eq. (19) for the secondary beam parameter.

From Eq. (13), we see that the secondary beam parameter, after propagating a distance L in the uniform medium, is given by

$$w_{\text{out}}^2 = \left(\frac{q_{\text{in}} + L}{q_{\text{in}}}\right)^2 w_{\text{in}}^2 + \frac{4iL}{k}\left(\frac{q_{\text{in}} + L}{q_{\text{in}}}\right), \quad (21)$$

where q_{in} and w_{in} are the initial beam parameters. (It is assumed here that the primary parameter q_{in} has been determined. This involves the methods discussed in Ref. 1.) The other optical elements considered in this paper affect only the primary parameter q within the plane of the element. Therefore, we expect

$$w_{\text{out}}^2 = w_{\text{in}}^2 \quad (22)$$

for thin lenses, mirrors, interfaces, and Gaussian apertures. Note that Eqs. (21) and (22) may both be written as

$$w_{\text{out}}^2 = \alpha w_{\text{in}}^2 + \beta, \quad (23)$$

where

$$\alpha = \left(\frac{q_{\text{in}} + L}{q_{\text{in}}}\right)^2 \quad \text{and} \quad \beta = \frac{4iL}{k}\left(\frac{q_{\text{in}} + L}{q_{\text{in}}}\right)$$

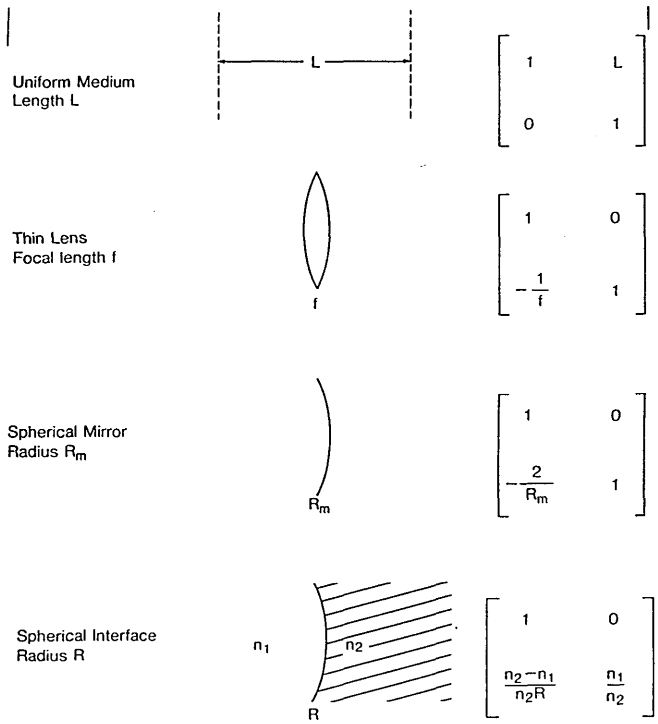

Fig. 1. ABCD matrices for a few simple optical elements. Gaussian beam is assumed to be incident from the left.

for length L in the uniform medium; and $\alpha = 1, \beta = 0$ for all other simple optical elements. Note that Eq. (23) may be cast in the same form as the transformation rule (19) for q[1,2,8] by writing

$$w_{\text{out}}^2 = \frac{\alpha w_{\text{in}}^2 + \beta}{\gamma w_{\text{in}}^2 + \delta},$$

where $\gamma = 0$ and $\delta = 1$. We, therefore, define for each optical element in the system an $\alpha\beta$ matrix given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix},$$

where α and β are the parameters used in Eq. (23). Using the same algebra that leads to the ABCD law for the primary beam parameter, we may show that the $\alpha\beta$ matrix for a sequence of N optical elements is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_N & \beta_N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_{N-1} & \beta_{N-1} \\ 0 & 1 \end{pmatrix}\cdots\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix}, \quad (24)$$

where α_n and β_n are the matrix elements of the individual optical elements, and the subscript n indicates the order in which the optical elements are encountered by the beam. We will refer to Eq. (24) as the $\alpha\beta$ law.

