# Symmetry-restricted phase transitions in two-dimensional solids

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Group-theoretical criteria which determine the possibility of group-subgroup phase transitions have been implemented on a computer. Lower-symmetry groups are determined by the subduction and chain criteria. We list all possible such symmetry-restricted transitions in two dimensions corresponding to $\vec{k}$ points of symmetry. We indicate relative origins and orientations of the prototype group and subgroup sufficient to obtain a wide class of useful experimental information.

## I. INTRODUCTION

Phase transitions in three-dimensional solids have been investigated theoretically and experimentally for many years. Mean-field (e.g., Landau theory[1]) and renormalization-group (RG) methods yield a well-developed approach to the analysis of space-group changes and the critical phenomena[2] associated with these transitions. The correspondence between models and physical systems yields similarity (universality) in critical properties for a variety of physical systems. For example, a wide class of ferromagnetic and ferroelectric transitions have the same critical exponents.

More recently, two-dimensional solids have become much more accessible to the experimentalist. Physically and chemically adsorbed atoms and molecules on crystal surfaces, surface reconstruction, domain growth on surfaces, etc., are being actively investigated with new techniques. In some cases, transitions associated with the solid surfaces have been investigated and correspondence made to model systems,[3,4] e.g., the Ising model, the three- and four-state Potts model, etc., thus providing a physical system to realize the theory. Barber[5] has written a review of phase transitions in two dimensions, discussing recent developments in theoretical and experimental methods. Based upon group-theoretical criteria, Rottman[6] has obtained Landau-Ginzburg-Wilson (LGW) Hamiltonians for the irreducible representations of the two-dimensional space groups obeying the Lifshitz criterion. Thus universality classes for these symmetry systems were indicated, and renormalization-group methods could subsequently be used for the description of the associated critical phenomena.

Deonarine and Birman[7] and independently Maksimov *et al.*[8] classified all possible symmetry changes allowed in two dimensions under the restriction that the phase transition correspond to $\vec{k}$ points of symmetry. It would be useful for the experimentalist to know origin and orientation relationships between the prototype space group and its lower symmetry phase, the change in cell size, order-parameter transformation properties, the corresponding representation of the order parameter which can then be used to obtain information about microscopic displacements and orderings, possible rotation and antiphase domains, etc. The tables listed by Maksimov *et al.* and those by Deonarine and Birman do not give sufficient information for the above properties to be obtained. Both papers indicate the irreducible representation (irrep) corresponding to the transition but do not give origin and relative orientations. In addition, both papers contain errors as well as omissions in their tables.

In the present paper, we list all possible group-subgroup transitions in two dimensions corresponding to $\vec{k}$ points of symmetry. The list is of isotropy subgroups and thus include both first- and second-order transitions. This complete list of isotropy groups is helpful in understanding the full phase diagram obtained from terms contributing to arbitrary order in the LGW Hamiltonian. We indicate space-group changes as well as information concerning the relationship of the lower-symmetry space group to the prototype space group sufficient to obtain the experimental information discussed above.

## II. LANDAU THEORY AND DIRECT GROUP-THEORETICAL METHODS

Classical Landau theory[1,9] presents a description of continuous phase transitions in which possible lower-symmetry phases can be predicted from the space group $G_0$. Here $G_0$ and the lower-symmetry group $G$ are each one of the 17 two-dimensional space groups. $G$ is a subgroup of $G_0$ with possible changes in cell size (lost translational symmetry), as well as possible lost local "point" symmetry.

The classical Landau theory has been reviewed elsewhere[10,11] and we indicate only the essential steps of that theory. Single irreps drive the transition. Only those irreps which satisfy the Landau and Lifshitz criteria can be active for commensurate transitions. The nonequilibrium (generalized) free energy $\Phi$ for the system is to be an invariant function of the order-parameter components $C_i^\alpha$ corresponding to irrep $D^\alpha$ of $G_0$. To lowest order in $C_i^\alpha$ (usually fourth order) solutions to extrema are obtained from the equations $\partial\Phi/\partial C_i^\alpha=0$ and subsequently absolute minima are retained. The density function

$$\delta\rho(\vec{r})=\sum_i C_i^\alpha \psi_i^\alpha(\vec{r})$$

 

corresponds to the nonzero order-parameter contribution in the lower phase, where $\psi_i^\alpha(\vec{r})$ are basis functions of irrep $D^\alpha$. The set of all transformations (of $G_0$) which leaves $\delta\rho(\vec{r})$ invariant is then the lower-symmetry space group.

Associated with the above formalism, certain direct group-theoretical criteria have been introduced which are necessary conditions for a continuous transition. We only briefly state them here. See Ref. 11 for a more complete discussion.

*(1) Subduction criterion.* The irrep $D^\alpha$ of $G_0$ must subduce into the identity irrep of $G$, that is,

$$i(G)=\frac{1}{|G|}\sum_g \chi^\alpha(g)\neq 0 ,$$

where $i(G)$ is the subduction frequency and $\chi^\alpha(g)$ is the character of $D^\alpha$ for the element $g$ of $G$. The summation is taken over all elements $g$ of $G$.

*(2) Chain criterion.* If $G$ is a subgroup of another subgroup $G'$ of $G_0$ [where $i(G')$ is also nonzero for irrep $D^\alpha$], then $i(G)$ must be greater than $i(G')$.

*(3) Landau criterion.* The symmetrized triple Kronecker product of the irrep $D^\alpha$ of $G_0$ does not contain the identity irrep of $G_0$, that is,

$$n_1=\frac{1}{|G_0|}\sum_{g_0}\{\tfrac{1}{3}\chi^\alpha(g_0^3)+\tfrac{1}{2}\chi^\alpha(g_0)\chi^\alpha(g_0^2)+\tfrac{1}{6}[\chi^\alpha(g_0)]^3\}=0 .$$

The summation is taken over all elements $g_0$ of $G_0$.

*(4) Lifshitz criterion.* The antisymmetrized double Kronecker product of $D^\alpha$ does not contain the vector representation of $G_0$, that is,

$$n_2=\frac{1}{|G_0|}\sum_{g_0}\tfrac{1}{2}\{[\chi^\alpha(g_0)]^2-\chi^\alpha(g_0^2)\}\chi^v(g_0)=0 ,$$

where $\chi^v(g_0)$ is the character of the vector representation of $G_0$. Note that in the equations above $\chi^\alpha$ is the character of a *physically* irreducible representation $D^\alpha$ of $G_0$. By physical, we mean that if a representation is complex, we consider the direct sum of it and its complex conjugate.

