PHYSICAL REVIEW LETTERS

VOLUME 41 30 OCTOBER 1978 NUMBER 18

Bäcklund Transformation for the Ernst Equation of General Relativity

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(Received 11 September 1978)

A Bäcklund transformation for the Ernst equation arising in general relativity in connection with several physical problems is derived, using the pseudopotential method of Wahlquist and Estabrook. A prolongation structure is also constructed, using a method of writing the equations in terms of differential forms, and an equation in the spirit of Lax is constructed, somewhat different from that given by Maison. Possible uses of the Bäcklund transformation to generate new solutions are mentioned.

A problem receiving much attention in recent years is that of solving the Ernst equation $[(\mathrm{Re}E)\nabla^2 E = (\nabla E)^2]$[1] for axially symmetric stationary vacuum gravitational fields.[2] The existence of an infinite number of potentials for the equations,[3] together with other interesting features, suggests[4] that the equation should admit a Bäcklund transformation (BT). A BT has now been found. The work of Maison on this problem[5] provided a valuable clue in the search.

I write the metric in the following form:

$$ds^2 = \lambda T(dx^1 + Q\,dx^2)^2 + S^2 T^{-1}(dx^2)^2 + e^{2\gamma}T^{-1}[(dx^3)^2 - \lambda(dx^4)^2], \tag{1}$$

where $\lambda = \pm 1$ and S, T, Q, and γ are functions of x^3 and x^4 only. Physical problems represented by this metric are axially symmetric stationary fields[1] ($\lambda = -1$, $x^1 = t$, $x^2 = \varphi$, $x^3 = \rho$, $x^4 = z$), cylindrical waves[6] ($\lambda = 1$, $x^1 = z$, $x^2 = \varphi$, $x^3 = \rho$, $x^4 = t$), or colliding plane waves[7] ($\lambda = 1$, $x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$, $x^4 = t$). S is taken equal either to x^3 or x^4. Coordinates are written so as to appear quasicylindrical. Electromagnetic problems may also be treated by letting $Q = 0$ in the metric, but by including an electromagnetic potential which occurs in the same place as Q in the equations.[8]

In order to treat several physical situations with the same equation, I define throughout this paper $k = \sqrt{\lambda}$ (i.e., 1 if $\lambda = 1$; i if $\lambda = -1$), and define new coordinates x and y as follows: If $S = x^3$, $x = \frac{1}{2}(x^3 + kx^4)$ and $y = \frac{1}{2}(x^3 - kx^4)$; if $S = x^4$, $x = \frac{1}{2}(k^{-1}x^3 + x^4)$ and $y = \frac{1}{2}(-k^{-1}x^3 + x^4)$. Thus, $y = \bar{x}$ if $k = i$ ($\lambda = -1$). In either case, $S = x + y$. Define $R = k^{-1}(x - y)$. I also define a linear Hodge star operator $*$ by

$$*dx = k^{-1}dx, \quad *dy = -k^{-1}dy.$$

Then $** = \lambda$.

The equation for Q may be satisfied by introducing a potential φ such that $*d\varphi = S^{-1}T^2 dQ$. I define $E = T + i\varphi$. Then both E and $\bar{E}$ satisfy the Ernst equation, which now takes the form (subscripts represent differentiation)

$$E_{xy} + \tfrac{1}{2}S^{-1}(E_x + E_y) = T^{-1}E_x E_y, \tag{2}$$

where $S = x + y$.

I define variables t, u, v, and w as follows:

$$t = T^{-1}E_x - S^{-1}, \quad u = T^{-1}E_y - S^{-1},$$
$$v = T^{-1}\bar{E}_x - S^{-1}, \quad w = T^{-1}\bar{E}_y - S^{-1}. \tag{3}$$

Note that $v=\bar{t}$ and $w=\bar{u}$ if $\lambda=1$; $v=\bar{u}$ and $w=\bar{t}$ if $\lambda=-1$. Equation (2), for E and $\bar{E}$, together with the integrability conditions for Eqs. (3), can be written as four 2-forms which are to vanish for a solution:

$$\alpha=-dt\wedge dx+\tfrac{1}{2}[tu-tw-S^{-1}(t+w)]\,dy\wedge dx\,, \tag{4a}$$

$$\beta=-du\wedge dy+\tfrac{1}{2}[tu-vu-S^{-1}(u+v)]\,dx\wedge dy\,, \tag{4b}$$

$$\gamma=-dv\wedge dx+\tfrac{1}{2}[vw-vu-S^{-1}(u+v)]\,dy\wedge dx, \tag{4c}$$

$$\delta=-dw\wedge dy+\tfrac{1}{2}[vw-tw-S^{-1}(t+w)]\,dx\wedge dy. \tag{4d}$$

(See discussion on writing differential equations as differential forms by Harrison and Estabrook[9].)

