Normal Vibration Frequencies of a Stiff Piano String

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This paper deals with the normal modes of vibration of a stiff piano string. The equations that govern the vibration of a solid string are developed along traditional lines. They are modified to apply to strings having a solid-steel core upon which are wrapped one or two copper windings. The bass strings of most pianos are made this way. Two boundary conditions are considered: namely, (1) pinned by sharp knife edges at both ends and (2) clamped at both ends. Formulas for the partial frequencies for both of these conditions are developed. The partial frequencies that are calculated by these formulas are compared to the experimental values obtained on an upright Hamilton piano. The experimental values appear to agree somewhat better with the pinned boundary condition rather than the clamped boundary condition, although the differences are not much greater than the observational error. It was found that the formula $f_n = nf_1[(1+Bn^2)/(1+B)]^{\frac{1}{2}}$ gives values of the partial frequencies that agree with the experimental ones where n is the number of the partial, f the fundamental frequency, and B a constant that can be calculated from the dimensions of the wire.

USUALLY, the frequencies of the partials of a piano tone are considered to be harmonic, that is, integer multiples of the fundamental frequency. Many college courses in physics still treat piano strings as having no stiffness, and this leads to the conclusion that the partials are harmonic.

For nearly one hundred years, this has been known to be only approximately true and a poor approximation for the bass strings. For some of these strings, the 40th or 50th partial may depart from the corresponding harmonic by as much as two full-tones sharp. This paper reviews the theoretical aspects of the problem of solid strings and modifies the equations so that they will apply to the wrapped strings in the bass section.

Lord Rayleigh,[1] Seebeck, and Donkin[2] worked on this problem seventy-five or eighty years ago. More recently, Morse,[3] Shankland,[4] Schuck,[5] Young[6,7] and others[8–11] have made contributions, but there seems to be no complete coverage of both the theoretical and experimental aspects of this problem in a single publication. This paper tries to do this, and adds some additional material on wrapped strings.

When a piano string is displaced a distance y at the position x, the restoring force due to the tension T is known to be

$$T(\delta^2 y/\delta x^2).$$

It is not so well-known, although it was given by Lord Rayleigh[1] more than eighty-five years ago, that the restoring force due to the elastic stiffness is

$$-QSK^2(\delta^4 y/\delta x^4),$$

where Q is Young's modulus of elasticity, S the area of cross section of the wire, and K its radius of gyration.

Let σ be the linear density and t the time. Then, the equation governing the motion of the piano string is

$$-QSK^2(\delta^4 y/\delta x^4)+T(\delta^2 y/\delta x^2)=\sigma(\delta^2 y/\delta x^2). \qquad (1)$$

This is the form of the equation originally set up by Lord Rayleigh.

A frictional term of the form

$$R(\delta y/\delta t),$$

should be introduced on the left-hand side of this equation. However, for the piano strings as they are now made and used, this term produces only a very small effect upon the values of the partial frequencies. As its introduction makes the solution of Eq. (1) much more

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[2] W. F. Donkin, *Acoustics* (Clarendon Press, Oxford, England, 1884), 2nd ed., p. 187.
[3] P. M. Morse, *Vibration and Sound* (McGraw–Hill Book Co., Inc., New York, 1948), 2nd ed., pp. 127–131.
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[10] Otto Schaefer, Ann. Phys. **62**, 156–164 (1920).
[11] G. E. Allan, Phil. Mag. **4**, 1324–1337 (1937).

complicated, it is not considered here. But it must be remembered that it is the principal term that measures the decay of the vibration of the string after it has been struck.

Let l be the length of the string between its supports and d its diameter. It is convenient to make the following substitutions:

$$B=(\pi^2 QSK^2/Tl^2), \tag{2}$$

$$f_0=(1/2l)(T/\sigma)^{\frac{1}{2}}. \tag{3}$$

For a string that has no stiffness and negligible frictional retarding force, the value of f_0 is the fundamental vibration of such a string. As we shall see, it is a close approximation to the fundamental vibration of actual piano strings. If the string is driven by an external periodic force, then f_0 is exactly the resonance frequency for a string having no stiffness.

