Microscopic Mechanism for the Macroscopic Asymmetry of Ferromagnetism

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Some of the physical implications involved in self-consistently selecting a ferromagnetic (inequivalent) representation for the Heisenberg exchange Hamiltonian are developed and discussed. This is done by comparing the spin-rotation symmetry of our system in original variables with that same symmetry when written in terms of physical variables. It is shown explicitly that Goldstone's theorem is satisfied and that dynamical rearrangement of symmetry has taken place in going from original to physical varibles. Thus it is found that the original spin-rotation symmetry transformation is taken up by physical "massless" fields and that the Bose-Einstein condensations of these fields in the physical (ferromagnetic) ground state produce the asymmetry by "printing" the spin quantum number on that state.

I. INTRODUCTION

In a previous paper[1] (referred to hereafter as I) it was concluded that the Heisenberg magnetic-exchange model which is described by the Hamiltonian

$$H=-\sum_{\vec{l}\neq\vec{l}}\sum J_{\vec{l}-\vec{l}'}\vec{S}_{\vec{l}}\cdot\vec{S}_{\vec{l}} \qquad (1.1)$$

does describe a ferromagnetic system when one is in the appropriate (inequivalent) representation. This was shown by writing the spin operators $\vec{S}_{\vec{l}}$ and $\vec{S}_{\vec{l}'}$ in second-quantized form in terms of fermion annihilation and creation operators (the a's) and then applying Umezawa's self-consistent-field theory techniques[2] to transform to the physical spin operators $\vec{s}_{\vec{l}}$ and $\vec{s}_{\vec{l}'}$, which were written in terms of physical annihilation and creation operators (the b's). All the higher-order terms were accounted for, and this was accomplished without recourse to the adiabatic theorem. Validity for using an exchange integral depending only on relative distance between lattice sites and, in particular, on nearest neighbors was a further result. Thus, using quantum-field-theory methods, we were able to bilinearize the Heisenberg exchange Hamiltonian. As noted at the end of I, this present paper is an attempt to explain the physical implications involved in picking out the ferromagnetic representation for the Heisenberg-exchange model. In particular, this will be done by looking at the symmetry of our system in original variables as compared with that same symmetry when written in physical variables. This will be the content of Sec. IV. However, in order to form a foundation for what will come, we will briefly review Goldstone's theorem in Sec. II and the dynamical rearrangement of symmetry in Sec. III.

II. GOLDSTONE'S THEOREM AND BROKEN SYMMETRIES

Goldstone's theorem, briefly stated for nonrelativistic systems, asserts that (i) if the interactions are sufficiently well behaved at large distances (this depends on the model being considered) and (ii) if the physical ground state is not an eigenstate of the time-independent generators G_V of symmetry transformations on the original Hamiltonian of the system (in other words, if a broken symmetry exists), there must exist an excitation mode with no energy gap present in the physical spectrum (referred to as "massless" particles).[3] This certainly seems understandable, physically speaking, since such massless particles, like the vacuum or ground state, can have zero energy and thus provide a mechanism for constructing various null eigenstates of the energy-momentum four-vector. These new ground states are in general not eigenstates of the symmetry generators, and so this idea provides a mechanism for obtaining broken symmetries, i.e., a way for obtaining solutions possessing a lower symmetry than a given Hamiltonian. These ground states are referred to as asymmetric ground states. Of course, this is of great importance in nonrelativistic quantum-field theory, since such systems as superconductors and ferromagnets have perfectly acceptable solutions to their field equations, which have less symmetry than that of the Hamiltonian.

The theorem was first conjectured by Goldstone[4] in 1961 on the basis of only the Goldstone model (complex-scalar-field model) and the model considered by Jona-Lasinio and Nambu.[5] Since then, many proofs have been proposed and given,[6] but probably the most understandable and applicable proof for nonrelativistic systems is essentially that of Lange,[7] which we will now outline.