Following Wahlquist and Estabrook,[10] we now attempt to find a pseudopotential q, such that the 1-form

$$\sigma=-dq+F\,dx+G\,dy\,, \tag{5}$$

where F and G are functions of x, y, t, u, v, w, and q, satisfies $d\sigma=0 \bmod (\sigma, \alpha, \beta, \gamma, \delta)$. This condition yields the equations

$$F_u=F_w=G_t=G_v=0\,, \tag{6}$$

$$2(FG_q-GF_q+G_x-F_y)+[tu-vu-S^{-1}(u+v)]\,G_u+[vw-tw-S^{-1}(t+w)]G_w-[vw-vu-S^{-1}(u+v)]\,F_v-[tu-tw-S^{-1}(t+w)]F_t=0. \tag{7}$$

Motivated by other work, we take $F=At+Bv$ and $G=Cu+Dw$, where A, B, C, and D are functions of x, y, and q. Solution and simplification give

$$\sigma=-dq+\tfrac{1}{2}[t(q+q^2\zeta)-v(q+\zeta)]\,dx+\tfrac{1}{2}[w(q+q^2\zeta^{-1})-u(q+\zeta^{-1})]\,dy\,, \tag{8}$$

where

$$\zeta\equiv\pm[(kl-y)(kl+x)^{-1}]^{1/2} \tag{9}$$

and l is a real constant.

To find the BT, we assume the existence of new solutions of Eq. (4), t', u', v', and w', which are functions of x, y, t, u, v, w, and q. We substitute into Eqs. (4) (with primed dependent variables), expand the differential dt', etc., in terms of dt, etc., use the old equations (4) and (8) to eliminate $dt\wedge dx$, etc., so far as possible, and equate coefficients to zero. This gives us a set of differential equations for t', etc. Solution yields the BT:

$$t'=-q(q+\zeta)^{-1}[(\zeta q+1)t+S^{-1}(1-\zeta^2)]\,, \tag{10a}$$

$$u'=-(q+\zeta)^{-1}[q^{-1}(\zeta q+1)u+S^{-1}\zeta^{-1}(\zeta^2-1)], \tag{10b}$$

$$v'=-(\zeta q+1)^{-1}[q^{-1}(q+\zeta)v+S^{-1}(1-\zeta^2)]\,, \tag{10c}$$

$$w'=-q(\zeta q+1)^{-1}[(q+\zeta)w+S^{-1}\zeta^{-1}(\zeta^2-1)]\,. \tag{10d}$$

Thus, knowledge of an old solution of (2) enables us to find a new solution, with arbitrary parameter l, by means of quadratures. Note that q must first be found from Eq. (8) (annulled, $\sigma=0$).

Maison[5] demonstrated the existence of an equation "in the spirit of Lax"[11] and thus implied the existence of a pseudopotential. This is easily seen by writing that equation in the form $d\psi_1=\tau_1\psi_1+\tau_2\psi_2$, $d\psi_2=\mu_1\psi_1+\mu_2\psi_2$, where the τ_i and μ_i are 1-forms in the variables of the system. If we put $z=\psi_2/\psi_1$, then we see immediately that $dz=\mu_1+(\mu_2-\tau_1)z-\tau_2z^2$, so that z serves as a pseudopotential. The quadratic form in z shows the SL(2, R) group structure discussed by many other authors.[12] Maison's paper was also helpful in identifying variables useful in working out the pseudopotential equations and their solutions. Maison's variables A, $\bar{A}$, α, σ, τ, and ξ are related to the variables t, u, v, w, S, R, and y of this paper by the equations

$$t=\bar{B}e^{-i\bar{\theta}},\quad u=Be^{-i\theta}, \tag{11a}$$

$$v=\bar{B}e^{i\bar{\theta}},\quad w=Be^{i\theta}, \tag{11b}$$

$$S=\tau,\qquad R=\sigma, \tag{11c}$$

where θ is related to α by $\theta+\bar{\theta}=\alpha$; $y=y(\xi)$; and $B=(d\xi/dy)A$.