To solve Eq. (1), one assumes y is a sum of terms of the form

$$y=Ce^{2\pi kx}e^{-2\pi jft}, \tag{4}$$

where C, k, and f are constants to be determined from Eq. (1) and the boundary and initial conditions. If the value of y from (4) is substituted in (1), the following equation results:

$$k^4-(1/4Bl^2)k^2-(f^2/16Bl^4f_0^2)=0. \tag{5}$$

This shows that for any possible frequency f there are four possible values of k as follows:

$$k=\pm k_1 \quad \text{where} \quad k_1^2=\frac{1}{8Bl^2}\left[\left(1+\frac{4Bf^2}{f_0^2}\right)^{\frac{1}{2}}+1\right], \tag{6}$$

and

$$k=\pm jk_2 \quad \text{where} \quad k_2^2=\frac{1}{8Bl^2}\left[\left(1+\frac{4Bf^2}{f_0^2}\right)^{\frac{1}{2}}-1\right]. \tag{7}$$

It will be noted that k_1 and k_2 are related as follows:

$$k_1^2-k_2^2=(1/4Bl^2). \tag{8}$$

The general solution of Eq. (1) is then

$$y=e^{-j2\pi ft}(K_1\cosh 2\pi k_1 x+K_3\cos 2\pi k_2 x +K_2\sin 2\pi k_1 x+K_4\sin 2\pi k_2 x). \tag{9}$$

All these relations are independent of the boundary conditions. So, for every possible value of k_1, there is a corresponding k_2 obtained from Eq. (8) and a corresponding value of f obtained from Eq. (5). If we choose the origin of the x axis at the center of the piano string, then one end will be at $l/2$ and the other at $-l/2$, and the boundary conditions will be symmetrical. Then, the even functions are built from the first two terms of Eq. (9) and the odd functions from the last two terms. For both of these, if the boundary conditions fit at $l/2$ they will also fit at $-l/2$.

PINNED BOUNDARY CONDITION

If both ends are pinned by knife edges, then $y=\delta^2 y/\delta x^2=0$ at $x=l/2$ and $-l/2$. If we use only the even functions, that is, take K_2 and K_4 equal to zero, these boundary conditions will fit if $K_1=0$ and $\cos\pi kl=0$; that is,

$$k=n/2l, \tag{10}$$

where n can be 1, 3, 5, 7, or any odd integer.

For each of these values of k, there is a corresponding frequency f of the odd partials obtained from Eq. (5) as

$$f_n=nf_0(1+Bn^2)^{\frac{1}{2}}. \tag{11}$$

For these frequencies,

$$y=K_3\cos(\pi nx/l)\cos 2\pi f_n t, \tag{12}$$

where K_3 takes on a different value for each odd partial and is determined by the amplitude of that partial.

If we use only the odd functions, then K_1, K_2, and K_3 are zero and $\sin\pi kl=0$, or

$$k=(n/2l), \tag{10A}$$

where $n=2$, 4, 6, or any even integer.

The frequencies corresponding to these values of k are also given by Eq. (11), and the values of y are

$$y=K_4\sin(\pi nx/l)\cos 2\pi f_n t, \tag{13}$$

where K_4 takes a different value for each even partial and is determined by the amplitude of that partial.

CLAMPED BOUNDARY CONDITIONS

For the case when the two ends are clamped, the solution is more complicated. The boundary conditions at $x=l/2$ are $y=0=\delta y/\delta x$. Applying these two conditions to the even functions gives

$$K_1\cosh(\pi k_1 l)=-K_3\cos(\pi k_2 l)$$

and

$$K_1 k_1\sinh(\pi k_1 l)=K_3 k_2\sin(\pi k_2 l).$$

Dividing one equation by the other gives, for the condition for determining the values of k_2 from the even functions,

$$-\tanh(\pi k_1 l)=(k_2/k_1)\tan(\pi k_2 l). \tag{14}$$

Similarly, applying the boundary conditions to the odd functions,

$$\tanh(\pi k_1 l)=(k_1/k_2)\tan(\pi k_2 l). \tag{15}$$

If the value of k_1 from Eq. (8) is substituted in Eqs. (14) and (15), the allowed values of k_2 for both the odd and the even functions can be obtained by numerical solution. These allowed values of k_2 can then be subsubstituted in Eq. (6) to obtain the values of the normal frequencies f_n as

