PHYSICAL REVIEW D VOLUME 21, NUMBER 6 15 MARCH 1980

New large family of vacuum solutions of the equations of general relativity

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(Received 2 July 1979)

A Bäcklund transformation for the Ernst equation of general relativity, published earlier by this author, is used to derive a new large family of vacuum metrics with two commuting Killing vectors from the family of Weyl or Einstein-Rosen metrics. Thus, any solution of the axially symmetric Laplace or wave equation yields a solution of the Ernst equation. Asymptotically flat Weyl metrics yield new asymptotically flat metrics. The solutions are nonstationary and may exhibit solitonlike behavior.

General solutions of linear partial differential equations may be characterized by an infinite number of parameters (power-series coefficients) or by arbitrary functions in an integral formulation. It is rare for this situation to occur for nonlinear equations. In general relativity, the sole examples for many years were the Weyl and Einstein-Rosen (WER) diagonal metrics.[1] This paper presents a new set of such metrics, which are solutions of the Ernst and related equations and are derived from the WER metrics by a Bäcklund transformation (BT).[2] The new metrics are nondiagonal and nonstationary. Thus they are different from the Hoenselaers, Kinnersley, and Xanthopoulos (HKX)[3] metrics—which themselves exhibit an infinite number of parameters.

In using a BT, one begins with an initial solution and finds a new solution from it by quadratures. In this paper, the initial solution is the WER metric

$$ds_0{}^2 = \lambda T_0(dx^1)^2 + S^2 T_0{}^{-1}(dx^2)^2 + \sigma e^{2\gamma_0} T_0{}^{-1}(dS^2 - \lambda\, dR^2)\,, \tag{1}$$

where $\lambda = \pm 1$, $\sigma = 1$ or $-\lambda$, and T_0 and γ_0 are functions of S and R only.[4] Axially symmetric metrics like Weyl's require $\lambda = -1$, wave metrics like Einstein-Rosen require $\lambda = +1$. (See I for details.) For the Kasner metric, which can be represented in the above form, $\lambda = -\sigma = 1$. We put, as in I, $k = \sqrt{\lambda} = 1$ or i, $x = \frac{1}{2}(S + kR)$, and $y = \frac{1}{2}(S - kR)$.

If we put $T_0 = e^{2V}$, Einstein's equations give

$$V_{SS} + S^{-1} V_S - \lambda V_{RR} = 0\,, \tag{2}$$

the axially symmetric Laplace or wave equation, which is

$$V_{xy} + \tfrac{1}{2} S^{-1}(V_x + V_y) = 0 \tag{3}$$

in terms of x and y. γ is determined by quadrature:

$$d\gamma = S(V_x{}^2 dx + V_y{}^2 dy)\,. \tag{4}$$

We now apply the BT to find a new metric of the form

$$ds^2 = \lambda T(dx^1 + Q\, dx^2)^2 + S^2 T^{-1}(dx^2)^2 + \sigma e^{2\gamma} T^{-1}(dS^2 - \lambda\, dR^2)\,, \tag{5}$$

where T, Q, and γ are functions of S and R only and other symbols have their previous meaning. Thus metric (5) has two commuting Killing vectors. The quadratures referred to above can be done entirely in terms of the functions T_0 (or V) and γ_0 above, and a new function U, defined below. (In fact, Cosgrove[5] has shown that the equations for this BT can be integrated completely, in terms of initial functions.) The BT procedure follows that given in I.

We define a real function U by

$$dU = \zeta V_x dx + \zeta^{-1} V_y dy\,, \tag{6}$$

where

$$\zeta = [(kl - y)(kl + x)^{-1}]^{1/2} \tag{7}$$

as in I, where l is a finite real constant. Equation (3) is the integrability condition for Eq. (6). We note that

$$d\zeta = \tfrac{1}{2} S^{-1}(\zeta^2 - 1)(\zeta dx + \zeta^{-1} dy)\,. \tag{8}$$