Let us now determine the transverse modes of a resonator which may include Gaussian apertures or mirrors. The self-consistency requirement for resonator modes[1,2] implies that both the primary and secondary beam parameter must repeat themselves after each round trip through the resonator. (If the resonator is symmetric, the beam parameters must repeat themselves after each single pass as well as after each round trip.) Let q_s and w_s be the values of the self-consistent primary and secondary beam parameters in some chosen reference plane. Then, since the beam parameter q must reproduce itself after each round trip, we require

$$q_s = \frac{Aq_s + B}{Cq_s + D} \quad \text{or} \quad \frac{1}{q_s} = \frac{C + D/q_s}{A + B/q_s}, \tag{25}$$

where we have used Eq. (19), and A, B, C, and D are the ray matrix elements for the round trip. Solving Eq. (25) for $1/q_s$,

$$\frac{1}{q_s} = \frac{D - A}{2B} \pm \frac{1}{B}\left[\left(\frac{A + D}{2}\right)^2 - 1\right]^{1/2}, \tag{26}$$

where we have used the relation AD − BC = 1, which applies to all round-trip ray matrices, and the arbitrary sign is chosen so that the spot size W is real (i.e., the imaginary part of $1/q_s$ must be negative). Now let w_s be the self-consistent secondary beam parameter in the chosen reference plane, and let α and β be the round-trip matrix elements for this parameter as determined from the $\alpha\beta$ law. Then, according to the self-consistency requirement and Eq. (23), we have

$$w_s^2 = \alpha w_s^2 + \beta \quad \text{or} \quad \omega_s^2 = \frac{\beta}{1 - \alpha}. \tag{27}$$

At this point, we have determined the self-consistent beam parameters q_s and w_s in a chosen reference plane. If we define $z = 0$ as our chosen reference plane, we may use Eqs. (17) and (18) to obtain the transverse resonator modes.

Consider now a simple symmetrical resonator consisting of two identical Gaussian mirrors facing each other along a common optical axis. Since the resonator is symmetrical, we consider only a single pass of the beam rather than a complete round trip. Casperson and Lunman[1] show that the ray matrix for a single pass of the beam is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{2L}{R_m} - \dfrac{2iL}{kW_m^2} & L \\ -\dfrac{2}{R_m} - \dfrac{2i}{kW_m^2} & 1 \end{pmatrix}, \tag{28}$$

where R_m and W_m are the radius of curvature and Gaussian reflectivity spot size, respectively, of the mirrors, and L is the mirror separation distance. This matrix is obtained by applying the ABCD law to the matrices given in Fig. 1 and Eq. (20). Substituting the matrix elements from Eq. (28) into Eq. (26), we have

$$\frac{L}{q_s} = \frac{L}{R_s} - \frac{2iL}{kW_s^2} = \rho + \frac{i}{2\pi N} \pm \left[\left(1 - \rho - \frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}, \tag{29}$$

where we have defined the Fresnel number as

$$N = \frac{kW_m^2}{2\pi L}$$

and normalized mirror curvature as $\rho = L/R_m$. If we consider Eq. (29) in the uniform mirror limit (i.e., the limit as $N \rightarrow \infty$), we see that a real beam spot size W_s exists only if

$$|1 - \rho| = \left|1 - \frac{L}{R_m}\right| < 1. \tag{30}$$

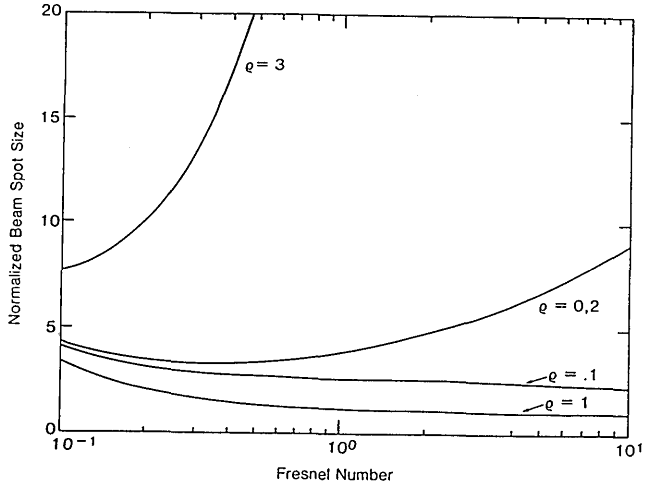

Fig. 2. Plots of the normalized beam spot size $kW_s^2/2L$ vs Fresnel number N for various values of $\rho = L/R_m$.