The subduction and chain criteria are valid in a more general transition (first order) if we restrict attention to a group-subgroup transition. These two criteria yield isotropy subgroups of $G_0$ corresponding to a vector (order parameter) of the representation space. The Landau and Lifshitz criteria in three dimensions are then additional conditions necessary for continuous commensurate transitions. These criteria select from the list of isotropy groups and if one then looks at absolute minima of the free energy a further selection is made. It has been proposed that minimization is equivalent to restricting attention to maximal isotropy groups,[12] i.e., isotropy groups of a given irrep which are not a subgroup of any other for that irrep regardless of subduction frequency. However, a counterexample to this principle has recently been presented.[13]

Although the Landau criterion appears to be valid in three dimensions, it is not necessary in two dimensions.[4,5] Thus we will not limit our consideration to only those irreps satisfying the Landau criterion. The Lifshitz criterion restricts our considerations to $\vec{k}$ points of symmetry only, since our interest here is in commensurate transitions.

We have recently implemented on computer an algorithm for obtaining *all* isotropy subgroups of space groups in *two and three dimensions.*[14] The details of the method and the algorithm with the resulting tables for three-dimensional space groups are to be published elsewhere. We have also tabulated by computer the Landau and Lifshitz criteria. We do not impose minimization of the lowest-order free-energy expansion which, as mentioned above, would additionally discard possible transitions.

TABLE I. The basis vectors, $\vec{t}_1$ and $\vec{t}_2$, of the two-dimensional Bravais lattices. The vectors are given in terms of $(x,y)$ Cartesian coordinates. We use the convention of Cracknell (Ref. 15).

| Lattice | $\vec{t}_1$ | $\vec{t}_2$ |
|---|---|---|
| Oblique, $p$ | $(a,0)$ | $(b\cos\theta, b\sin\theta)$ |
| Rectangular, $p$ | $(a,0)$ | $(0,b)$ |
| Rectangular, $c$ | $(\frac{1}{2}a,\frac{1}{2}b)$ | $(-\frac{1}{2}a,\frac{1}{2}b)$ |
| Square, $p$ | $(a,0)$ | $(0,a)$ |
| Hexagonal, $p$ | $(0,-a)$ | $(\frac{1}{2}\sqrt{3}a,\frac{1}{2}a)$ |

## III. PHASE TRANSITIONS IN TWO DIMENSIONS

Irreps of the two-dimensional space groups are obtained by induction from the irreps of the little group of $\vec{k}$. As stated earlier, we restrict considerations to $\vec{k}$ points of symmetry in the first Brillouin zone since others will not satisfy the Lifshitz criterion and would thus disallow commensurate continuous transitions. Our labeling and choice of irreps will follow that of Cracknell.[15] The short symbol of the space groups is used, and $k_i$ indicates the $i$th irrep of the vector $\vec{k}$, e.g., $\Gamma_2$ is the second irrep of $\vec{k}=0$. If an irrep is complex, we take the physically irreducible representation and, for example, indicate $X_1+X_2$ as the physical irrep.

So that this paper may be as self-contained as possible and include sufficient detail to obtain the experimental properties listed earlier, we repeat some of the information contained in Cracknell.[15] Tables I and II give the basis vectors of the two-dimensional Bravais and reciprocal lattices. Table III lists the two-dimensional space groups

TABLE II. The basis vectors, $\vec{g}_1$ and $\vec{g}_2$, of the two-dimensional reciprocal lattice. The vectors are given in terms of $(x,y)$ Cartesian coordinates.

| Lattice | $\vec{g}_1$ | $\vec{g}_2$ |
|---|---|---|
| Oblique, $p$ | $(2\pi/a)(1,-\cot\theta)$ | $(2\pi/b)(0,\csc\theta)$ |
| Rectangular, $p$ | $(2\pi/a)(1,0)$ | $(2\pi/b)(0,1)$ |
| Rectangular, $c$ | $2\pi(1/a,1/b)$ | $2\pi(-1/a,1/b)$ |
| Square, $p$ | $(2\pi/a)(1,0)$ | $(2\pi/a)(0,1)$ |
| Hexagonal, $p$ | $(2\pi/a)(1/\sqrt{3},-1)$ | $(2\pi/a)(2/\sqrt{3},0)$ |

TABLE III. The 17 two-dimensional space groups and their generating elements. Vectors are given in terms of components of $\vec{t}_1$, $\vec{t}_2$. We use the convection of Cracknell (Ref. 15).

| System | Number | Symbol | Generating elements |
|---|---|---|---|
| Oblique | 1 | $p1$ | $\{E\mid 00\}$ |
| | 2 | $p2$ | $\{C_{2z}\mid 00\}$ |
| Rectangular | 3 | $pm$ | $\{\sigma_y\mid 00\}$ |
| | 4 | $pg$ | $\{\sigma_y\mid \frac{1}{2}0\}$ |
| | 5 | $cm$ | $\{\sigma_x\mid 00\}$ |
| | 6 | $pmm$ | $\{C_{2z}\mid 00\},\{\sigma_y\mid 00\}$ |
| | 7 | $pmg$ | $\{C_{2z}\mid 00\},\{\sigma_y\mid \frac{1}{2}0\}$ |
| | 8 | $pgg$ | $\{C_{2z}\mid 00\},\{\sigma_x\mid \frac{1}{2}\frac{1}{2}\}$ |
| | 9 | $cmm$ | $\{C_{2z}\mid 00\},\{\sigma_x\mid 00\}$ |
| Square | 10 | $p4$ | $\{C_{4z}^{+}\mid 00\}$ |
| | 11 | $p4m$ | $\{C_{4z}^{+}\mid 00\},\{\sigma_x\mid 00\}$ |
| | 12 | $p4g$ | $\{C_{4z}^{+}\mid 00\},\{\sigma_x\mid \frac{1}{2}\frac{1}{2}\}$ |
| Hexagonal | 13 | $p3$ | $\{C_3^{+}\mid 00\}$ |
| | 14 | $p3m1$ | $\{C_3^{+}\mid 00\},\{\sigma_{v1}\mid 00\}$ |
| | 15 | $p31m$ | $\{C_3^{+}\mid 00\},\{\sigma_{d1}\mid 00\}$ |
| | 16 | $p6$ | $\{C_6^{+}\mid 00\}$ |
| | 17 | $p6m$ | $\{C_6^{+}\mid 00\},\{\sigma_{d1}\mid 00\}$ |

