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Selection of stable fixed points by the Toledano-Michel symmetry criterion: Six-component example

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Applications of symmetry to the renormalization-group method are discussed. The six-dimensional representations of space groups and their associated Hamiltonians are investigated using the Toledano-Michel symmetry criterion for stability. It is found that only two potentials have stable fixed points. One of these arises from a newly identified space-group image.

I. INTRODUCTION

The renormalization-group (RG) method[1] in reciprocal space yields a set of first-order differential equations (recursion relations). A stable fixed point of the Hamiltonian flow determined by these equations characterizes the critical behavior at the continuous transition for the associated physical system. There are as many types of initial effective Hamiltonians and RG recursion relations as there are types of quartic potentials. The Hamiltonian is a fourth-degree polynomial expansion in the order parameter and usually only includes isotropic gradient terms. (Some systems also allow anisotropic gradient terms.[2] We restrict our considerations here to contributions from isotropic terms only.) A natural generalization of the potential obtained in the Landau theory gives this initial Hamiltonian to which RG methods are to be applied.

The Landau theory[3] assumes the existence of an order parameter $\boldsymbol{\phi}$, which is an n-component vector in the carrier space E of an active physically irreducible representation (irrep) $\Gamma \rightarrow D(\Gamma)$ of the higher symmetry group Γ. The matrices $D(\gamma)$, representing $\gamma \in \Gamma$, are orthogonal matrices in n dimensions which satisfy the Landau[3] and Lifshitz[4] conditions ("active" irrep). The Landau potential is obtained by constructing invariant polynomials in the components of $\boldsymbol{\phi}$. To fourth degree the potential can be written

$$V = \frac{r}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + P_4(\boldsymbol{\phi}) \ , \tag{1}$$

with $P_4(\boldsymbol{\phi})$ of the general form

$$P_4 = \sum_{ijkl} u_{ijkl}\phi_i\phi_j\phi_k\phi_l = \sum_{\nu=1}^{p} u_\nu I_\nu(\boldsymbol{\phi}) \ . \tag{2}$$

Each $I_\nu(\boldsymbol{\phi})$ is an invariant polynomial and the u_ν are arbitrary coefficients carrying the temperature and pressure dependence of the potential. Including isotropic gradient terms generalizes this potential to give the effective Hamiltonian density for RG considerations:

$$H = \sum_{i=1}^{n} (\nabla\phi_i)^2 + \frac{r}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + P_4(\boldsymbol{\phi}) \ . \tag{3}$$

The RG method associates to a selected vector in P_4 (i.e., specific constants $u_\nu = u_\nu^0$) a flow of polynomials depending on the same invariants I_ν but with varying coefficients. Thus it determines a flow in the space P_4 spanned by the independent invariants I_ν. The characteristics of the flow are determined by p recursion relations which take the form

$$\frac{du_\nu}{d\ln\lambda} = \beta_\nu(u_{\nu'}) \ . \tag{4}$$

Critical properties are obtained from stable fixed points as $\lambda \rightarrow \infty$. A fixed point u_ν^* satisfies the p nonlinear equations

$$\beta_\nu(u_{\nu'}^*) = 0 \ , \tag{5}$$

and will be stable if in addition $(\partial\beta_\nu/\partial u_{\nu'})$ is a positive matrix at the fixed point. Michel has shown that to two-loop order, β_ν in Eq. (4) can be expressed as a symmetric product and that a stable fixed point must be unique.[5–7]

Recently, the covariance of the recursion relations and the unicity of the stable fixed point has been exploited. As a result Toledano and Michel[6–8] have introduced a symmetry criterion as a necessary condition for a stable fixed point. It allows a significant reduction in the actual number of Hamiltonians that need to be considered in the complete classification of stable fixed points associated with four-component order parameters.[7]

Here we make use of this symmetry criterion. We have recently obtained an active six-dimensional space-group image which has not been previously reported. Using this image as an example, we will show that the symmetry criterion immediately restricts considerations to selected subspaces of P_4. We will also indicate the presence or absence of stable fixed points for all densities which arise from six-dimensional active irreps of the 230 space groups.