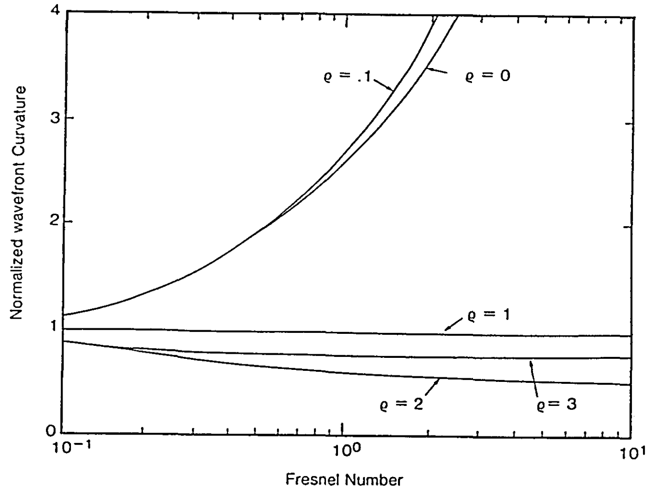

Fig. 3. Plots of the normalized wave-front curvature R_s/L vs N.

Therefore, in the case of a simple symmetrical resonator with uniform mirrors, a confined beam exists only if the inequality (30) is satisfied. Siegman[10] and Yariv[11] give a similar confinement criterion for nonsymmetrical resonators. Resonators satisfying such criterion are generally referred to as stable resonators. Figures 2 and 3 show plots from Eq. (29) of the secondary beam parameter for the simple symmetrical resonator considered in this section. The $\alpha\beta$ matrix for a single pass of the beam is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \left(\dfrac{q_1 + L}{q_1}\right)^2 & \dfrac{4iL}{q_1}\left(\dfrac{q_1 + L}{q_1}\right) \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \dfrac{q_1 + L}{q_1} & \dfrac{4iL}{k}\left(\dfrac{q_1 + L}{q_1}\right) \\ 0 & 1 \end{bmatrix}, \tag{31}$$

assumed to be a uniform mirror immediately followed by a Gaussian aperture. The parameter q_1 is the primary beam parameter immediately before propagation of the beam across the resonator. This parameter is related to q_s by

$$q_s = q_1 + L \quad \text{or} \quad q_1 = q_s - L, \tag{32}$$

since, according to our choice of reference plane, q_s is the primary beam parameter immediately before reflection (or immediately after propagation from one mirror to the other mirror). Substituting elements of the $\alpha\beta$ matrix from Eq. (32) into Eq. (31) gives us

$$w_s^2 = \frac{2iL}{k}\left[\left(\frac{L}{q_1}\right)^{-1} + \left(\frac{L}{q_1} + 2\right)^{-1}\right]$$

$$\text{or} \tag{33}$$

$$\frac{kw_s^2}{2L} = -i\left\{\left(\frac{L}{q_s}\right)^{-1} + \left\{\left[\left(\frac{L}{q_s}\right)^{-1} - 1\right]^{-1} - 2\right\}^{-1} - 1\right\},$$

where we have used Eq. (32), and L/q_s is given in Eq. (29). We will refer to the complex quantity $(kw_s^2)/2L$ as the normalized secondary beam parameter. Plots of the real and imaginary parts of the normalized secondary beam parameter are shown in Figs. 4 and 5. From the plots, we see that for resonators with values of ρ satisfying the inequality (30), the real part of w_s approaches the beam spot size W_s, and the imaginary part of w_s approaches zero as N increases. Hence, in the uniform mirror limit, $w(z)$ and $W(z)$ are equal in the same reference plane as well as satisfying the same first-order differential equation governing propagation along the resonator axis. Therefore, $w(z)$ must equal $W(z)$ for stable (i.e., confined mode) optical resonators with uniform mirrors. This is expected since the conventional Gaussian modes apply in this case.