Proof. We begin with the assumption that there exists a conserved current $J^\mu(x)$,

$$\partial_\mu J^\mu=0 \quad (\mu=1,\ 2,\ 3,\ 0)\ . \qquad (2.1)$$

Next, we define the generator of symmetry transformations in volume V, $G_V(t)$, as

$$G_V(t)=\int_V d^3x\,J^0(\vec{x}\ ,\ t)=\int_V d\vec{x}\,J^0(x)\ . \qquad (2.2)$$

Now in order to incorporate the effect of current conservation, let us consider

$$\int_V d\vec{x}\,[\partial_\mu J^\mu(x),\ \phi(x')] = 0 \quad , \tag{2.3}$$

where $\phi(x')$ is any appropriate operator of the field theory being considered. When Eq. (2.3) is written out, it becomes

$$\begin{aligned}[\partial_0 G_V(t),\ \phi(x')] + [\textstyle\int_V d\vec{x}\,\vec{\nabla}\cdot\vec{J}(\vec{x}),\ \phi(x')] = 0 \\ = \partial_t \textstyle\int_V d\vec{x}\,[J^0(x),\ \phi(x')] \\ + \textstyle\int_{S(V)} [d\vec{S}\cdot\vec{J}(x),\ \phi(x')] \quad ,\end{aligned} \tag{2.4}$$

where $S(V)$ is the surface bounding V. If for some sufficiently large volume V (may have to have $V\to\infty$), we have

$$\lim_{V\to\infty}\{\textstyle\int_{S(V)} [d\vec{S}\cdot\vec{J}(x),\ \phi(x')]\} = 0 \quad , \tag{2.5}$$

then

$$\lim_{V\to\infty}\{\partial_t [G_V(t),\ \phi(x')]\} = 0 \tag{2.6}$$

or

$$\lim_{V\to\infty}\{[G_V(t),\ \phi(x')]\} = C \quad , \tag{2.7}$$

where $dC/dt = 0$, and where C may or may not be zero.

Now let us impose the broken-symmetry condition,

$$\langle 0|\,C\,|0\rangle \neq 0 \quad , \tag{2.8}$$

where $|0\rangle$ is the translationally invariant physical vacuum or ground state. Taking the vacuum-expectation value of Eq. (2.4), inserting a complete set of states $|n\rangle$, and letting $V\to\infty$, so that Eq. (2.5) is satisfied, we have

$$\lim_{V\to\infty}\Big(\sum_n [\langle 0|G_V(t)|n\rangle\langle n|\phi(x')|0\rangle - \langle 0|\phi(x')|n\rangle \times\langle n|G_V(t)|0\rangle]\Big) = \langle 0|C|0\rangle \neq 0 \quad . \tag{2.9}$$

Assuming $J^0(x)$ to have the following translational behavior, $J^0(x) = e^{-ip_x} J^0(0)\, e^{ip_x}$ and also that $e^{ip_x}|0\rangle = |0\rangle$, Eq. (2.9) becomes

$$\begin{aligned}\lim_{V\to\infty}\Big[\sum_n\Big(\int_V d\vec{x}\,[\langle 0|J^0(0)|n\rangle\langle n|\phi(x')|0\rangle\, e^{ip_n x} \\ - \langle 0|\phi(x')|n\rangle\langle n|J^0(0)|0\rangle\, e^{-ip_n x}]\Big)\Big] \\ = \Big(\sum_n (2\pi)^3\,\delta(\vec{p}_n)\,[\langle 0|J^0(0)|n\rangle\langle n|\phi(x')|0\rangle\, e^{-ip_n^0 x^0} \\ - \langle 0|\phi(x')|n\rangle\langle n|J^0(0)|0\rangle\, e^{ip_n^0 x^0}]\Big) \\ = \langle 0|C|0\rangle \neq 0 \quad .\end{aligned} \tag{2.10}$$

where p_n are the eigenvalues of the four-momentum operator p.

Now what are the conditions under which Eq. (2.10) is satisfied in the limit as $\vec{p}_n\to 0$? Since Eq. (2.10) is valid for all times x^0, and since $dC/dt = dC/dx^0 = 0$, it follows that the left-hand side of Eq. (2.10) must not depend on x^0. These conditions are consistent with each other only if the left-hand side vanishes, except for those states where $p_n^0 = \omega_n = 0$ in the limit as $\vec{p}_n\to 0$. Thus, we have shown that if (i) the condition in Eq. (2.5) holds (which limits the range of the interaction forces) and (ii) a symmetry is broken (the ground state is not an eigenstate of the symmetry generator), then there must be excitation modes in the spectrum of the symmetry generator whose energy vanishes in the limit that the momentum of these modes vanishes.