II. IMAGES AND HAMILTONIAN DENSITIES

The process of obtaining irreps of the 230 space groups is well known.[9] The condition that the transition be commensurate imposes the Lifshitz condition, which in turn allows only $\mathbf{k}$ points of symmetry to be used in the construction of the irreps. The set G of distinct matrices of an irrep $D(\Gamma)$ is a subgroup of $O(n)$ and this same image (to within equivalence) may appear many times within the collection of irreps of the space groups.[10]

P_4 is obtained as the most general homogeneous quartic polynomial invariant under Γ. But the transformation prop-

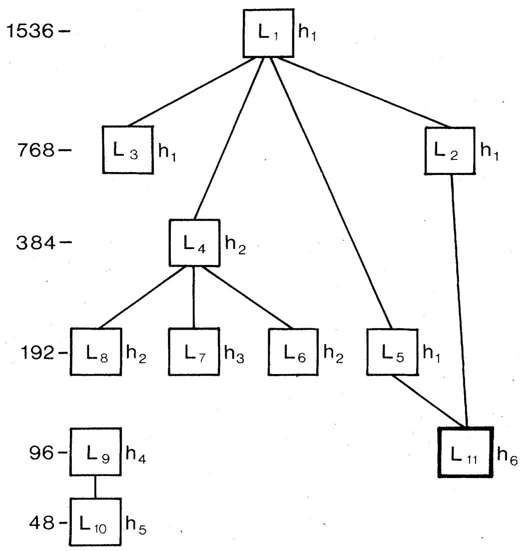

FIG. 1. Lattice tree of active six-dimensional images for the 230 space groups. A solid line indicates that the subgroup is a direct subgroup. The order of the image is indicated on the left. The active image L_{11} has not been previously listed.

erties of ϕ are entirely determined by G and therefore also the transformation properties of P_4. Thus, with respect to a selected basis in E, the same fourth-degree polynomial can be obtained for many of the irreps of space groups. Moreover, even though different images yield different potentials in higher-order expansions, truncations at fourth order may yield the same P_4.

For the six-dimensional irreps (arising from **k** points of symmetry) Toledano and Toledano[11] listed ten active images (labeled $L_1, \ldots, L_{10}$) and constructed a lattice (tree) of these images. We have also considered the lattice of images of six dimensions, both active and inactive. There is no standard of reference at this time for labeling the finite subgroups of $O(6)$. We thus use the labeling of Ref. 11 for active images. We indicate the lattice tree in Fig. 1. A solid line in Fig. 1 indicates that the group is a direct subgroup (not just equivalent to a subgroup) of the higher-order

TABLE I. Basic invariant polynomials which appear in the six-dimensional irreducible representations of space groups.

$$I_0=\eta_1^2+\zeta_1^2+\eta_2^2+\zeta_2^2+\eta_3^2+\zeta_3^2$$

$$I_1=\eta_1^4+\zeta_1^4+\eta_2^4+\zeta_2^4+\eta_3^4+\zeta_3^4$$

$$I_2=(\eta_1^2+\zeta_1^2)^2+(\eta_2^2+\zeta_2^2)^2+(\eta_3^2+\zeta_3^2)^2$$

$$I_3=\eta_1\zeta_1\eta_2\zeta_2+\eta_2\zeta_2\eta_3\zeta_3+\eta_3\zeta_3\eta_1\zeta_1$$

$$I_4=\eta_1^2\eta_2^2+\zeta_1^2\zeta_2^2+\eta_2^2\eta_3^2+\zeta_2^2\zeta_3^2+\eta_3^2\eta_1^2+\zeta_3^2\zeta_1^2$$

$$I_5=\eta_1^2\zeta_2^2+\eta_2^2\zeta_3^2+\eta_3^2\zeta_1^2$$

$$I_6=\eta_1\zeta_1(\eta_2^2+\zeta_2^2-\eta_3^2-\zeta_3^2)+\eta_2\zeta_2(\eta_3^2+\zeta_3^2-\eta_1^2-\zeta_1^2)+\eta_3\zeta_3(\eta_1^2+\zeta_1^2-\eta_2^2-\zeta_2^2)$$

$$I_7=\eta_1\zeta_1(\eta_1^2-\zeta_1^2)+\eta_2\zeta_2(\eta_2^2-\zeta_2^2)+\eta_3\zeta_3(\eta_3^2-\zeta_3^2)$$

$$I_8=\eta_1\zeta_1(\eta_2^2-\zeta_3^2)+\eta_2\zeta_2(\eta_3^2-\zeta_1^2)+\eta_3\zeta_3(\eta_1^2-\zeta_2^2)$$

$$I_9=\eta_1\zeta_1(\zeta_2^2-\eta_3^2)+\eta_2\zeta_2(\zeta_3^2-\eta_1^2)+\eta_3\zeta_3(\zeta_1^2-\eta_2^2)$$

TABLE II. Fourth-degree polynomials associated with the six-dimensional images.