III. Mode Stability

In this section, we discuss the stability of the resonator modes derived in the previous sections. These modes will be stable if, and only if, small deviations in both the primary and secondary beam parameters for self-consistency do not grow with each round trip of the beam.

Consider first the primary beam parameter. Let $\delta(1/q)$ be the deviation in $1/q$ from its self-consistent value in some chosen reference plane. Then we may write

$$\frac{1}{q} = \frac{1}{q_s} + \delta\frac{1}{q}, \tag{34}$$

where q_s and q are the self-consistent and perturbed beam parameters, respectively. Let $\Delta(1/q)$ be the deviation of $1/q$ from $1/q_s$ after one round trip of the beam through the resonator. Then we may write

$$\frac{1}{q_s} + \Delta\frac{1}{q} = \frac{\mathrm{C} + \mathrm{D}\left(\frac{1}{q_s} + \delta\frac{1}{q}\right)}{\mathrm{A} + \mathrm{B}\left(\frac{1}{q_s} + \delta\frac{1}{q}\right)}, \tag{35}$$

where A, B, C, and D are the elements of the round-trip matrix from the chosen reference plane. Using a linear analysis in which we assume small deviations in $1/q$ from self-consistency, Eq. (35) becomes[1,2]

$$\Delta\frac{1}{q} = \frac{1}{\left(\mathrm{A} + \frac{\mathrm{B}}{q_s}\right)^2}\,\delta\frac{1}{q}. \tag{36}$$

Therefore, we may write

$$\left|\Delta\frac{1}{q}\right| = \frac{1}{F_q^2}\left|\delta\frac{1}{q}\right|, \tag{37}$$

where

$$F_q = \left|\mathrm{A} + \frac{\mathrm{B}}{q_s}\right|. \tag{38}$$

By examining Eq. (37), we see that small deviations in $1/q$ from self-consistency will grow with each round trip if $F_q < 1$ and dampen with each round trip if $F_q > 1$.

We now consider the stability of the secondary beam parameter. As with the primary parameter we let δw^2 be the deviation in w^2 from self-consistency in some chosen reference plane. Then we may write

$$w_s^2 + \Delta w^2 = \alpha(w_s^2 + \delta w^2) + \beta, \tag{39}$$

where Δw^2 is the deviation in w^2 from self-consistency after one round trip, and α and β are the elements of the round-trip $\alpha\beta$ matrix. Subtracting Eq. (27) from Eq. (39) gives us

$$\Delta w^2 = \alpha(\delta w^2). \tag{40}$$

Therefore, we may write

$$|\Delta w^2| = |\alpha||\delta w^2| = \frac{1}{F_w^2}|\delta w^2|, \tag{41}$$

where

$$F_w = \frac{1}{\sqrt{|\alpha|}}. \tag{42}$$

From Eq. (41), we see that there is instability in w^2 if $F_w < 1$ and stability in w^2 if $F_w > 1$.

In deriving Eqs. (41) and (42), we have not used any approximations that require δw^2 to be small. Therefore, these equations may be used to analyze unstable behavior of w^2, even if $\delta w^2 \gg w^2$, provided the primary parameter q is stable. Let us consider the growth of the perturbation of w^2 for an optical resonator with a stable primary beam parameter and an unstable secondary parameter. Since the primary parameter q is stable, we may use Eq. (41) to calculate the growth of δw^2 regardless of its size. From Eq. (41) we see that δw^2 grows by a factor of F_w^{-2} with each round trip of the beam and that this growth is unbounded. By the triangle inequality

$$|w^2| \geq |\delta w^2| - |w_s^2|,$$

we see that as δw^2 grows, so also does w^2. Since w^2 grows with each pass, the argument x/w occurring in the Hermite polynomials in Eq. (17) approaches zero with a large number of beam round trips. Therefore, after a large number of round trips, the intensity profile of each of the higher-order modes approaches that of the fundamental mode or zero. Hence this instability is desirable if the beam is to have the intensity