The pseudopotential equation may also be studied by an alternative formulation worked out previously by this author.[13] Define a potential η by $*dT=S^{-1}T(d\eta+Q\,d\varphi)$, and define 1-forms ξ_i by $\xi_1=T^{-1}d\varphi$, $\xi_2=*\xi_1=S^{-1}T\,dQ$, $\xi_3=S^{-1}(d\eta+Q\,d\varphi)$, $\xi_4=T^{-1}dT=\lambda*\xi_3$, $\xi_5=S^{-1}dS$, and $\xi_6=S^{-1}dR=*\xi_5$. These 1-forms satisfy the following identities:

$$d\xi_1=\xi_1\wedge\xi_4,\quad d\xi_2=\xi_2\wedge(\xi_5-\xi_4),\quad d\xi_6=\xi_6\wedge\xi_5,$$
$$d\xi_3=\xi_3\wedge\xi_5-\xi_1\wedge\xi_2,\quad d\xi_4=d\xi_5=0. \tag{12}$$

The following 2-forms are annulled by virtue of the field equations (2):

$$\xi_3 \wedge \xi_1 - \xi_2 \wedge \xi_4, \quad \xi_3 \wedge \xi_2 - \lambda \xi_1 \wedge \xi_4,$$
$$\xi_5 \wedge \xi_2 - \xi_1 \wedge \xi_6, \quad \lambda \xi_5 \wedge \xi_1 - \xi_2 \wedge \xi_6, \tag{13}$$
$$\xi_5 \wedge \xi_3 - \xi_4 \wedge \xi_6, \quad \lambda \xi_4 \wedge \xi_5 - \xi_6 \wedge \xi_3.$$

It is easily seen that

$$t dx = \omega_1 + \omega_2 + \omega_3 + \omega_4, \tag{14a}$$
$$u dy = \omega_1 + \omega_2 - \omega_3 - \omega_4, \tag{14b}$$
$$v dx = \omega_1 - \omega_2 + \omega_3 - \omega_4, \tag{14c}$$
$$w dy = \omega_1 - \omega_2 - \omega_3 + \omega_4, \tag{14d}$$

where $\omega_1 = \frac{1}{2}(\xi_4 - \xi_5)$, $\omega_2 = \frac{1}{2} i \xi_1$, $\omega_3 = \frac{1}{2} k (\xi_3 - \xi_6)$, and $\omega_4 = \frac{1}{2} i k \xi_2$. Equations (8) and (10a) become

$$\sigma = -dq + \tfrac{1}{2}\zeta^{-1}(\zeta^2+1)(q^2-1)\omega_1 + \tfrac{1}{2}\zeta^{-1}(\zeta^2-1)(q^2+1)\omega_2 + \tfrac{1}{2}\zeta^{-1}(\zeta^2-1)(q^2-1)\omega_3 + \tfrac{1}{2}\zeta^{-1}[(\zeta^2+1)(q^2+1)+4q\zeta]\omega_4, \tag{15}$$

$$\omega_1' + \omega_2' + \omega_3' + \omega_4' = -q(q+\zeta)^{-1}[\zeta q + 1)(\omega_1+\omega_2+\omega_3+\omega_4) + \tfrac{1}{2}(1-\zeta^2)(\xi_5 + k\xi_6)], \tag{16}$$

since $S^{-1}dx = \frac{1}{2}(\xi_5 + k\xi_6)$. Equations (10b)–(10d) yield equations similar to (16). Addition of the four equations gives an equation for $\omega_1' = \frac{1}{2}(T'^{-1}dT' - S^{-1}dS)$ which, with elimination of ω_4 by means of Eq. (15), enables us to integrate explicitly for T'. We find

$$T' = khSq(\zeta^2-1)(q+\zeta)^{-1}(1+q\zeta)^{-1}T^{-1}, \tag{17}$$

where h is a real constant. I also write an equation for φ' (from these equations):

$$d\varphi' = -\tfrac{1}{2} i T'[(t'-v')dx + (u'-w')dy]. \tag{18}$$

From the field equations we get an equation for γ:

$$d(\gamma - \tfrac{1}{2}\ln T + \tfrac{1}{4}\ln S) = \tfrac{1}{2}S(tv\,dx + uw\,dy); \tag{19}$$

the same equation holds for primed variables.