and their generating elements. Table IV gives the $\vec{k}$ points of symmetry for each Brillouin zone. Table V gives the irreps of the two-dimensional space groups at the $\vec{k}$ points of symmetry. This table uses notation similar to Table 5.7 in Bradley and Cracknell[16] for the three-dimensional space groups. The irreps and their reality are listed in numerical order. For example, the irreps $X_1$, $X_2$, $X_3$, and $X_4$ of space group *pmm* (labeling and numbering follow the convention of Cracknell[15]) correspond to the irreps $R_1$, $R_3$, $R_2$, and $R_4$, respectively, of the abstract group $G_4^2$ listed in Table 5.1 of Bradley and Cracknell.[16] They each have reality of type 1.

In Table VI we list the results of the computer calculation. We give the high-symmetry space group $G_0$, its irrep, the Landau criterion $n_1$, the Lifshitz criterion $n_2$, the number of arms of the star of $\vec{k}$, the change in primitive cell size (volume), the lower-symmetry space group $G$, its subduction frequency $i(G)$, the primitive basis vectors of $G$ in terms of the primitive basis vectors of $G_0$, and the change of space-group origin (from $G_0$ to $G$) in terms of the primitive basis vectors of $G_0$. In our tables, we do not indicate which subgroups are maximal since for real irreps the maximal subgroups have subduction frequency $i(G)=1$, while for complex irreps, the maximal subgroups have subduction frequency $i(G)=2$. We have compared our listing of isotropy subgroups with the listing of subgroups given by Sayari *et al.*[17] We find complete agreement since all of our isotropy subgroups are contained in the appropriate allowed listing as given in their table.

## IV. COMPARISON WITH PREVIOUS LISTINGS

Of the many papers written concerning phase transitions in two dimensions, three are notable in that they seek to classify all possible commensurate transitions consistent with mean-field theory (or RG). These are the papers by Rottman,[6] Maksimov *et al.*,[8] and Deonarine and Birman.[7]

TABLE IV. The $\vec{k}$ points of symmetry for the two-dimensional lattices. The vectors are given in terms of $\vec{g}_1$ and $\vec{g}_2$. We use the convention of Cracknell (Ref. 15).

| Lattice | $\vec{k}$ |
|---|---|
| Oblique, $p$ | $\Gamma(00)$ |
| | $X(\frac{1}{2}0)$ |
| | $Y(0\frac{1}{2})$ |
| | $A(\frac{1}{2}\frac{1}{2})$ |
| Rectangular, $p$ | $\Gamma(00)$ |
| | $X(\frac{1}{2}0)$ |
| | $Y(0\frac{1}{2})$ |
| | $S(\frac{1}{2}\frac{1}{2})$ |
| Rectangular, $c$ | $\Gamma(00)$ |
| | $S(\frac{1}{2}0)$ |
| | $X(\frac{1}{2}\frac{1}{2})$ |
| Square, $p$ | $\Gamma(00)$ |
| | $X(\frac{1}{2}0)$ |
| | $M(\frac{1}{2}\frac{1}{2})$ |
| Hexagonal, $p$ | $\Gamma(00)$ |
| | $M(0\frac{1}{2})$ |
| | $K(-\frac{1}{3}\frac{2}{3})$ |

TABLE V. The irreps of the two-dimensional space groups $G_0$ at the $\vec{k}$ points of symmetry. The abstract groups are given in Table 5.1 of Bradley and Cracknell (Ref. 16). The generating elements of the abstract groups are $P,Q,R$ (as needed). The irreps are given in numerical order, using the convention of Cracknell (Ref. 15). Each pair of numbers $i,j$ refers to the irrep $R_i$ of the abstract group and reality type equal to $j$. This table uses notation similar to Table 5.7 in Bradley and Cracknell (Ref. 16) for the three-dimensional space groups.