III. DYNAMICAL REARRANGEMENT OF SYMMETRIES

Looking for the microscopic mechanism producing the asymmetric ground states noted in Sec. II, Umezawa and others began a more detailed study of certain models, and they discovered the phenomenon of dynamical rearrangement of symmetry.

During 1965 and 1966, Umezawa, Leplae, Sen, and Nakagawa made more detailed calculations for the Nambu model,[8] the complex-scalar-field model,[9] and for neutral superconductivity.[10] They discovered that the symmetry transformations for the original fields (in terms of which the interacting field equations are expressed) took entirely different forms for the physical fields (the free-field operators for the observed particles). For example, the original symmetry of the Nambu Hamiltonian $\psi\to e^{i\alpha\gamma_5}\psi$ becomes in physical variables, $B\to B+\chi\theta$, where ψ is the original fermion field, χ is a constant, B is the physical "pion" field and is massless, and θ is a transformation parameter.[11] In other words, the symmetries can be dynamically rearranged. When this occurred, certain broken symmetries were observed even when the Hamiltonian was fully symmetric. In other words, we might think of broken symmetries as having a dynamical origin.

After dealing with the above models, Umezawa, Leplae, and Sen[12] called upon their physical intuitions to make general statements which appeared to be independent of any model. From their results, they concluded that for a fully symmetric Hamiltonian, dynamical rearrangement of symmetry takes place if and only if the ground state is asymmetric, and that an asymmetric ground state can only be realized by the Bose-Einstein condensation of massless physical fields in the physical ground state.[13] (Given a field ψ, Bose-Einstein condensation is said to take place when

$$\langle 0|\psi|0\rangle = c \text{ number} \neq 0 \quad , \tag{3.1}$$

where $|0\rangle$ is the ground state.[14]) Thus, the existence of massless physical fields is required, and it is these fields which take over the original symmetry transformation, as indicated above for the Nambu model. Since the commutation relations, and not the anticommutation relations, are invariant under the inhomogeneous transformation, $B \to B + c$ number, the only canonical device for the massless fields are Bose fields. The c numbers coming from the ground-state expectation values of these symmetry-preserving massless fields (indicating Bose-Einstein condensation) are related directly to the symmetry quantum numbers associated with the corresponding symmetry transformations. An explanation for this is that the symmetry quantum numbers are torn off the original particles and "printed" on the physical ground state by the Bose-Einstein condensation of the massless fields, thus producing an asymmetric ground state. Certain physical fields are left frozen; that is, they do not respond at all to the original symmetry transformations when written in physical variables.

A good example, considered by Umezawa,[15] is the neutral superconductor. We denote the original fermion field by ψ, and the Hamiltonian is symmetric under the electron phase transformation, $\psi \to e^{i\theta}\psi$. In physical variables this symmetry is taken up by the phononlike field B as $B \to B + \eta\theta$, where η is a c number constant. The c number η indicates Bose-Einstein condensation and is directly proportional to the number of electrons. This can be interpreted by saying that the total fermion number is carried by the Bose-Einstein condensation of the massless particles in the ground state, thus resulting in asymmetry of that state with respect to the phase transformation.

We will now use the results of these last two sections and show explicitly what takes place physically in selecting the ferromagnetic representation for Eq. (1.1). (How that representation is selected was the content of I.)

IV. DYNAMICAL MAP, MASSLESS PARTICLES, BOSE-EINSTEIN CONDENSATIONS, etc.

Let us begin by looking at the symmetry of our system in original variables, as compared with that same symmetry written in physical variables. As is well known, our original Hamiltonian (written in terms of the a's) [Eq. (1.1)], is invariant under spin rotations generated by the unitary operators

$$U_V(\vec\theta) = e^{i\vec\theta\cdot\vec S} = \exp\left(i\vec\theta\cdot\sum_{\vec l}\vec S_{\vec l}\right) . \tag{4.1}$$