I now attempt to find a "prolongation structure" as defined by Wahlquist and Estabrook,[14] by considering 1-forms for multiple pseudopotentials q_α, involving the ξ_i:

$$\sigma_\alpha = -dq_\alpha + A_\alpha{}^i \xi_i \quad \text{(summed on } i\text{)}. \tag{20}$$

In the spirit of Eq. (15) I take the $A_\alpha{}^i$ to be functions only of ζ and the q_α, also $A_\alpha{}^6 = -A_\alpha{}^3$ and $A_\alpha{}^5 = -A_\alpha{}^4$. If we assume the $A_\alpha{}^i$ to be linear in the q_α, then

$$A_\alpha{}^i = B_\alpha{}^{i\beta} q_\beta. \tag{21}$$

Then the $B^i = [B_\alpha{}^{i\beta}]$ are matrix functions of ζ. Solution of the pseudopotential equations that arise gives $B^1 = \zeta^{-1}(\zeta^2-1)a$, $B^2 = k\zeta^{-1}(\zeta^2+1)a + b$, $B^3 = \zeta^{-1}(\zeta^2-1)c$, and $B^4 = k^{-1}\zeta^{-1}(\zeta^2+1)c + d$, where a, b, c, and d are constant matrices satisfying the prolongation structure $[c,d] = [a,d] = 0$, $[b,c] = -\lambda a$, $[a,b] = -c$, and $[b,d] + 4[a,c] = -b$. Use of the Jacobi identity shows that we have a complete Lie algebra:

$$[c,d] = [a,d] = [e,d] = [c,f] = [a,f] = [e,f] = 0,$$
$$[a,c] \equiv e, \quad [d,f] = f, \tag{22}$$
$$[a,e] = \tfrac{1}{2}c, \quad [c,e] = -\tfrac{1}{2}\lambda a,$$

where $f = b + 4e$. Representations of this algebra can be obtained by setting $f = d = 0$ and by taking a, c, and e proportional to angular momentum matrices.

Equation (20) (with σ_α annulled) can be written as a generalized Lax-type equation $dz = Mz$, where $z = [q_\alpha]$ and M is the matrix of 1-forms $M_\alpha{}^\beta = B_\alpha{}^{i\beta}\xi_i$. If M is 2×2, we have a typical equation in the spirit of Lax, with M linear in t, u, v, and w. This does not appear to be the same as Maison's, which has these variables appearing raised to fractional powers.

If we choose the usual flat-space metric in cylindrical coordinates as a beginning solution, we find a new solution which also turns out to be flat. If, however, we take the Kasner cosmology as a starting point, we get a metric which appears to be a generalization of that found by similar methods by Belinsky and Zakharov,[15] which represents solitonlike motions in cosmology. Details of some of these calculations will be published elsewhere.

The author appreciates very helpful discussions with Hugo Wahlquist.

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Measurement of *np* Charge Exchange for Neutron Energies 150–800 MeV

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The s and u variations of the np charge-exchange ($np \to pn$) cross section are measured to be relatively smooth and without structure at intermediate energies—in sharp contast to previous results.

During the 1960's it was noted[1] that the shape of the np charge-exchange (CEX) cross section could be fitted by an empirical double exponential in the square of the invariant four-momentum transfer u: $d\sigma/du = \alpha_1 \exp(\beta_1 u) + \alpha_2 \exp(\beta_2 u)$. Although it was certain that the very sharp peak at the extreme back angles ($-u \leq 0.02$) was due to one-pion exchange (OPE), in Born approximation the OPE amplitude yields a dip at $u=0$ instead of the observed peak. Phillips[2] suggested that the sharp peak could be caused by a destructive interference between the OPE amplitude and a slowly varying background term. Further developments of this idea considered absorption corrections[3] to the OPE amplitude in both the initial and final states caused by competing inelastic channels. These improvements indeed turned the dip into a spike but also predicted a secondary maximum in the cross section which was simply not observed. Other ways of handling the background terms have been developed,[4] but none have been completely successful in fitting the s and u variations of the np CEX cross section at medium energies (s is the square of the total c.m. energy).

During the past few years two experiments have produced large amounts of new data relating to np CEX at medium energy. In 1969, the group[5] from the Princeton-Pennsylvania Accelerator (PPA) reported a large peak in both the cross section and its logarithmic derivative at $u=0$. The peak was centered about an incident neutron momentum (P_n) of about 850 MeV/c, and the experiment covered the range $600 < P < 1730$. In 1975 the data from an experiment[6] at Saclay were published and, while disagreeing with PPA data[5] rather markedly for $P_n > 1.2$ GeV/c, the data for the lower momenta (down to their minimum 0.98