| $G$ | $\vec{k}$ | Abstract group | $P,Q,R$ | irreps,reality |
|---|---|---|---|---|
| $p1$ | $\Gamma$ | $G_1^1$ | $\{E\,\|\,00\}$ | 1,1 |
| | $X$ | $G_1^1\times T_2$ | $\{E\,\|\,00\}$ | 1,1 |
| | $Y$ | $G_1^1\times T_2$ | $\{E\,\|\,00\}$ | 1,1 |
| | $A$ | $G_1^1\times T_2$ | $\{E\,\|\,00\}$ | 1,1 |
| $p2$ | $\Gamma$ | $G_2^1$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| | $X$ | $G_2^1\times T_2$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| | $Y$ | $G_2^1\times T_2$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| | $A$ | $G_2^1\times T_2$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| $pm$ | $\Gamma$ | $G_2^1$ | $\{\sigma_y\,\|\,00\}$ | 1,1;2,1 |
| | $X$ | $G_2^1\times T_2$ | $\{\sigma_y\,\|\,00\}$ | 1,1;2,1 |
| | $Y$ | $G_2^1\times T_2$ | $\{\sigma_y\,\|\,00\}$ | 1,1;2,1 |
| | $S$ | $G_2^1\times T_2$ | $\{\sigma_y\,\|\,00\}$ | 1,1;2,1 |
| $pg$ | $\Gamma$ | $G_2^1$ | $\{\sigma_y\,\|\,\frac{1}{2}0\}$ | 1,1;2,1 |
| | $X$ | $G_4^1$ | $\{\sigma_y\,\|\,\frac{1}{2}0\}$ | 2,3;4,3 |
| | $Y$ | $G_2^1\times T_2$ | $\{\sigma_y\,\|\,\frac{1}{2}0\}$ | 1,1;2,1 |
| | $S$ | $G_4^1$ | $\{\sigma_y\,\|\,\frac{1}{2}0\}$ | 2,3;4,3 |
| $cm$ | $\Gamma$ | $G_2^1$ | $\{\sigma_x\,\|\,00\}$ | 1,1;2,1 |
| | $S$ | $G_1^1\times T_2$ | $\{E\,\|\,00\}$ | 1,1 |
| | $X$ | $G_2^1\times T_2$ | $\{\sigma_x\,\|\,00\}$ | 1,1;2,1 |
| $pmm$ | $\Gamma$ | $G_4^2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| | $X$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| | $Y$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| | $S$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| $pmg$ | $\Gamma$ | $G_4^2$ | $\{C_{2z}\,\|\,00\},\{\sigma_x\,\|\,\frac{1}{2}0\}$ | 1,1;4,1;2,1;3,1 |
| | $X$ | $G_8^4$ | $\{\sigma_y\,\|\,\frac{1}{2}0\},\{\sigma_x\,\|\,\frac{1}{2}0\}$ | 5,1 |
| | $Y$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_x\,\|\,\frac{1}{2}0\}$ | 1,1;4,1;2,1;3,1 |
| | $S$ | $G_8^4$ | $\{\sigma_y\,\|\,\frac{1}{2}0\},\{\sigma_x\,\|\,\frac{1}{2}0\}$ | 5,1 |
| $pgg$ | $\Gamma$ | $G_4^2$ | $\{C_{2z}\,\|\,00\},\{\sigma_x\,\|\,\frac{1}{2}\frac{1}{2}\}$ | 1,1;3,1;2,1;4,1 |
| | $X$ | $G_8^4$ | $\{\sigma_y\,\|\,\frac{1}{2}\frac{1}{2}\},\{C_{2z}\,\|\,00\}$ | 5,1 |
| | $Y$ | $G_8^4$ | $\{\sigma_x\,\|\,\frac{1}{2}\frac{1}{2}\},\{C_{2z}\,\|\,00\}$ | 5,1 |
| | $S$ | $G_8^2$ | $\{\sigma_x\,\|\,\frac{1}{2}\frac{1}{2}\},\{C_{2z}\,\|\,00\}$ | 6,3;8,3;4,3;2,3 |
| $cmm$ | $\Gamma$ | $G_4^2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| | $S$ | $G_2^1\times T_2$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| | $X$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;3,1;2,1;4,1 |
| $p4$ | $\Gamma$ | $G_4^1$ | $\{C_{4z}^+\,\|\,00\}$ | 1,1;3,1;2,3;4,3 |
| | $X$ | $G_2^1\times T_2$ | $\{C_{2z}\,\|\,00\}$ | 1,1;2,1 |
| | $M$ | $G_4^1\times T_2$ | $\{C_{4z}^+\,\|\,00\}$ | 1,1;3,1;2,3;4,3 |
| $p4m$ | $\Gamma$ | $G_8^4$ | $\{C_{4z}^+\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;2,1;3,1;4,1;5,1 |
| | $X$ | $G_4^2\times T_2$ | $\{C_{2z}\,\|\,00\},\{\sigma_y 00\}$ | 1,1;3,1;2,1;4,1 |
| | $M$ | $G_8^4\times T_2$ | $\{C_{4z}^+\,\|\,00\},\{\sigma_y\,\|\,00\}$ | 1,1;2,1;3,1;4,1;5,1 |
| $p4g$ | $\Gamma$ | $G_8^4$ | $\{C_{4z}^+\,\|\,00\},\{\sigma_y\,\|\,\frac{1}{2}\frac{1}{2}\}$ | 1,1;2,1;3,1;4,1;5,1 |
| | $X$ | $G_8^4$ | $\{\sigma_y\,\|\,\frac{1}{2}\frac{1}{2}\},\{\sigma_x\,\|\,\frac{1}{2}\frac{1}{2}\}$ | 5,1 |
| | $M$ | $G_{16}^{10}$ | $\{C_{4z}^+\,\|\,00\},\{C_{2z}\,\|\,01\},\{\sigma_{db}\,\|\,\frac{1}{2}\frac{1}{2}\}$ | 8,3;5,3;6,3;7,3;9,1 |
| $p3$ | $\Gamma$ | $G_3^1$ | $\{C_3^+\,\|\,00\}$ | 1,1;2,3;3,3 |
| | $M$ | $G_1^1\times T_2$ | $\{E\,\|\,00\}$ | 1,1 |
| | $K$ | $G_3^1\times T_3$ | $\{C_3^+\,\|\,00\}$ | 1,$x$;2,$x$;3,$x$ |
| $p3m1$ | $\Gamma$ | $G_6^2$ | $\{C_3^+\,\|\,00\},\{\sigma_{v1}\,\|\,00\}$ | 1,1;2,1;3,1 |
| | $M$ | $G_2^1\times T_2$ | $\{\sigma_{v1}\,\|\,00\}$ | 1,1;2,1 |
| | $K$ | $G_3^1\times T_3$ | $\{C_3^+\,\|\,00\}$ | 1,1;2,1;3,1 |
| $p31m$ | $\Gamma$ | $G_6^2$ | $\{C_3^+\,\|\,00\},\{\sigma_{d1}\,\|\,00\}$ | 1,1;2,1;3,1 |

TABLE V. (*Continued*).

| G | $\vec{k}$ | Abstract group | P,Q,R | irreps,reality |
|---|---|---|---|---|
| | M | $G_2^1 \times T_2$ | $\{\sigma_{d1}\|00\}$ | 1,1;2,1 |
| | K | $G_6^2 \times T_3$ | $\{C_3^+\|00\},\{\sigma_{d1}\|00\}$ | 1,$x$;2,$x$;3,$x$ |
| $p6$ | Γ | $G_6^1$ | $\{C_6^+\|00\}$ | 1,1;5,3;3,3;4,1;2,3;6,3 |
| | M | $G_2^1 \times T_2$ | $\{C_{2z}\|00\}$ | 1,1;2,1 |
| | K | $G_3^1 \times T_3$ | $\{C_3^+\|00\}$ | 1,1;2,3;3,3 |
| $p6m$ | Γ | $G_{12}^3$ | $\{C_6^+\|00\},\{\sigma_{v1}\|00\}$ | 1,1;2,1;4,1;3,1;6,1;5,1 |
| | M | $G_4^2 \times T_2$ | $\{C_{2z}\|00\},\{\sigma_{v1}\|00\}$ | 1,1;3,1;2,1;4,1 |
| | K | $G_6^2 \times T_3$ | $\{C_3^+\|00\},\{\sigma_{d1}\|00\}$ | 1,1;2,1;3,1 |

TABLE VI. Isotropy subgroups $G$ of the 17 two-dimensional space groups $G_0$ for the $\vec{k}$ points of symmetry. We give the Landau criterion $n_1$, the Lifshitz criterion $n_2$, the number of arms of the star of $\vec{k}$, the change in primitive cell size (volume), the subduction frequency $i(G)$, the primitive basis vectors of $G$ in terms of the primitive basis vectors of $G_0$, and the change in space-group origin (from $G_0$ to $G$) in terms of the primitive basis vectors of $G_0$.