For metric (1), the functions t, u, v, and w of I become $t_0 = v_0 = 2V_x - S^{-1}$ and $u_0 = w_0 = 2V_y - S^{-1}$. [Subscripts "0" refer to metric (1).] Equation (8) in I for the pseudopotential q becomes, after rearrangement,

$$(q^2 - 1)^{-1} dq = dU - (\zeta^2 - 1)^{-1} d\zeta\,, \tag{9}$$

which integrates to

$$q = -(F + \zeta)^{-1}(\zeta F + 1)\,, \tag{10}$$

where

$$F = (e^{2U} - i\epsilon)(e^{2U} + i\epsilon)^{-1}\,, \tag{11}$$

$\epsilon = \pm 1$, and we have used the facts that if $\lambda = +1$ (-1),

ζ (or q) is real and $\bar{q}=q^{-1}$ ($\bar{\zeta}=\zeta^{-1}$) (ignoring reality considerations relating to the domain of definition of ζ).

Substitution of Eq. (10) into Eqs. (10) of I gives

$$t=-(\zeta F+1)[t_0 F(F+\zeta)^{-1}+S^{-1}], \quad (12a)$$

$$u=-(F+\zeta)[u_0 F(\zeta F+1)^{-1}+S^{-1}\zeta^{-1}], \quad (12b)$$

$$v=-F^{-1}(F+\zeta)[v_0(\zeta F+1)^{-1}+S^{-1}], \quad (12c)$$

$$w=-F^{-1}(\zeta F+1)[w_0(F+\zeta)^{-1}+S^{-1}\zeta^{-1}]. \quad (12d)$$

Equation (17) of I yields

$$T=khS(\zeta F+1)(F+\zeta)(\zeta^2-1)^{-1}F^{-1}T_0^{-1}, \quad (13)$$

where h is a real constant.

We now use Eqs. (12) and (13) to find Q and γ. We first integrate Eq. (18) of I for the twist potential ϕ:

$$\phi=-ihkS\zeta(F^2-1)(\zeta^2-1)^{-1}F^{-1}T_0^{-1}. \quad (14)$$

To find Q, we integrate $dQ=ST^{-2}*d\phi$, where $*$ is linear and $*dx=k^{-1}dx$, $*dy=-k^{-1}dy$ (see I). We get

$$Q=-\lambda i h^{-1}\zeta(F^2-1)(\zeta F+1)^{-1}(F+\zeta)^{-1}T_0. \quad (15)$$

We write $d\gamma-d\gamma_0$ from Eq. (19) in I and obtain terms linear in t, u, v, w, which integrate to give (K constant)

$$\gamma=\gamma_0+V+\tfrac{1}{2}\ln[KTT_0^{-1}(\zeta^2-1)\zeta^{-1}S^{-1}\cosh 2U]. \quad (16)$$

Equations (13) and (14) now give the Ernst potential $E=T+i\phi$:

$$E=khST_0^{-1}(\zeta^2-1)^{-1}(1+\zeta^2+2\zeta F). \quad (17)$$

For substitution into the new metric, it is convenient to introduce new real independent variables r and θ:

$$R=-2l+r\cosh k\theta, \quad (18)$$

$$S=-kr\sinh k\theta. \quad (19)$$

Then $\zeta=e^{k\theta}$, $r^2=(R+2l)^2-\lambda S^2$, $*dr=-r\,d\theta$, and $*d\theta=-\lambda r^{-1}dr$. Thus,

$$T=-\lambda hrT_0^{-1}(\cosh k\theta+\tanh 2U), \quad (20a)$$

$$\phi=\epsilon\lambda hrT_0^{-1}\operatorname{sech}2U, \quad (20b)$$

$$Q=-\epsilon\lambda h^{-1}T_0\operatorname{sech}2U(\cosh k\theta+\tanh 2U)^{-1}, \quad (20c)$$

and

$$e^{2\gamma}=e^{2\gamma_0}T_0^{-1}(\sinh 2U+\cosh\theta\cosh 2U). \quad (20d)$$

The new metric (where we put $T_0=e^{2V}$) becomes

$$\begin{aligned} ds^2=&-hre^{-2V}(\cosh k\theta+\tanh 2U)(dx^1)^2\\ &+2\epsilon\lambda r\operatorname{sech}2U\,dx^1\,dx^2\\ &-h^{-1}re^{2V}(\cosh k\theta-\tanh 2U)(dx^2)^2\\ &+\sigma e^{2\gamma_0}\cosh 2U(r\,d\theta^2-\lambda r^{-1}dr^2). \end{aligned} \quad (21)$$

It is clear that, by virtue of the general nature of V and U, Eq. (21) may include a large variety of solutions.