Potential	Fourth-degree invariants
h_1	I_1, I_2
h_2	I_1, I_2, I_3
h_3	I_1, I_2, I_3, I_6
h_4	I_1, I_2, I_3, I_4, I_5
h_5	$I_1, I_2, I_3, I_4, I_5, I_7, I_8, I_9$
h_6	I_1, I_2, I_7

group. Moreover, all group-subgroup relationships in the tree are simultaneously satisfied. The orders of the subgroups are indicated on the left. Notice that we claim the existence of an active image (which we here label as L_{11}) that was not listed in Ref. 11. This image arises from irreps of O^6 and O^7.

The fourth-degree independent invariants can be constructed by use of conventional projection-operator techniques. We have selected the basis of the irreps so that the form of the quartic terms is essentially the same as Ref. 11, although we express the invariant polynomials in an alternate form. In Table I we list the basic invariant polynomials which arise in connection with the active images, and in Table II we indicate the six quartic potentials P_4 which occur for these images. The potential associated with each image is shown in Fig. 1.

III. SYMMETRY OF P_4 AND RG FLOW

Let us quickly review some terminology[6,8,12] before we state and apply the symmetry criterion to the six-dimensional image L_{11}. For more details and a comprehensive application to four-dimensional images, see Ref. 7.

Several images $G < O(n)$ may yield the same space of quartic invariants, e.g., L_1, L_2, L_3, and L_5. The centralizer G_c of P_4 is the largest subgroup of $O(n)$ leaving simultaneously invariant every polynomial in P_4. Thus G_c leaves invariant every vector of the space spanned by the I_ν. The centralizer contains, as a subgroup, each image group which generates the same space P_4. A given vector in P_4 has an invariance group G_0 which is called its little group. G_c is the intersection of all little groups. The largest subgroup of $O(n)$ which contains G_c as an invariant subgroup is called the normalizer G_n of G_c.

The RG recursion relations of Eq. (4) are covariant under transformations of $O(n)$.[6,13,14] Thus the little group G_0 for a vector in P_4 is not decreased along the flow trajectory and may increase at a fixed point.[14] Each trajectory then has a little group G_0 associated with it. The group G_c leaves each point of every trajectory invariant. The normalizer transforms any polynomial in P_4 into another polynomial of the same form, but generally with different coefficients. Thus it preserves P_4 as a whole. Each element g_n in G_n transforms a given trajectory with little group G_0 into a physically equivalent trajectory with little group $G_0'^{n}=g_n G_0 g_n^{-1}$. Also, a fixed point is transformed into a physically equivalent fixed point with a conjugated symmetry group and with the same stability and critical behavior.

As mentioned in the Introduction, it has been shown that if a stable fixed point exists it must be unique.[5–7] This result follows not from symmetry conditions alone, but from the symmetric product form of the recursion relations, namely, the $\beta_\nu(u_{\nu'})$ of Eq. (4) can be written at one loop order as

$$\beta(u)=\epsilon u-\tfrac{3}{2}u\wedge u \quad . \tag{6}$$

Here the symmetric product $\wedge$ is defined as

$$u\wedge v=\frac{1}{144}\left(\sum_{i,k}\frac{\partial^2 P_4^u}{\partial\phi_i\,\partial\phi_k}\frac{\partial^2 P_4^v}{\partial\phi_i\,\partial\phi_k}\right) \quad . \tag{7}$$

Similar results are found at two-loop order[7] for the case $n=4$. The uniqueness of the stable fixed point has led to two symmetry criteria on stable fixed points.[6–8] We state one criterion which will be of particular use to us here: A stable fixed point is necessarily characterized by the coincidence of the centralizer and the normalizer; i.e., $G_c^*=G_n^*$.

IV. SYMMETRY CRITERION APPLIED

There are six Hamiltonians that are of interest for the active six-dimensional images (see Fig. 1). We will use the new image L_{11} as an example of the application of the Toledano-Michel symmetry criterion. Generators for this image are given in Table III. The quartic potential obtained from this image is

$$P_4^{(1,2,7)}=u_0I_0^2+u_1I_1+u_2I_2+u_7I_7 \quad . \tag{8}$$

We use superscripts to indicate the invariant polynomials spanning P_4. I_0^2 is always considered present. Image L_{11} yields the set of invariants given in Eq. (8), but it is not the centralizer of $P_4^{(1,2,7)}$. The centralizer $G_c^{(1,2,7)}$ of Eq. (8) has four times as many elements as the image L_{11}. The two additional generators for G_c are given in Table IV. The centralizer $G_c^{(1,2,7)}$ is not equal to its normalizer $G_n^{(1,2,7)}$, since the direct product $A\otimes B$, where $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and B is the three-dimensional identity matrix, is an element of $G_n^{(1,2,7)}$ but not of $G_c^{(1,2,7)}$. Thus no stable fixed point can exist for a generic vector of $P_4^{(1,2,7)}$. It is possible that a stable fixed point might exist at a more symmetric vector of P_4. Thus we consider little groups, or equivalently, centralizers of a subspace of $P_4^{(1,2,7)}$.