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | G | $i(G)$ | Basis vectors | | Origin |
|---|---|---|---|---|---|---|---|---|---|---|
| $p1$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $p1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | $p1$ | 1 | 2,0 | 0,1 | 0,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | $p1$ | 1 | 1,0 | 0,2 | 0,0 |
| | $A_1$ | 0 | 0 | 1 | 2 | $p1$ | 1 | 1,$\bar{1}$ | 1,1 | 0,0 |
| $p2$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $p2$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | $p1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 2,0 | 0,1 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_2$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $A_1$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 1,$\bar{1}$ | 1,1 | 0,0 |
| | $A_2$ | 0 | 0 | 1 | 2 | $p2$ | 1 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| $pm$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $pm$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | $p1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | $pm$ | 1 | 2,0 | 0,1 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | $pg$ | 1 | 2,0 | 0,1 | 0,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | $pm$ | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_2$ | 0 | 0 | 1 | 2 | $pm$ | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $S_1$ | 0 | 0 | 1 | 2 | $cm$ | 1 | $\bar{1}$,1 | $\bar{1}$,$\bar{1}$ | 0,0 |
| | $S_2$ | 0 | 0 | 1 | 2 | $cm$ | 1 | $\bar{1}$,1 | $\bar{1}$,$\bar{1}$ | $\frac{1}{2}$,$\frac{1}{2}$ |
| $pg$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $pg$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | $p1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $X_1+X_2$ | 0 | 1 | 1 | 2 | $p1$ | 2 | 2,0 | 0,1 | 0,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | $pg$ | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_2$ | 0 | 0 | 1 | 2 | $pg$ | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $S_1+S_2$ | 0 | 1 | 1 | 2 | $p1$ | 2 | 1,$\bar{1}$ | 1,1 | 0,0 |
| $cm$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $cm$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | $p1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $S_1$ | 0 | 0 | 1 | 2 | $p1$ | 1 | 1,0 | 0,2 | 0,0 |
| | | | | | | $p1$ | 1 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | $cm$ | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | $p1$ | 2 | 2,0 | 0,2 | 0,0 |

TABLE VI. (*Continued*).

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | $G$ | $i(G)$ | Basis vectors | | Origin |
|---|---|---|---|---|---|---|---|---|---|---|
| | $X_1$ | 0 | 0 | 1 | 2 | *pm* | 1 | $\bar{1},\bar{1}$ | $1,\bar{1}$ | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | *pg* | 1 | $\bar{1},\bar{1}$ | $1,\bar{1}$ | $\frac{1}{4},\frac{3}{4}$ |
| *pmm* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *pmm* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *pm* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *p*2 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *pm* | 1 | $0,\bar{1}$ | 1,0 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 2,0 | 0,1 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | $X_3$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 2,0 | 0,1 | 0,0 |
| | $X_4$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_2$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 0,2 | $\bar{1}$,0 | 0,$\frac{1}{2}$ |
| | $Y_3$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 0,2 | $\bar{1}$,0 | 0,0 |
| | $Y_4$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $S_1$ | 0 | 0 | 1 | 2 | *cmm* | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | $S_2$ | 0 | 0 | 1 | 2 | *cmm* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,0 |
| | $S_3$ | 0 | 0 | 1 | 2 | *cmm* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2},\frac{1}{2}$ |
| | $S_4$ | 0 | 0 | 1 | 2 | *cmm* | 1 | 1,1 | $\bar{1}$,1 | 0,$\frac{1}{2}$ |
| *pmg* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *pmg* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *pg* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *p*2 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *pm* | 1 | $0,\bar{1}$ | 1,0 | $\frac{1}{4}$,0 |
| | $X_1$ | 0 | 1 | 1 | 2 | *pm* | 1 | $0,\bar{1}$ | 2,0 | $\frac{1}{4}$,0 |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | 0,0 |
| | | | | | | *p*1 | 2 | 2,0 | 0,1 | 0,0 |
| | $Y_1$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_2$ | 0 | 0 | 1 | 2 | *pgg* | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $Y_3$ | 0 | 0 | 1 | 2 | *pgg* | 1 | 1,0 | 0,2 | 0,0 |
| | $Y_4$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | $S_1$ | 0 | 1 | 1 | 2 | *cm* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{4},\frac{1}{4}$ |
| | | | | | | *p*2 | 1 | $1,\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| | | | | | | *p*2 | 1 | $1,\bar{1}$ | 1,1 | 0,0 |
| | | | | | | *p*1 | 2 | $1,\bar{1}$ | 1,1 | 0,0 |
| *pgg* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *pgg* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *pg* | 1 | 1,0 | 0,1 | 0,$\frac{1}{4}$ |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *p*2 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *pg* | 1 | $0,\bar{1}$ | 1,0 | $\frac{1}{4}$,0 |
| | $X_1$ | 0 | 1 | 1 | 2 | *pg* | 1 | $0,\bar{1}$ | 2,0 | $\frac{1}{4}$,0 |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | 0,0 |
| | | | | | | *p*1 | 2 | 2,0 | 0,1 | 0,0 |
| | $Y_1$ | 0 | 1 | 1 | 2 | *pg* | 1 | 1,0 | 0,2 | 0,$\frac{1}{4}$ |
| | | | | | | *p*2 | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | | | | | | *p*2 | 1 | 1,0 | 0,2 | 0,0 |
| | | | | | | *p*1 | 2 | 1,0 | 0,2 | 0,0 |

TABLE VI. (*Continued.*)