Since $dV=V_x dx+V_y dy$ and $*dV=k^{-1}(V_x dx-V_y dy)$, we have $V_x dx=\frac{1}{2}(dV+k*dV)$ and $V_y dy=\frac{1}{2}(dV-k*dV)$. Using these results and $\zeta=e^{k\theta}$ in Eq. (6), we find

$$dU=\cosh k\theta\,dV+k\sinh k\theta*dV. \quad (22)$$

The Laplace-wave equation (3) has solutions of the form, in terms of r and $\mu=\cosh k\theta$,

$$V=\sum_{n=0}^{\infty}(A_n r^{-n-1}+B_n r^n)P_n(\mu) \quad (23)$$

or of the integral form

$$V=\int f(\alpha)(\alpha^2-2\alpha\mu r+r^2)^{-1/2}d\alpha, \quad (24)$$

where $f(\alpha)$ is an arbitrary function and one takes appropriate account of the singularity. Using a version of Eq. (22),

$$U_r=\mu V_r+r^{-1}(1-\mu^2)V_\mu, \quad (25)$$

$$U_\mu=-rV_r+\mu V_\mu, \quad (26)$$

helps us find expressions for U corresponding to the above V:

$$U=\sum_{n=0}^{\infty}(A_{n-1}r^n+B_{n+1}r^{n+1})P_n(\mu) \quad (27)$$

or

$$U=\int f(\alpha)\alpha^{-1}[r(\alpha^2-2\alpha\mu r+r^2)^{-1/2}-1]d\alpha \quad (28)$$

(A_{-1} is an integration constant).

Asymptotic behavior. Metric (1) is asymptotically flat if V and $\gamma_0\to 0$ as $r\to\infty$. If $U\to 0$ as $r\to\infty$, then metric (21) also is asymptotically flat. We make two successive variable transformations to show this explicitly.

First set $r=\frac{1}{4}\tau^2$, $\theta=2\delta$, $x^1=\xi+\eta$, $x^2=\xi-\eta$, and $\theta=1$, $h=-\lambda$, $U=V=\gamma_0=0$. Equation (21) then takes the asymptotic form

$$\begin{aligned} ds^2=&\lambda\tau^2(d\xi^2\cosh^2 k\delta+d\eta^2\sinh^2 k\delta)\\ &+\sigma(\tau^2 d\xi^2-\lambda\,d\tau^2). \end{aligned} \quad (29)$$

Now put $m=\sqrt{\sigma}$ ($=1$ or i) and introduce the real variables

$$a=\tau\cosh k\delta\cosh m\xi, \quad (30a)$$

$$b=m\tau\cosh k\delta\sinh m\xi, \quad (30b)$$

$$f=k\tau\sinh k\delta\cos m\eta, \quad (30c)$$

$$g=km\tau\sinh k\delta\sin m\eta. \quad (30d)$$

Then Eq. (29) becomes the manifestly flat form

$$ds^2=dg^2+\sigma df^2+\lambda\,db^2-\lambda\sigma\,da^2. \quad (31)$$

For the three possible choices of σ and λ, this metric has the correct Minkowski signature. By choosing a, b, f, and g as in Table I, we can express Eq. (31) as $ds^2=d\rho^2+\rho^2 d\phi^2+dz^2-dt^2$ in the usual cylindrical variables.