Of course, this can be effected by the $3\times\underset{\to}{3}$ matrix describing the rotation of the spin vector $\vec S_{\vec l}$ around a given axis by an angle $\vec\theta$. Thus, using the Euler[16] matrix representation for this rotation, H_{original} (written in second-quantized form) remains invariant in form when we let

$$\begin{aligned}
S_{\vec l}^{(x)} \to{} & (\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)\, S_{\vec l}^{(x)} \\
& + (\sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi)\, S_{\vec l}^{(y)} \\
& + (\sin\theta\sin\psi)\, S_{\vec l}^{(z)} , \\
S_{\vec l}^{(y)} \to{} & (-\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi)\, S_{\vec l}^{(x)} \\
& + (-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)\, S_{\vec l}^{(y)} \\
& + (\sin\theta\cos\psi)\, S_{\vec l}^{(z)} \\
S_{\vec l}^{(z)} \to{} & (\sin\theta\sin\phi)\, S_{\vec l}^{(x)} \\
& - (\sin\theta\cos\phi)\, S_{\vec l}^{(y)} \\
& + (\cos\theta)\, S_{\vec l}^{(z)}
\end{aligned} \tag{4.2}$$

where ψ, θ, and ϕ are angles as defined in Ref. 16.

In order to see what effect the transformations in Eq. (4.2) have on the physics, we need to know the relationship between our original symmetry generators ($S_{\vec l}^{(x)}$, $S_{\vec l}^{(y)}$, and $S_{\vec l}^{(z)}$) and our corresponding physical symmetry generators ($\mathcal{S}_{\vec l}^{(x)}$, $\mathcal{S}_{\vec l}^{(y)}$, and $\mathcal{S}_{\vec l}^{(z)}$). This is obtained from our dynamical map, Eq. (3.11) of I, [which is

$$\begin{aligned}
a_{\vec l\uparrow}^\dagger &= u_{\vec l}\, b_{\vec l\uparrow}^\dagger - v_{\vec l}\, b_{\vec l\downarrow}^\dagger , \\
a_{\vec l\uparrow} &= u_{\vec l}\, b_{\vec l\uparrow} - v_{\vec l}\, b_{\vec l\downarrow} , \\
a_{\vec l\downarrow}^\dagger &= v_{\vec l}\, b_{\vec l\uparrow}^\dagger + u_{\vec l}\, b_{\vec l\downarrow}^\dagger , \\
a_{\vec l\downarrow}^\dagger &= v_{\vec l}\, b_{\vec l\uparrow} + u_{\vec l}\, b_{\vec l\downarrow} ,
\end{aligned} \tag{4.3}$$

where $u_{\vec l}$ and $v_{\vec l}$ are Hermitian parameters which have to be determined self-consistently, and $(u_{\vec l}^2 + v_{\vec l}^2) = 1$ for this to be a canonical transformation] and from Eq. (3.2) of I, namely,

$$\begin{aligned}
S_{\vec l}^{(z)} &= \tfrac{1}{2}\,(a_{\vec l\uparrow}^\dagger a_{\vec l\uparrow} - a_{\vec l\downarrow}^\dagger a_{\vec l\downarrow}) , \\
S_{\vec l}^{(y)} &= -\tfrac{1}{2} i\, (a_{\vec l\uparrow}^\dagger a_{\vec l\downarrow} - a_{\vec l\downarrow}^\dagger a_{\vec l\uparrow}) , \\
S_{\vec l}^{(x)} &= \tfrac{1}{2}\,(a_{\vec l\uparrow}^\dagger a_{\vec l\downarrow} + a_{\vec l\downarrow}^\dagger a_{\vec l\uparrow}) ,
\end{aligned} \tag{4.4}$$