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | $G$ | $i(G)$ | Basis vectors | | Origi |
|---|---|---|---|---|---|---|---|---|---|---|
| | $S_1+S_2$ | 0 | 0 | 1 | 2 | *p*2 | 2 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,( |
| | $S_3+S_4$ | 0 | 0 | 1 | 2 | *p*2 | 2 | 1,$\bar{1}$ | 1,1 | 0,0 |
| *cmm* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *cmm* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *cm* | 1 | 0,1 | $\bar{1}$,0 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *p*2 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *cm* | 1 | 1,0 | 0,1 | 0,0 |
| | $S_1$ | 0 | 0 | 1 | 2 | *p*2 | 1 | 1,0 | 0,2 | 0,0 |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | *cmm* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p*2 | 2 | 2,0 | 0,2 | 0,0 |
| | $S_2$ | 0 | 0 | 1 | 2 | *p*2 | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | | | | | | *p*2 | 1 | 2,0 | 0,1 | $\frac{1}{2}$,C |
| | | | | 2 | 4 | *cmm* | 1 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *p*2 | 2 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $X_1$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 1,$\bar{1}$ | 1,1 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 1,1 | $\bar{1}$,1 | 0,$\frac{1}{2}$ |
| | $X_3$ | 0 | 0 | 1 | 2 | *pgg* | 1 | 1,$\bar{1}$ | 1,1 | 0,0 |
| | $X_4$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| *p*4 | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p*4 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *p*2 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3+\Gamma_4$ | 0 | 0 | 1 | 1 | *p*1 | 2 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | *p*2 | 1 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | *p*4 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p*2 | 2 | 2,0 | 0,2 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | *p*2 | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | 2 | 4 | *p*4 | 1 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *p*2 | 2 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $M_1$ | 0 | 0 | 1 | 2 | *p*4 | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | *p*4 | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $M_3+M_4$ | 0 | 0 | 1 | 2 | *p*2 | 2 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| *p*4*m* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p*4*m* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *p*4 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *pmm* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *cmm* | 1 | 0,1 | $\bar{1}$,0 | 0,0 |
| | $\Gamma_5$ | 0 | 0 | 1 | 1 | *cm* | 1 | 0,1 | $\bar{1}$,0 | 0,0 |
| | | | | | | *pm* | 1 | 0,$\bar{1}$ | 1,0 | 0,0 |
| | | | | | | *p*1 | 2 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | *p*4*m* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *pmm* | 2 | 2,0 | 0,2 | 0,0 |
| | $X_2$ | 0 | 0 | 1 | 2 | *pmm* | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | 2 | 4 | *p*4*m* | 1 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *pmm* | 2 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $X_3$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | *p*4*g* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *pgg* | 2 | 2,0 | 0,2 | 0,0 |

TABLE VI. (*Continued*).

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | $G$ | $i(G)$ | Basis vectors | | Origin |
|---|---|---|---|---|---|---|---|---|---|---|
| | $X_4$ | 0 | 0 | 1 | 2 | *pmg* | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | 2 | 4 | *p4g* | 1 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *pgg* | 2 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $M_1$ | 0 | 0 | 1 | 2 | *p4m* | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | *p4g* | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | $M_3$ | 0 | 0 | 1 | 2 | *p4g* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $M_4$ | 0 | 0 | 1 | 2 | *p4m* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | $M_5$ | 0 | 0 | 1 | 2 | *cmm* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,0 |
| | | | | | | *pmg* | 1 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| | | | | | | *p2* | 2 | 1,$\bar{1}$ | 1,1 | $\frac{1}{2}$,0 |
| *p4g* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p4g* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *p4* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | *pgg* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *cmm* | 1 | 0,1 | $\bar{1}$,0 | 0,$\frac{1}{2}$ |
| | $\Gamma_5$ | 0 | 0 | 1 | 1 | *cm* | 1 | 0,1 | $\bar{1}$,0 | 0,$\frac{1}{2}$ |
| | | | | | | *pg* | 1 | 0,$\bar{1}$ | 1,0 | $\frac{1}{4}$,0 |
| | | | | | | *p1* | 2 | 1,0 | 0,1 | 0,0 |
| | $X_1$ | 0 | 1 | 1 | 2 | *pg* | 1 | 0,$\bar{1}$ | 2,0 | $\frac{1}{4}$,0 |
| | | | | | | *p2* | 1 | 2,0 | 0,1 | $\frac{1}{2}$,0 |
| | | | | | | *p2* | 1 | 2,0 | 0,1 | 0,0 |
| | | | | | | *p1* | 2 | 2,0 | 0,1 | 0,0 |
| | | | | 2 | 4 | *p4* | 1 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *p4* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *cmm* | 1 | 0,2 | $\bar{2}$,0 | 0,$\frac{1}{2}$ |
| | | | | | | *cm* | 2 | 0,2 | $\bar{2}$,0 | 0,$\frac{1}{2}$ |
| | | | | | | *p2* | 2 | 2,0 | 0,2 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *p2* | 2 | 2,0 | 0,2 | 0,$\frac{1}{2}$ |
| | | | | | | *p2* | 2 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p1* | 4 | 2,0 | 0,2 | 0,0 |
| | $M_1+M_4$ | 0 | 0 | 1 | 2 | *pmm* | 2 | 1,1 | $\bar{1}$,1 | 0,$\frac{1}{2}$ |
| | $M_2+M_3$ | 0 | 0 | 1 | 2 | *pgg* | 2 | 1,1 | $\bar{1}$,1 | 0,$\frac{1}{2}$ |
| | $M_5$ | 0 | 0 | 1 | 2 | *p4* | 1 | 1,1 | $\bar{1}$,1 | $\frac{1}{2}$,$\frac{1}{2}$ |
| | | | | | | *p4* | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | | | | | | *pmg* | 1 | 1,1 | $\bar{1}$,1 | 0,0 |
| | | | | | | *p2* | 2 | 1,$\bar{1}$ | 1,1 | 0,0 |
| *p3* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p3* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2+\Gamma_3$ | 2 | 0 | 1 | 1 | *p1* | 2 | 1,0 | 0,1 | 0,0 |
| | $M_1$ | 1 | 0 | 1 | 2 | *p1* | 1 | 1,0 | 0,2 | 0,0 |
| | | | | 3 | 4 | *p3* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p1* | 3 | 2,0 | 0,2 | 0,0 |
| | $K_1+K_1^*$ | 2 | 0 | 1 | 3 | *p3* | 2 | 2,1 | $\bar{1}$,1 | 0,0 |
| | $K_2+K_2^*$ | 2 | 0 | 1 | 3 | *p3* | 2 | 2,1 | $\bar{1}$,1 | $\frac{1}{3}$,$\frac{2}{3}$ |
| | $K_3+K_3^*$ | 2 | 0 | 1 | 3 | *p3* | 2 | 2,1 | $\bar{1}$,1 | $\frac{2}{3}$,$\frac{1}{3}$ |
| *p3m1* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p3m1* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *p3* | 1 | 1,0 | 0,1 | 0,0 |

TABLE VI. (*Continued*).