We note from Eqs. (18), (19), and (30) and the definition of r, θ, and μ that

$$4(R+2l)=4\mu r=a^2-\sigma b^2+\lambda f^2+\lambda\sigma g^2 \tag{32}$$

and

$$4r=\tau^2=a^2-\sigma b^2-\lambda f^2-\lambda\sigma g^2. \tag{33}$$

We now see from Table I that for all cases

$$r=\tfrac{1}{4}\lambda\sigma(t^2-z^2-\rho^2) \tag{34}$$

and

$$\mu r=\tfrac{1}{4}\lambda(t^2-z^2+\rho^2). \tag{35}$$

Thus, wherever r and θ appear in the new metric (21) we may use Eqs. (34) and (35) to express it in terms of the variables t, z, and ρ, for which (21) is manifestly flat at infinity. Thus it is clear that the new metric is nonstationary. If we impose the reasonable condition $r(=\frac{1}{4}\tau^2)>0$, we see from (34) that we are restricted to the interior or exterior of the light cone in the new metric, depending on the sign of $\lambda\sigma$.

The expression $r^2-2\alpha\mu r+\alpha^2$, which occurs in Eqs. (24) and (28), takes the form

$$\tfrac{1}{16}[t^2-z^2-(\rho-B)^2][t^2-z^2-(\rho+B)^2],$$

TABLE I. Choices of coordinates a, b, f, and g, for various λ and σ, which give the usual Minkowski cylindrical metric.

λ	σ	t	z	$\rho\sin\phi$	$\rho\cos\phi$
−1	1	b	a	g	f
1	1	a	b	g	f
1	−1	f	g	b	a

with $B=2\sqrt{\lambda\alpha}$ in terms of the variables ρ, z, and t. Where the expression vanishes—true singularities in the new metric—we have $t^2=z^2+(\rho\pm B)^2$, displaced light cones. Thus the singularities are suggestive of solitonlike behavior.

We choose as an example the Kasner metric. We write it in the form

$$ds^2=t^{s_1^2+s_2^2-1}(dz^2-dt^2)+t^{2s_1}dx^2+t^{2s_2}dy^2,$$

where $s_1+s_2=1$, in usual coordinates. Set $K=2s_1$. Then $S=t$, $T=S^K$, $V=\frac{1}{2}K\ln S$, and $\gamma_0=\frac{1}{4}K^2\ln S$.

We obtain $e^{2U}=(\tanh\delta)^K$. Investigation shows that this yields the Belinsky-Zakharov metric,[6] essentially as noted in I. (This metric has soliton behavior, as noted by Belinsky and Zakharov.) Similar solutions have recently been found by Cosgrove[5] and Neugebauer.[7]

The author appreciates helpful discussions with W. Kinnersley, C. Hoenselaers, and C. Cosgrove.

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[3]C. Hoenselaers, W. Kinnersley, and B. Xanthopoulos, J. Math. Phys. 20, 2530 (1979); see also the article by the same authors in Phys. Rev. Lett. 42, 481 (1979).

[4]The usual notation for this metric and metric (5) uses f instead of T, ω for Q, and ρ and z for S and R, and $\lambda=-1$. The current notation is used for consistency with paper I, and also to emphasize the fact that these findings apply to more than just the $\lambda=-1$ case. Also, we note, in comparison with I, that Eq. (1) of that paper shows that $(dx^3)^2-\lambda(dx^4)^2=\sigma(dS^2-\lambda dR^2)$, where $\sigma=1$ if $S=x^3$ $(R=x^4)$ and $\sigma=-\lambda$ if $S=x^4$ $(R=\lambda x^3)$. Thus, three separate cases obtain: $\lambda=\sigma=1$, $\lambda=-\sigma=1$, and $-\lambda=\sigma=1$.

[5]C. M. Cosgrove, in lecture presented at the Second Marcel Grossmann Meeting on the Recent Developments of General Relativity, Trieste, Italy, 1979 (unpublished).

[6]V. A. Belinsky and V. E. Zakharov, Zh. Eksp. Teor. Fiz. 75, 1953 (1978) [Sov. Phys.-JETP 48, 985 (1978)]; 77, 3 (1979) [50, 1 (1979)].

[7]G. Neugebauer (unpublished).