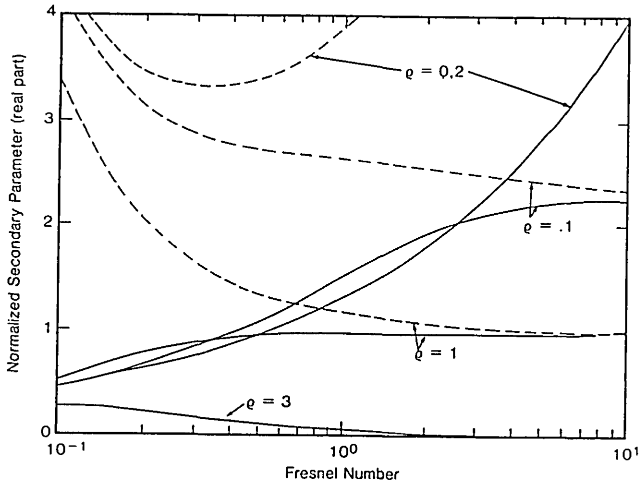

Fig. 4. Plots of the real part of the normalized secondary beam parameter $kw_s^2/2L$ vs N. For resonators that have confined beam modes at large N, these curves approach each other as N increases.

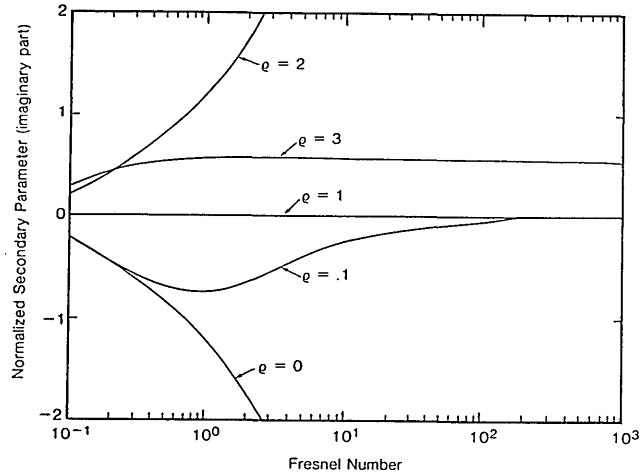

Fig. 5. Plots of the imaginary part of the normalized beam parameter vs N. For resonators with confined beam modes at large N, these curves approach zero as N increases.

profile and focusing properties of the fundamental mode.

Let us now calculate the stability of the beam parameters for the case of the simple symmetrical resonator. As indicated before, we need only consider a single pass of the beam in a symmetrical resonator rather than a complete round trip. Substituting the values of A and B from Eq. (28) into Eq. (38), we have

$$F_q = \left|1 - \frac{2L}{R_m} - \frac{2Li}{kW_m^2} + \frac{L}{q_s}\right|$$

$$= \left|1 - 2\rho - \frac{i}{\pi N} + \frac{L}{q_s}\right| , \qquad (43)$$

where L/q_s is given by Eq. (29). Substituting L/q_s from Eq. (29) into Eq. (43) gives us

$$F_q = \left|1 - \rho - \frac{i}{2\pi N} \pm \left[\left(1 - \rho \frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}\right| , \qquad (44)$$

where the sign was chosen earlier for the beam spot size to be real. Substituting α from Eq. (31) into Eq. (42) gives us

$$F_w = \frac{1}{1 + \dfrac{L}{q_1}} = 1 - \frac{L}{q_s}$$

$$= \left|1 - \rho - \frac{i}{2\pi N} \mp \left[\left(1 - \rho - \frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}\right| . \qquad (45)$$

The stability factors F_q and F_w are plotted in Figs. 6 and 7 for the values of ρ chosen in the previous sections. From these plots, we see that F_q is generally greater than unity, and F_w is generally less than unity. Therefore, the primary beam parameter q is stable, while the secondary parameter w is unstable. Hence we may conclude that perturbations in the secondary beam parameter will grow and cause the intensity profile of the higher-order modes to evolve into that of the fundamental mode. This is desirable for most applications including the design of multipass or regenerative laser amplifiers, where nearly uniform beams are desired.