$$f_n=2k_2 f_0(1+4B^2k_2^2)^{\frac{1}{2}}. \tag{16}$$

An approximate formula for k_2 in terms of the constants of the string can be obtained as follows. From the previous analysis, one should expect that the value of k_2 would be close to $n/2l$. So

$$k_2 = n/2l(1+\epsilon), \tag{17}$$

where ϵ is a small quantity as compared to unity. The equations for determining ϵ are now derived. Since k_2 is approximately equal to $n/2l$, the quantity k_1l_1 is approximately equal to $\pi/2[n^2+(1/B)]^{\frac{1}{2}}$. The smallest value of this quantity will be when $n=1$ and B is the largest value found in piano strings. The largest experimental value of B was found to be 0.024 for the highest note on the piano. This gives for $\pi k_1 l$ a value of 10. Therefore, the quantity $\tanh(\pi k_1 l)$ can always be taken as unity. Then, Eq. (14) reduces to

$$-\tan\pi k_2 l = -\tan\left(\frac{\pi n\epsilon}{2}+\frac{\pi n}{2}\right)$$

$$= \cot\frac{\pi n\epsilon}{2} = \frac{[k_2{}^2+(1/4B^2)]^{\frac{1}{2}}}{k_2}. \tag{18}$$

If $n/2l(1+\epsilon)$ is substituted for k_2 and $1/\pi n\epsilon$ for $\cot \pi n\epsilon/2$, the following equation in ϵ is obtained:

$$[(\pi n\epsilon)^2/2]\{[(\pi n\epsilon)^3/2]Bn^2(1+\epsilon)^2\} = Bn^2(1+\epsilon)^2. \tag{19}$$

If higher powers of ϵ greater than ϵ^2 are neglected, the solution of (19) is

$$\epsilon = (2/\pi)B^{\frac{1}{2}} + (4/\pi^2)B. \tag{20}$$

If this value of ϵ is substituted in Eq. (16) and terms containing B to powers greater than unity are neglected, then

$$f_n = nf_0[1+(2/\pi)B^{\frac{1}{2}}+(4/\pi^2)B](1+Bn^2)^{\frac{1}{2}}. \tag{21}$$

This shows that clamping the ends, instead of pinning them, increases the partial frequencies by the factor

$$[1+(2/\pi)B^{\frac{1}{2}}+(4/\pi^2)B].$$

Equation (21) can be written

$$f_n = nf_0\{1+(4/\pi)B^{\frac{1}{2}}+[(12/\pi^2)+n^2]B\}^{\frac{1}{2}}, \tag{22}$$

which is the formula first deduced by Seebeck.[2]

If we start with Eq. (21) and retain only the square-root expansion, then this equation reduces to

$$f_n = nf_0\{1+(2/\pi)B^{\frac{1}{2}}+[(4/\pi^2)+(n^2/2)]B\}. \tag{23}$$

This is the formula given by Morse[3] in his book.

This was deduced for the even functions where $n=1$, 3, 5, 7, etc. For the odd functions $n=2$, 4, 6, 8, etc., Eq. (15) reduces to

$$\tan(\pi k_2 l) = k_2/k_1.$$

Again, let $k_2=(n/2l)(1+\epsilon)$. Then,

$$\tan(\pi k_2 l) = \tan(\pi n\epsilon/2) = k_2/k_1,$$

when n is an even integer. This equation is the same as Eq. (18). Therefore, the above equations hold for n being any integer, either even or odd.

An examination of the supports for the wires in a piano shows that the boundary condition lies somewhere between the two conditions treated above. In general, for either boundary conditions, one sees that

$$f_n = nF(1+Bn^2)^{\frac{1}{2}}, \tag{24}$$

where F and B are two constants that can be obtained from an accurate measurement of the frequencies of any two partials. The frequency of all the other partials can be obtained from Eq. (24).