Any subspace of $P_4^{(1,2,7)}$ containing a component along I_7 (for example $P_4^{(1,7)}$) yields the same centralizer, namely, $G_c^{(1,2,7)}$. Thus we can restrict our attention to the subspace consisting of invariants I_0^2, I_1, and I_2. But this is the space of invariants of image L_1. The stable fixed points of $P_4^{(1,2)}$ have been determined by conventional methods in Ref. 15. However, because we are demonstrating the use of the symmetry criterion, we will discuss $P_4^{(1,2)}$ from this approach.

TABLE III. Generators of the image L_{11}.

0	0	0	0	1	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0	1	0	0
1	0	0	0	0	0	0	−1	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	−1
0	0	0	1	0	0	0	0	0	0	1	0

TABLE IV. Generators additional to those of Table III, which generate the centralizer G_c of $P_4^{(1,2,7)}$.

−1	0	0	0	0	0	1	0	0	0	0	0
0	−1	0	0	0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	0	1	0	0	0	0	0	0	−1
0	0	0	0	0	1	0	0	0	0	1	0

The image L_1 is not the centralizer of $P_4^{(1,2)}$. $G_c^{(1,2)}$ is the wreath product $B_2 ⓦ S_3$ and is of order 3072. Here we use the notation of Coxeter and Moser,[16] where $B_n=Z_2 ⓦ S_n$. $G_c^{(1,2)}$ is not equal to its normalizer, since $A\otimes B$ leaves $G_c^{(1,2)}$ invariant when $A=1/\sqrt{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ and B is the three-dimensional identity matrix. Thus no stable fixed point is possible in this space of three invariant polynomials.

We thus have reduced our considerations to only two subspaces and their centralizers. The first is the subspace of the quartic potential

$$P_4^{(1)}=u_0I_0^2+u_1I_1 \quad . \tag{9}$$

Its centralizer $G_c^{(1)}$ is B_6. The second is the potential

$$P_4^{(2)}=u_0I_0^2+u_2I_2 \quad . \tag{10}$$

Its centralizer $G_c^{(2)}$ is denoted Γ_{23} in Ref. 6. The fixed points of both of these potentials have been well studied.[12,17] There is a stable fixed point of the potential $P_4^{(1)}$ at $(\frac{1}{3},\frac{2}{9})$ and a stable fixed point of $P_4^{(2)}$ at $(\frac{3}{11},\frac{3}{11})$. That these fixed points are stable for their respective potentials does not guarantee stability in the space $P_4^{(1,2,7)}$. We must check each fixed point, $(\frac{1}{3},\frac{2}{9},0,0)$ and $(\frac{3}{11},0,\frac{3}{11},0)$, for stability by conventional methods. The symmetric product discussed in Refs. 6 and 12 is useful in constructing the recursion relations and the associated Hessian matrix to check for stability. Of these fixed points only $(\frac{3}{11},0,\frac{3}{11},0)$ remains stable in $P_4^{(1,2,7)}$.

A similar approach can be used for all of the active six-dimensional images. The symmetry criterion, together with the use of conventional methods to check stability, allow stable fixed points in only two of the six potentials. Only stable fixed points can correspond to continuous transitions and thus determine critical exponents.[18] These stable fixed points arise from the same subspace in both cases. For h_1 we obtain the stable fixed point $(\frac{3}{11},0,\frac{3}{11})$ and for h_6 we obtain the stable fixed point $(\frac{3}{11},0,\frac{3}{11},0)$. All fixed points in the potentials h_2, h_3, h_4, and h_5 are not stable. The critical exponents can be obtained from the knowledge of two exponents through scaling laws. The general expression for the two exponents η and ν have been given in Ref. 19. The exponents for h_1 and h_6 are the same and thus of the same universality class. The exponents η and ν for this class have been previously given in Ref. 15.

V. CONCLUSION

The symmetry criterion,[6–8] combined with conventional methods, has been used to obtain the stable fixed points of Hamiltonians generated from six-dimensional images.

There is no listing of the finite subgroups of $O(6)$. As a result, determining centralizers and normalizers of subspaces in P_4 is very tedious. This was not the case in Ref. 7, where all finite subgroups of $O(4)$ were known. Under those more favorable conditions the symmetry criterion is a very useful tool.

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