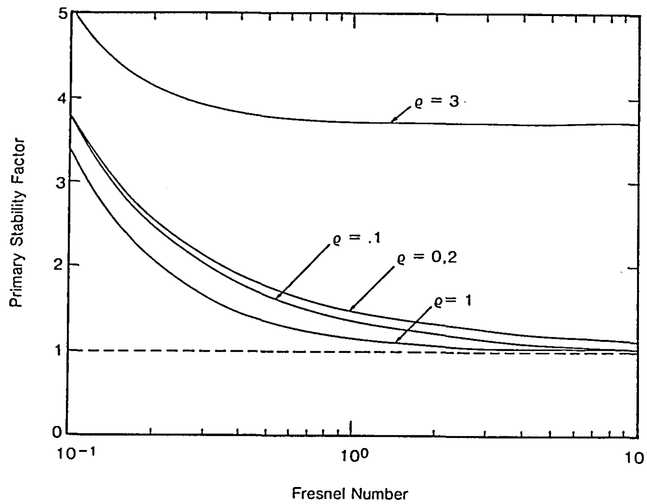

Fig. 6. Plots of the stability factor F_q of the primary beam parameter vs Fresnel number for the selected values of ρ.

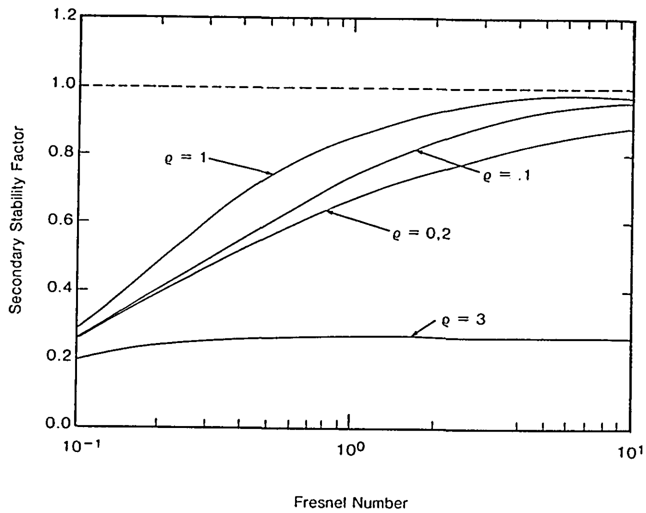

Fig. 7. Plots of the stability factor F_w of the secondary beam parameter vs Fresnel number for the selected values of ρ.

IV. Mode Losses and Discrimination

As indicated previously, it is important to eliminate the higher-order modes in high-output regenerative amplifiers. Such mode discrimination can be obtained with the use of Gaussian mirrors, because with such mirrors the losses of the higher-order modes are greater than those of the fundamental mode. In this section, we derive expressions for the power losses of the resonator modes due to transmission through the Gaussian mirrors.

We define the reflection coefficient $\gamma_{n,m}$ of the $TEM_{n,m}$ mode as the ratio of the reflected mode power to the incident mode power in the plane of the mirror. For the Hermite-Gaussian modes, this is given by

$$\gamma_{n,m} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|H_n\left(\frac{\sqrt{2}x}{w_s}\right)H_m\left(\frac{\sqrt{2}2y}{w_s}\right)\right|^2 \exp[-2(x^2-y^2)/W_s'^2]dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|H_n\left(\frac{\sqrt{2}x}{w_s}\right)H_m\left(\frac{\sqrt{2}2y}{w_s}\right)\right|^2 \exp[-2(x^2-y^2)/W_s^2]dxdy}, \tag{46}$$

where W_s and W_s' are the beam spot sizes immediately before and after reflection, respectively, w_s is the secondary beam parameter in the plane of the mirror, and R_0 is the reflectivity at the center of the mirror. Note that W_s and W_s' are related by

$$\frac{1}{W_s'^2} = \frac{1}{W_s^2} + \frac{1}{W_m^2}, \tag{47}$$

where W_m is the reflectivity spot size of the mirror. If we wish to work with the Laguerre-Gaussian modes, we have

$$\gamma_{n,m} = R_0^2 \frac{\int_0^{\infty} r^{2m}\left|L_n^m\left(\frac{2r^2}{w_s^2}\right)\right|^2 \exp(-2r^2/W_s'^2 r dr}{\int_0^{\infty} r^{2m}\left|L_n^m\left(\frac{2r^2}{w_s^2}\right)\right|^2 \exp(-2r^2/W_s^2 r dr}, \tag{48}$$

where the $L_{n,m}$ are the associated Laguerre polynomials.