If f_n is the frequency of the nth partial and f_m the frequency of the mth partial, then it follows from Eq. (24) that

$$F^2 = \frac{[(m/n)f_n]^2-[(n/m)f_m]^2}{m^2-n^2}, \tag{25}$$

$$B = \frac{(rm/n)^2-1}{n^2-(rm/n)^2m^2}, \tag{26}$$

where r is the ratio of the frequency of the nth partial to the frequency of the mth partial.

The value of f_0 is between the value of F and F divided by

$$1+(2/\pi)B^{\frac{1}{2}}+(4/\pi^2)B.$$

Since B depends upon f_0, there will be two calculated values of B corresponding to the two boundary conditions.

COMPARISON OF CALCULATED AND EXPERIMENTAL VALUES FOR B FOR SOLID PIANO WIRES

It will be seen from Eqs. (2) and (3) that the calculated value of B is given by

$$B = \pi^2 QSK^2/4l^4\sigma f_0{}^2. \tag{27}$$

Solid piano strings are round and made of steel. If we use cgs units for all numerical work, then the volume density of steel is 7.7, the value of $Q=19.5\times10^{11}$,

$$S=\pi d^2/4,\ K=d/4,\ \text{and}\ \sigma=7.7S.$$

Therefore,

$$B = 3.95\times10^{10}(d^2/l^4f_0{}^2). \tag{27a}$$

The values of d and l can be measured directly on an installed piano string, but the value of f_0 must be obtained from the fundamental frequency f_1 of the string. The relationship between f_1 and f_0 is different for the two boundary conditions considered. The value of f_1 is the same under each of the boundary conditions.

Let f_p represent f_0, calculated from f_1 with pinned boundary conditions, and f_c the corresponding value for the clamped condition. Also, let B_p and B_c be the corresponding calculated values of B from Eq. (27a).

For the fundamental frequency, $n=1$ so

$$f_1=f_p(1+B_p)^{\frac{1}{2}}, \tag{28}$$

and

$$f_1=f_c[1+(2/\pi)B_c^{\frac{1}{2}}+(4/\pi)B_c^2](1+B_c)^{\frac{1}{2}}. \tag{28a}$$

All of the quantities in Eq. (28), except f_0, are the same for both boundary conditions. Therefore,

$$B_c/B_p=(f_p/f'_c)^2$$
$$=[1+(4/\pi)B_c+(12/\pi^2)B_c][(1+B_c)/(1+B_p)]. \tag{28b}$$

Since the values of B_c and B_p range from 0.01 to 0.0001 and are never far apart, it is safe to put the last factor equal to unity. Also, the third term is usually negligible. For example, for notes below high C, the value of B is less than 0.01, and so the last term is always 500 or more times the second term. Therefore,

$$B_c=[1+(4/\pi)B_c^{\frac{1}{2}}]B_p. \tag{28c}$$

The values of f_1 can be obtained arbitrarily from a table or calculated from the equation

$$f_1=27.5\times 2^{(N-1)/12}, \tag{29}$$

where N is the number of the key starting with the first key on the left side of the keyboard as number 1 and the last key on the right as number 88.

The piano used for this comparison was a Hamilton upright with key system 763. The dimensions of the wires are given in Table I for the solid strings, starting with key No. 31. The values of B calculated from Eq. (28b) are given by the solid points in Fig. 1. The sudden breaks in the curve are due to sudden changes in the gauge of the piano wire as shown by underlines in Table I. The one exception to this is from key No. 58 to No. 59. The jump in the value of B at this point is due to a change in the frame that supports the wires, thus producing a sudden large change in the length of the wire.