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | $G$ | $i(G)$ | Basis vectors | | Origin |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\Gamma_3$ | 1 | 0 | 1 | 1 | *cm* | 1 | 1,1 | 0,1 | 0,0 |
| | | | | | | *p*1 | 2 | 1,0 | 0,1 | 0,0 |
| | $M_1$ | 1 | 0 | 1 | 2 | *pm* | 1 | $\bar{1},\bar{2}$ | 1,0 | 0,0 |
| | | | | 3 | 4 | *p*3*m*1 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *cm* | 2 | 2,2 | 0,2 | 0,0 |
| | | | | | | *p*1 | 3 | 2,0 | 0,2 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | *pg* | 1 | $\bar{1},\bar{2}$ | 1,0 | $\frac{1}{4}$,0 |
| | | | | 3 | 4 | *p*3 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *cm* | 1 | 2,2 | 0,2 | 0,0 |
| | | | | | | *p*1 | 3 | 2,0 | 0,2 | 0,0 |
| | $K_1$ | 1 | 0 | 2 | 3 | *p*31*m* | 1 | 2,1 | $\bar{1}$,1 | 0,0 |
| | | | | | | *p*3 | 2 | 2,1 | $\bar{1}$,1 | 0,0 |
| | $K_2$ | 1 | 0 | 2 | 3 | *p*31*m* | 1 | 2,1 | $\bar{1}$,1 | $\frac{1}{3},\frac{2}{3}$ |
| | | | | | | *p*3 | 2 | 2,1 | $\bar{1}$,1 | $\frac{1}{3},\frac{2}{3}$ |
| | $K_3$ | 1 | 0 | 2 | 3 | *p*31*m* | 1 | 2,1 | $\bar{1}$,1 | $\frac{2}{3},\frac{1}{3}$ |
| | | | | | | *p*3 | 2 | 2,1 | $\bar{1}$,1 | $\frac{2}{3},\frac{1}{3}$ |
| *p*31*m* | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p*31*m* | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | *p*3 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 1 | 0 | 1 | 1 | *cm* | 1 | 0,1 | $\bar{1},\bar{1}$ | 0,0 |
| | | | | | | *p*1 | 2 | 1,0 | 0,1 | 0,0 |
| | $M_1$ | 1 | 0 | 1 | 2 | *pm* | 1 | 1,0 | 1,2 | 0,0 |
| | | | | 3 | 4 | *p*31*m* | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *cm* | 2 | 0,2 | $\bar{2},\bar{2}$ | 0,0 |
| | | | | | | *p*1 | 3 | 2,0 | 0,2 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | *pg* | 1 | 1,0 | 1,2 | $\frac{1}{4},\frac{1}{2}$ |
| | | | | 3 | 4 | *p*3 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *cm* | 1 | 0,2 | $\bar{2},\bar{2}$ | 0,0 |
| | | | | | | *p*1 | 3 | 2,0 | 0,2 | 0,0 |
| | $K_1+K_1^*$ | 2 | 0 | 1 | 3 | *p*3*m*1 | 2 | 2,1 | $\bar{1}$,1 | 0,0 |
| | $K_2+K_2^*$ | 0 | 0 | 1 | 3 | *p*3 | 2 | 2,1 | $\bar{1}$,1 | 0,0 |
| | $K_3+K_3^*$ | 2 | 1 | 1 | 3 | *p*3 | 2 | 2,1 | $\bar{1}$,1 | $\frac{2}{3},\frac{1}{3}$ |
| | | | | | | *p*3 | 2 | 2,1 | $\bar{1}$,1 | $\frac{1}{3},\frac{2}{3}$ |
| | | | | | | *cm* | 2 | 1,$\bar{1}$ | 2,1 | 0,0 |
| | | | | | | *p*1 | 4 | 1,$\bar{1}$ | 2,1 | 0,0 |
| *p*6 | $\Gamma_1$ | 1 | 0 | 1 | 1 | *p*6 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2+\Gamma_3$ | 2 | 0 | 1 | 1 | *p*2 | 2 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | *p*3 | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_5+\Gamma_6$ | 0 | 0 | 1 | 1 | *p*1 | 2 | 1,0 | 0,1 | 0,0 |
| | $M_1$ | 1 | 0 | 1 | 2 | *p*2 | 1 | 1,0 | 0,2 | 0,0 |
| | | | | 3 | 4 | *p*6 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p*2 | 3 | 2,0 | 0,2 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | *p*2 | 1 | 1,0 | 0,2 | 0,$\frac{1}{2}$ |
| | | | | 3 | 4 | *p*3 | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | *p*2 | 2 | 2,0 | 0,2 | 0,$\frac{1}{2}$ |
| | | | | | | *p*1 | 3 | 2,0 | 0,2 | 0,0 |
| | $K_1$ | 1 | 0 | 2 | 3 | *p*6 | 1 | 2,1 | $\bar{1}$,1 | 0,0 |
| | | | | | | *p*3 | 2 | 2,1 | $\bar{1}$,1 | 0,0 |
| | $K_2+K_3$ | 2 | 2 | 2 | 3 | *p*3 | 2 | 2,1 | $\bar{1}$,1 | $\frac{1}{3},\frac{2}{3}$ |

TABLE VI. (*Continued*).