The loss coefficient $\wedge_{n,m}$ of a resonator mirror is defined as the ratio of the power lost from the $TEM_{n,m}$ mode on reflection to the incident mode power. Since the incident beam energy must be either reflected or lost due to transmission, we have

$$\wedge_{n,m} = 1 - \gamma_{n,m}, \tag{49}$$

where $\gamma_{n,m}$ is given by Eqs. (46) or (48).

To calculate the losses of the fundamental mode and the mode discrimination property of the resonator, we must have expressions for the reflection coefficients of the two lowest-order modes (i.e., the fundamental $TEM_{0,0}$ mode and the $TEM_{0,1}$ mode). Since the Hermite-Gaussian $TEM_{0,0}$ and $TEM_{0,1}$ modes are equal to the Laguerre-Gaussian $TEM_{0,0}$ and $TEM_{0,1}$ modes, we may use either Eqs. (46) or (48) to calculate the reflection coefficients $\gamma_{0,0}$, and $\gamma_{0,1}$. From Eq. (48) we have

$$\gamma_{0,0} = R_0^2 \frac{\int_0^{\infty} \exp(-2r^2/W_s'^2) r dr}{\int_0^{\infty} \exp(-2r^2/W_s^2) r dr} = R_0^2 \frac{W_s'^2}{W_s^2}, \tag{50}$$

$$\gamma_{0,1} = R_0^2 \frac{\int_0^{\infty} r^3 \exp(-2r^2/W_s'^2) dr}{\int_0^{\infty} r^3 \exp(-2r^2/W_s^2) dr} = R_0^2 \frac{W_s'^4}{W_s^4}. \tag{51}$$

Therefore, the loss coefficients $\wedge_{0,0}$ and $\wedge_{0,1}$ are given by

$$\wedge_{0,0} = 1 - R_0^2 \frac{W_s'^2}{W_s^2}, \tag{52}$$

$$\wedge_{0,1} = 1 - R_0^2 \frac{W_s'^4}{W_s^4}. \tag{53}$$

To evaluate the mode discrimination property of the resonator, let us define the discrimination parameter D as

$$D = 1 - \frac{\gamma_{0,1}}{\gamma_{0,0}}. \tag{54}$$

Thus, if $D = 0$, we have no mode discrimination; and if $D = 1$, we have total discrimination (i.e., all higher-order modes are lost after a single reflection). (Neither of these extremes will in fact be realized.) Substituting Eqs. (50) and (51) into Eq. (54) gives us

$$D = 1 - \frac{W_s'^2}{W_s^2}. \tag{55}$$

Let us now calculate the mirror loss coefficients $\wedge_{0,0}$ and $\wedge_{0,1}$ and the mode discrimination parameter D for the simple symmetrical resonator. For simplicity, we will assume that the mirror centers are totally reflecting, so that $R_0 = 1$. In this case, we have from Eqs. (52) and (53)

$$\wedge_{0,0} = D = 1 - \frac{W_s'^2}{W_s^2} = 1 - \frac{\left(\frac{2L}{kW_s^2}\right)}{\left(\frac{2L}{kW_s'^2}\right)}, \tag{56}$$

$$\wedge_{0,1} = 1 - \frac{W_s'^4}{W_s^4} = 1 - \frac{\left(\frac{2L}{kW_s^2}\right)^2}{\left(\frac{2L}{kW_s'^2}\right)^2}, \tag{57}$$

$$\frac{2L}{kW_s^2} = -\mathrm{Im}\left\{\rho + \frac{i}{2\pi N} \pm \left[\left(1-\rho-\frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}\right\}$$
$$= \left|\mathrm{Im}\left\{\left[\left(1-\rho-\frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}\right\} - \frac{1}{2\pi N}\right|, \tag{58}$$

and from Eq. (46) we have

$$\frac{2L}{kW_s'^2} = \frac{2L}{kW_s^2} + \frac{2L}{kW_m^2} = \frac{2L}{kW_s^2} + \frac{1}{\pi N}$$
$$= \left|\mathrm{Im}\left\{\left[\left(1-\rho-\frac{i}{2\pi N}\right)^2 - 1\right]^{1/2}\right\} - \frac{1}{2\pi N}\right|. \tag{59}$$