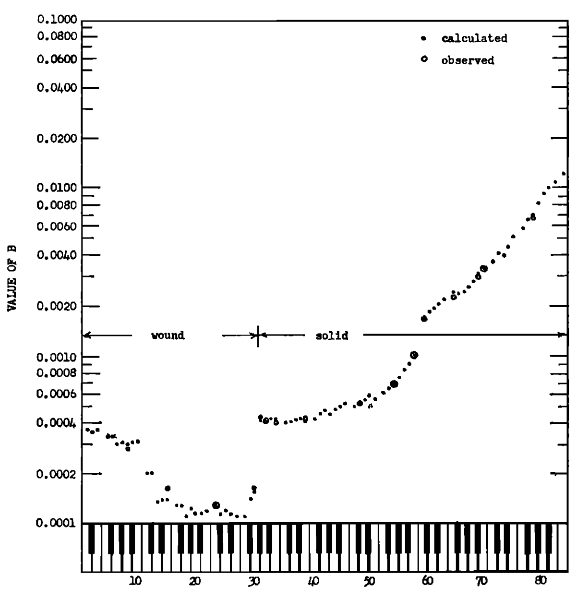

FIG. 1. Values of B in Hamilton upright piano (new model).

To find experimentally the values of B of this piano, one must find the frequencies of the partial tones. A method of doing this was described in **our** paper, "The Quality of Piano Tones."[12] This method was refined by obtaining a better analyzer with a passband only 4 cps wide. Another change was to pluck the string instead of hitting the key in the usual manner. This intensified the higher partials and, therefore, made it possible to obtain a greater accuracy in the measurement of their frequencies. The three strings corresponding to a single key in the upper register of the piano were not exactly in tune. In Table II are given the three frequencies as measured by a Stroboconn. For example, the frequency of the tenth partial of one of the G′ wires is more than twenty-three cps above one of the others. For this reason, it was difficult to identify the partial unless only one string was vibrating.

TABLE I. Dimensions of solid strings in Hamilton upright piano (new model).

Key No.	d (cm)	l (cm)	Key No.	d (cm)	l (cm)	Key No.	d (cm)	l (cm)
31	0.119	86.1	51	0.094	40.0	71	0.085	13.65
32	0.119	83.6	52	0.094	38.1	72	0.085	12.95
33	0.119	81.1	53	0.094	36.4	73	0.084	12.25
34	0.119	78.5	54	0.094	34.6	74	0.084	11.55
35	0.114	76.0	55	0.094	33.0	75	0.084	10.88
36	0.114	73.4	56	0.094	31.4	76	0.084	10.24
37	0.114	70.8	57	0.094	29.8	77	0.084	9.63
38	0.114	68.4	58	0.094	28.3	78	0.084	9.15
39	0.109	65.3	59	0.094	23.8	79	0.084	8.60
40	0.109	63.2	60	0.094	22.8	80	0.084	8.08
41	0.109	60.7	61	0.094	21.9	81	0.079	7.59
42	0.109	58.2	62	0.094	21.0	82	0.079	7.13
43	0.104	55.8	63	0.094	20.1	83	0.079	6.77
44	0.104	53.5	64	0.094	19.1	84	0.079	6.36
45	0.104	51.4	65	0.085	18.3	85	0.079	5.97
46	0.104	49.4	66	0.085	17.5	86	0.079	5.60
47	0.099	47.4	67	0.085	16.7	67	0.079	5.25
48	0.099	45.3	68	0.085	15.9	88	0.079	4.90
49	0.099	43.6	69	0.085	15.05			
50	0.099	41.7	70	0.085	14.35			

A third method, which gave a greater accuracy than either of the two mentioned above, is now described. An alternating magnetic driving force was produced by a small magnetic driving coil. When the frequency of the driving force was equal to one of the partial frequencies, a maximum sound was produced by the string vibrating in the mode corresponding to this frequency. A small microphone was placed near the string, which picked

[12] H. Fletcher, E. D. Blackham, and R. Stratton, "Quality of Piano Tones," J. Acoust. Soc. Am. 34, 749 (1962).

up the sound. The corresponding electrical current was sent to a voltmeter. The frequency was varied until the voltmeter read a maximum. This frequency was then measured either on the Stroboconn or an electronic counter. The former was used for the low frequencies and the latter for the high frequencies.