| $G_0$ | irrep | $n_1$ | $n_2$ | Arms | Size | $G$ | $i(G)$ | Basis vectors | | Origin |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $p2$ | 2 | $1,\bar{1}$ | 2,1 | 0,0 |
| | | | | | | $p1$ | 4 | $1,\bar{1}$ | 2,1 | 0,0 |
| $p6m$ | $\Gamma_1$ | 1 | 0 | 1 | 1 | $p6m$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_2$ | 0 | 0 | 1 | 1 | $p6$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_3$ | 0 | 0 | 1 | 1 | $p31m$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_4$ | 0 | 0 | 1 | 1 | $p3m1$ | 1 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_5$ | 0 | 0 | 1 | 1 | $cm$ | 1 | 1,1 | 0,1 | 0,0 |
| | | | | | | $cm$ | 1 | 0,1 | $\bar{1},\bar{1}$ | 0,0 |
| | | | | | | $p1$ | 2 | 1,0 | 0,1 | 0,0 |
| | $\Gamma_6$ | 1 | 0 | 1 | 1 | $cmm$ | 1 | 1,1 | 0,1 | 0,0 |
| | | | | | | $p2$ | 2 | 1,0 | 0,1 | 0,0 |
| | $M_1$ | 1 | 0 | 1 | 2 | $pmm$ | 1 | 1,0 | 1,2 | 0,0 |
| | | | | 3 | 4 | $p6m$ | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | $cmm$ | 2 | 2,2 | 0,2 | 0,0 |
| | | | | | | $p2$ | 3 | 2,0 | 0,2 | 0,0 |
| | $M_2$ | 0 | 0 | 1 | 2 | $pmg$ | 1 | 1,0 | 1,2 | $\frac{1}{2},\frac{1}{2}$ |
| | | | | 3 | 4 | $p3m1$ | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | $cmm$ | 1 | 2,2 | 0,2 | $\frac{1}{2},0$ |
| | | | | | | $cm$ | 2 | 2,2 | 0,2 | 0,0 |
| | | | | | | $p2$ | 2 | 2,0 | 0,2 | $0,\frac{1}{2}$ |
| | | | | | | $p1$ | 3 | 2,0 | 0,2 | 0,0 |
| | $M_3$ | 0 | 0 | 1 | 2 | $pgg$ | 1 | 1,0 | 1,2 | 0,0 |
| | | | | 3 | 4 | $p6$ | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | $cmm$ | 1 | 2,2 | 0,2 | 0,0 |
| | | | | | | $p2$ | 3 | 2,0 | 0,2 | 0,0 |
| | $M_4$ | 0 | 0 | 1 | 2 | $pmg$ | 1 | 1,2 | $\bar{1},0$ | $0,\frac{1}{2}$ |
| | | | | 3 | 4 | $p31m$ | 1 | 2,0 | 0,2 | 0,0 |
| | | | | | | $cmm$ | 1 | 2,2 | 0,2 | $\frac{1}{2},0$ |
| | | | | | | $cm$ | 2 | 0,2 | $\bar{2},\bar{2}$ | 0,0 |
| | | | | | | $p2$ | 2 | 2,0 | 0,2 | $0,\frac{1}{2}$ |
| | | | | | | $p1$ | 3 | 2,0 | 0,2 | 0,0 |
| | $K_1$ | 1 | 0 | 2 | 3 | $p6m$ | 1 | 2,1 | $\bar{1},1$ | 0,0 |
| | | | | | | $p3m1$ | 2 | 2,1 | $\bar{1},1$ | 0,0 |
| | $K_2$ | 0 | 0 | 2 | 3 | $p6$ | 1 | 2,1 | $\bar{1},1$ | 0,0 |
| | | | | | | $p31m$ | 1 | 2,1 | $\bar{1},1$ | 0,0 |
| | | | | | | $p3$ | 2 | 2,1 | $\bar{1},1$ | 0,0 |
| | $K_3$ | 1 | 1 | 2 | 3 | $p31m$ | 1 | 2,1 | $\bar{1},1$ | $\frac{2}{3},\frac{1}{3}$ |
| | | | | | | $cmm$ | 1 | 2,1 | $\bar{1},1$ | 0,0 |
| | | | | | | $p3$ | 2 | 2,1 | $\bar{1},1$ | $\frac{1}{3},\frac{2}{3}$ |
| | | | | | | $cm$ | 2 | 2,1 | $\bar{1},1$ | 0,0 |
| | | | | | | $cm$ | 2 | $1,\bar{1}$ | 2,1 | 0,0 |
| | | | | | | $p2$ | 2 | $1,\bar{1}$ | 2,1 | 0,0 |
| | | | | | | $p1$ | 4 | $1,\bar{1}$ | 2,1 | 0,0 |

Rottman[6] lists the irreps which satisfy the Lifshitz criterion ($n_2=0$) and gives their associated LGW Hamiltonians. (He does not list the low-symmetry space groups.) We compared his list with our results and found three differences. He lists two irreps for each of the following $\vec{k}$ points: $p4gm(j)$, $p3(l)$, and $p3ml(m)$, which correspond to our $\vec{k}$ points $p4g(M)$, $p3(K)$, and $p3m(K)$. In our Table VI, we list *three* different physical irreps for each of these $\vec{k}$ points.

Maksimov *et al.*[8] list the lower-symmetry space groups which arise from irreps which satisfy both the Landau and Lifshitz criteria ($n_1=0$ and $n_2=0$). Their results correspond to listing only maximal subgroups for a given irrep. In our list these are the isotropy subgroups for which the subduction frequency $i(G)=1$ if the irrep is real and $i(G)=2$ if the irrep is complex. We compared their result with ours and found several differences. For example, $p(\sqrt{2}\times\sqrt{2})45°$ of space group $p4$ should give rise to space groups $p2,p4$, not $p1,p4$ as they list. As a second example, $p(1\times1)$ of space group $p6$ does not give rise to $p2$ as they list. (Actually, one irrep, $\Gamma_2+\Gamma_3$ of $p6$, does give rise to $p2$, but this irrep fails the Landau criterion.)

Deonarine and Birman[7] list all the isotropy subgroups allowed by the subduction and chain criterion. Again, we find many differences with our results. These differences include omissions of points of symmetry (e.g., $\Gamma$, $X$, and $S$ points of $pg$), incorrect calculations of the Landau and Lifshitz criteria (e.g., irreps $E_1$ and $E_2$ of the $\Gamma$ point of $p6m$), and incorrect results of the subduction and chain criteria (e.g., incorrect isotropy subgroups arising from the irreps $B_1$ and $B_2$ of the $\vec{k}^*=[\frac{1}{2},0]$ point of $p2mm$).

The calculations for finding isotropy subgroups are very prone to error as we have seen in the listings of Deonarine and Birman[7] and also in those of Maksimov *et al.*[8] Computers are particularly well suited for these kinds of calculations, and up to the present time, we have found our computer-generated tables to be free of any errors.

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