Substituting Eqs. (58) and (59) into Eqs. (56) and (57), gives us

$$\wedge_{0,0} = 1 - \frac{|\mathrm{Im}\{[(2\pi N - 2\pi N\rho - i)^2 - 4\pi^2N^2]^{1/2}\}| - 1}{|\mathrm{Im}\{[(2\pi N - 2\pi N\rho - i)^2 - 4\pi^2N^2]^{1/2}\}| + 1}, \tag{60}$$

$$\wedge_{0,1} = 1 - \frac{(|\mathrm{Im}\{[(2\pi N - 2\pi N\rho - i)^2 - 4\pi^2N^2]^{1/2}\}| - 1)^2}{(|\mathrm{Im}\{[(2\pi N - 2\pi N\rho - i)^2 - 4\pi^2N^2]^{1/2}\}| + 1)^2}. \tag{61}$$

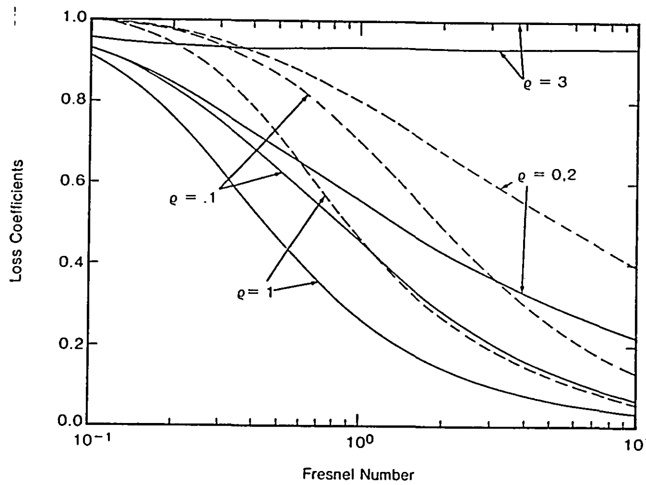

Fig. 8. Plots of the loss coefficients $\Lambda_{0,0}$ (solid curves) and $\Lambda_{0,1}$ (dashed curves) for the selected values of ρ.

Plots of these loss coefficients are shown in Fig. 8. As can be seen from the plots, confocal resonators ($\rho = 1$) have the lowest losses, and unstable resonators have high losses and, therefore, high-mode discrimination. This may be useful in the design of regenerative laser amplifiers since such amplifiers require mode discrimination. Also the large beam diameters associated with unstable resonators tend to fill the cavity and make much more efficient use of the gain medium than the narrow beams associated with stable resonators.

V. Conclusions

An analytical method has been developed for calculating the transverse modes of optical resonators with Gaussian mirrors. Also, we have considered stability and losses of these modes at the resonator mirrors.

To meet the self-consistency requirement for all resonator modes, it was necessary to define a secondary beam parameter w in addition to the conventional primary parameter q. Each higher-order mode in the resonator must have the property that the secondary beam parameter repeat itself after each round trip of the beam through the resonator as well as the primary parameter.

When considering the stability of the resonator modes, it was necessary to consider the stability of both the primary and secondary beam parameters. Calculations showed that the primary beam parameter is generally stable while the secondary parameter is generally unstable. Therefore, small perturbations in the secondary beam parameter would cause the intensity profile of each higher-order mode to evolve into that of the fundamental mode. Hence, this instability is desirable since a nearly uniform beam is generally desired.

The methods presented in this paper may be applied to any periodic optical system consisting of lenses, spherical mirrors, interfaces, and Gaussian apertures. By applying these methods to a simply symmetrical resonator, we were able to gain useful insight into the design of a regenerative laser amplifier.

Dwight Walsh is currently employed at the Air Force Weapons Laboratory, Kirtland AFB, NM.

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