Typical results obtained by this last method are shown in Tables III and IV. The method of making the calculations of F and B from these data is also indicated in these tables. For all solid strings in this piano, the values of Bn^2 are always so small that Eq. (24) can be written

$$f_n - nF = BFn^3/2. \tag{24a}$$

If we designate $f_n - nF$ as Δf, then it is seen that

$$B = (2/F)(\Delta f/n^3). \tag{26b}$$

For calculating F and B from such data as those in Tables III and IV, it is convenient to choose the two partials such that $m = 2n$, in which case Eqs. (25) and (26) reduce to

$$F = (8f_n - f_{2n})/6n, \tag{25a}$$

$$B = (2/n^2)[(r-2)/(8-r)]. \tag{26a}$$

These equations make the calculations more simple. For example, in Table III, the number n of the partial is given in Column 1, and the corresponding observed frequency in column 2. In column 3, the values of $\Delta f/n^3$ are given. It is seen that these values are very nearly the same for partials from 4 to 17. For the lower partials, an observational error of 0.2 or 0.3 cps will account for the variation. For the partials higher than 16, the intensity level is so low that the ability to pick up the partial in the background noise becomes much more difficult and, hence, a large observational error results. So only values corresponding to $n=4$ to $n=16$ were used in the average values shown. The data in Table IV were similarly treated.

In this way, the experimental values of B were obtained, which are shown by the circles in Fig. 1. It is seen that they agree very well with the calculated values. These calculations are for the pinned boundary condition. The difference between B_p and B_c is not much greater than the observational error, being 10% for the highest key No. 78 and less than 4% for the

TABLE II. Frequencies of the three strings associated with one key.

A#	477.9	474.3	474.9
G	403.5	402.1	401.2
D	570.6	570.6	566.0
F#	717.3	713.1	712.7
G	765.2	765.6	763.2
C	1018.5	1020.8	1019.1
F	1350.1	1343.1	1334.6
D	2251.0	2253.6	2245.8

TABLE III. Observations and calculations for key No. 31.

n	obs f_n	$\Delta f/n^3$
1	152.6	...
2	305.8	0.025
3	459.2	0.030
4	613.0	0.0281
5	767.9	0.0312
6	924.3	0.0346
7	1080.9	0.0328
8	1240.2	0.0346
9	1399.7	0.0335
10	1561.7	0.0337
11	1725.8	0.0338
12	1891.8	0.0337
13	2060.0	0.0335
14	2230.3	0.0332
15	2405.0	0.0334
16	2574.0	0.0318
17	2744.0	0.0297
20	3317.0	0.0327

Calculation of F and B

n	F	B
1 and 2	152.50	0.0013
2 and 4	152.62	0.00082
3 and 6	152.74	0.00058
4 and 8	152.66	0.00048
5 and 10	152.77	0.00045
6 and 12	152.85	0.00043
7 and 14	152.78	0.00043
8 and 16	153.07	0.00045
10 and 20	152.94	0.00053
Average	152.8	0.000448

Average $\Delta f/n^3 = 0.0337$
Corresponding $B = 0.000441$
Final $B = 0.000444$

TABLE IV. Observations and calculations for key No. 59.

n	obs. f_n	$\Delta f/n^3$
1	777.2	0.50
2	1558.1	0.59
3	2348.0	0.66
4	3148.7	0.66
5	3966.0	0.66
6	4800.0	0.65
7	5647.0	0.61
8	6544.0	0.65
9	7451.0	0.64

Calculation of F and B

n	F	B
1 and 2	776.6	0.00160
2 and 4	776.4	0.00175
3 and 6	776.9	0.00166
4 and 8	776.9	0.00166
Average	776.7	0.00167

Average $\Delta f/n^3 = 0.650$
Corresponding $B = 0.000167$
Final $B = 0.000167$

keys below No. 58. The data seem to indicate that the pinned condition is the one that governs, although the evidence is not conclusive. It also shows that Eq. (28b), with the numerical constant shown, will give accurate values of B for the piano wires used on this piano. It is our understanding that these same wires are used generally for pianos.

Having the values of B, one can calculate all the partial frequencies from the equation

$$f_n=\left[f_1\bigg/\left(1+\frac{B}{2}\right)\right]\left(n+\frac{B}{2}n^3\right), \tag{24b}$$

where f_1 is the fundamental frequency.

PIANO STRINGS WITH STEEL CORE WOUND WITH COPPER WIRE

The first 30 keys are associated with wound wires, some with one and some with two copper windings. Therefore, the above equations cannot be applied without modification. If these cases, the elastic-restoring torque is due almost entirely to the steel core, but the linear density is due to the core and the windings. Let us first consider the linear density.

Determination of the Linear Density

Let d be the diameter of the steel core and D the diameter of the string including the winding. If the copper winding were a sheaf, its cross-sectional area would be

$$(\pi D^2/4)-(\pi d^2/4).$$

If it were a round wire, its cross section would be reduced by $\pi/4$ of this value. By use of the volume density of steel as 7.7 and of copper as 8.8, the value of the linear density of the wound string (in cgs units) is then

$$\sigma=5.43D^2-0.62d^2. \tag{30}$$

It must be remembered that this is the linear density of only the part of the string that is fully covered.

There are two windings on the core of some of these strings so that the copper may be packed somewhat tighter than for a single winding; but the above formula agrees with direct measurements of σ.

A section of the A'''' string was cut and weighed and σ was found to be 2.086. For this string, the outer diameter is 0.617 cm and the core diameter is 0.141 cm. Equation (30) then gives $\sigma=2.083$.

Similarly, for the G''' string, the value of σ was measured to be 0.860. The values of D and d were measured to 0.395 cm and 0.125 cm, so Eq. (30) gives the value of 0.860. This close agreement indicates that Eq. (30) should give fairly accurate values for the linear density.

Experimental Determination of QSK^2

It is known that a rod of length l clamped tightly at one end has a period of vibration of T sec, such that

$$QSK^2=3.19(l^4\sigma/T^2). \tag{31}$$

It is also known that such a bar will be deflected a distance y by a weight w that is hung l centimeters from the support if

$$QSK^2=\tfrac{1}{3}(980wl^3/y). \tag{32}$$

For the A'''' string, the first bass string on the piano, $\sigma=2.08$ gm/cm. Vibration tests gave the following results:

$l=34.5$ cm, $T=0.477$ sec, so, $QSK^2=4.1\times10^7$;
$l=43.0$ cm, $T=0.755$ sec, so, $QSK^2=4.0\times10^7$.

Deflection tests gave the following results:

$l=10.85$ cm, $y/w=0.01035$, so, $QSK^2=4.0\times10^7$;
$l=11.9$ cm, $y/w=0.0135$, so, $QSK^2=4.1\times10^7$.

Deflection measurements upon the core alone gave the value $QSK=3.84\times10^7$. A value calculated for a steel wire of diameter 0.141 cm from Eq. (28) is 3.80 $\times10^7$. This indicates that the measured value of QSK^2 only about 7% higher than that for the core alone. A solid steel wire of the same size would have a value 38 times larger than the one given above.

Similar deflection measurements that were made on the G''' string gave a value of $QSK^2=2.4\times10^7$. For the core alone, such measurements gave a value of 2.30×10^7, which may be compared to a value calculated for $d=0.124$ cm of 2.27×10^7. This indicates that the G''' string has a value of QSK^2 that is only about 5% higher than its core. Similar deflection measurements of the G'' string yielded $QSK^2=0.99\times10^7$ and for the core alone a value of 0.96×10^7.

Calculation of B for Wrapped Strings

If we assume that the restoring elastic torque is all due to the steel core, then $Q=19.5\times10^{11}$, $S=\pi d^2/4$, and $K=d/4$, where all of these quantities refer to the steel core. If D is the outer diameter, then the linear density σ is given approximately by $\sigma=5.5D^2$. If it is assumed that the windings extend the entire length l of the core, and that the torque is 7% greater than that produced by the core, then Eq. (28) becomes

$$B=4.6\times10^{10}(d^4/D^2f_0{}^2l^4). \tag{28c}$$

Since the windings do not cover the entire length of the core, there are two lengths l and l_1 for those wires having one winding, and three lengths l, l_1, and l_2 for those having two windings. There must be a reflection of the wave on the wire at the places where each winding stops. Consequently, we would expect three fundamental modes with the corresponding frequencies close together. Measurement of the partials of these strings show that there are many weak partials besides those accounted for by Eq. (24b). In this paper, no attempt is made to account for all these partials, but only the most prominent ones. These can be calculated approximately by Eq. (28c) if l is taken as the distance between where the outside winding starts and where it ends.

The dimensions of the bass strings in this piano are given in Table V. The value of l is taken as the value of l_2 for the first ten strings and as the value of l_1 in the other bass strings. Then the calculated values of B from Eq. (28c) are shown by the curve in Fig. 1. It is seen that there is fair agreement between the observed B and that calculated by (28c).

TABLE V. Dimensions of wound strings in Hamilton upright piano (new model).

Key No.	f_1	d	D	l	l_1	l_2
1	27.5	0.140	0.600	121.6	118.1	115.4
2	29.1	0.140	0.585	120.6	117.1	114.3
3	30.9	0.140	0.550	119.6	115.8	113.5
4	32.7	0.135	0.521	118.6	114.8	112.3
5	34.6	0.135	0.500	117.6	113.8	111.3
6	36.7	0.130	0.470	116.6	112.8	110.3
7	38.9	0.130	0.449	115.6	111.8	109.3
8	41.2	0.130	0.432	114.6	110.8	108.5
9	43.7	0.130	0.413	113.3	109.7	107.2
10	46.3	0.130	0.397	112.5	108.7	106.0
11	49.0	0.114	0.345	111.5	107.7	...
12	51.9	0.114	0.332	110.5	106.7	...
13	55.0	0.104	0.318	109.5	105.7	...
14	58.3	0.104	0.305	108.5	104.7	...
15	61.7	0.104	0.295	107.5	103.7	...
16	65.4	0.102	0.285	106.5	102.7	...
17	69.3	0.102	0.271	105.4	101.6	...
18	73.4	0.102	0.261	104.4	100.6	...
19	77.8	0.102	0.256	103.4	99.6	...
20	82.4	0.099	0.247	102.4	98.6	...
21	87.3	0.099	0.236	101.4	97.6	...
22	92.5	0.099	0.223	100.4	96.6	...
23	98.0	0.099	0.207	99.3	95.5	...
24	103.8	0.094	0.191	98.3	94.5	...
25	110.0	0.094	0.179	97.3	93.5	...
26	116.5	0.094	0.174	96.3	92.5	...
27	123.5	0.094	0.171	95.3	91.5	...
28	130.8	0.094	0.166	94.3	90.5	...
29	138.6	0.094	0.156	91.6	86.2	...
30	146.8	0.094	0.150	88.6	82.6	...

TABLE VI. Comparison of calculated and observed values of partial frequencies.

	String No. 23		String No. 54	
n	Calc	Obs	Calc	Obs
1	97.3	...	581.5	...
2	194.6	...	1163.3	...
3	292.1	...	1449.5	...
4	389.6	388.5	2338.5	2338.5
5	487.2	487.9	2932.0	2932.0
6	584.1	583.6	3532.0	3532.0
7	683.1	...	4139.0	4138.0
8	781.4	780.0	4755.0	4755.0
9	880.0	879.5	5381.0	5377.0
10	978.8	976.4	6013.0	6010.0
11	1078.0	1076.0	6661.0	6654.0
12	1178.0	1176.0		
13	1277.0	1276.0		
14	1378.0	1377.0		
15	1479.0	1478.0		
16	1591.0	1581.0		
17	1683.0	1682.0		
18	1686.0	1682.0		
19	1786.0	1782.0		
20	1889.0	1889.0		

To show how close the calculated values of the frequencies agree with those observed using the values of B in Fig. 1, a table of observed and calculated values is given in Table VI for string No. 23 and No. 54.

It is concluded that for the usual piano strings, such as are used in the Hamilton piano, Eq. (28) gives an accurate value of B for solid strings, and that the frequencies of the partials can be calculated from Eq. (24b). For the wound strings, Eq. (28c) gives a good approximation for B and also Eq. (24b) gives the partial frequencies. As indicated in our paper,[12] the excellence of the tone from a piano can not be said to be greater or less as the value of B becomes greater or less. There must be an optimum value of B for each string and this value has not yet been found. It is certainly not $B=0$, which would mean that all the partials should be harmonic.

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