and from Eq. (3.42) of I, namely,

$$\begin{aligned}
\mathcal{S}_{\vec l}^{(z)} &= \tfrac{1}{2}(b_{\vec l\uparrow}^\dagger b_{\vec l\uparrow} - b_{\vec l\downarrow}^\dagger b_{\vec l\downarrow}) , \\
\mathcal{S}_{\vec l}^{(x)} &= \tfrac{1}{2}\,(b_{\vec l\uparrow}^\dagger b_{\vec l\downarrow} + b_{\vec l\downarrow}^\dagger b_{\vec l\uparrow}) , \\
\mathcal{S}_{\vec l}^{(y)} &= -\tfrac{1}{2} i\, (b_{\vec l\uparrow}^\dagger b_{\vec l\downarrow} - b_{\vec l\downarrow}^\dagger b_{\vec l\uparrow}) , \\
\mathcal{S}_{\vec l}^{+} &= \mathcal{S}_{\vec l}^{(x)} + i\mathcal{S}_{\vec l}^{(y)} = b_{\vec l\uparrow}^\dagger b_{\vec l\downarrow} , \\
\mathcal{S}_{\vec l}^{-} &= \mathcal{S}_{\vec l}^{(x)} - i\mathcal{S}_{\vec l}^{(y)} = b_{\vec l\downarrow}^\dagger b_{\vec l\uparrow} ,
\end{aligned} \tag{4.5}$$

resulting in

$$\begin{aligned}
\mathcal{S}_{\vec l}^{(z)} &= (u_{\vec l}^2 - v_{\vec l}^2)\, S_{\vec l}^{(z)} + u_{\vec l} v_{\vec l}\, (S_{\vec l}^{+} + S_{\vec l}^{-}) , \\
\mathcal{S}_{\vec l}^{+} &= u_{\vec l}^2 S_{\vec l}^{+} - v_{\vec l}^2 S_{\vec l}^{-} - 2u_{\vec l} v_{\vec l}\, S_{\vec l}^{(z)} , \\
\mathcal{S}_{\vec l}^{-} &= u_{\vec l}^2 S_{\vec l}^{-} - v_{\vec l}^2 S_{\vec l}^{+} - 2u_{\vec l} v_{\vec l}\, S_{\vec l}^{(z)} ,
\end{aligned} \tag{4.6}$$

or, vice versa,

$$\begin{aligned}
S_{\vec l}^{(z)} &= (u_{\vec l}^2 - v_{\vec l}^2)\,\mathcal{S}_{\vec l}^{(z)} - u_{\vec l} v_{\vec l}\, (\mathcal{S}_{\vec l}^{+} + \mathcal{S}_{\vec l}^{-}) , \\
S_{\vec l}^{+} &= 2u_{\vec l} v_{\vec l}\,\mathcal{S}_{\vec l}^{(z)} - v_{\vec l}^2 \mathcal{S}_{\vec l}^{-} + u_{\vec l}^2 \mathcal{S}_{\vec l}^{+} , \\
S_{\vec l}^{-} &= 2u_{\vec l} v_{\vec l}\,\mathcal{S}_{\vec l}^{(z)} + u_{\vec l}^2 \mathcal{S}_{\vec l}^{-} - v_{\vec l}^2 \mathcal{S}_{\vec l}^{+} .
\end{aligned} \tag{4.7}$$

Using Eq. (4.6) along with Eqs. (3.15)–(3.18), and (3.21) of I (which are the self-consistent equations for $u_{\vec{l}}$ and $v_{\vec{l}}$), we have under the transformations in Eq. (4.2) written now for physical variables,

$$\mathcal{S}_{\vec{l}}^{(z)} \to \mathcal{S}_{\vec{l}}^{(z)} ,$$

$$\begin{aligned}
\mathcal{S}_{\vec{l}}^{+} \to\ & [i \sin\theta \cos\psi (u_{\vec{l}}^2 - v_{\vec{l}}^2) - 2i u_{\vec{l}} v_{\vec{l}} (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)] \mathcal{S}_{\vec{l}}^{(z)} \\
& + [\tfrac{1}{2} (\cos\theta \cos\phi \cos\psi - \sin\phi \sin\psi) - \tfrac{1}{2} i (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)(u_{\vec{l}}^2 - v_{\vec{l}}^2) - i \sin\theta \cos\psi (u_{\vec{l}} v_{\vec{l}})] \mathcal{S}_{\vec{l}}^{+} \\
& + [-\tfrac{1}{2} (\cos\theta \cos\phi \cos\psi - \sin\phi \sin\psi) - \tfrac{1}{2} i (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)(u_{\vec{l}}^2 - v_{\vec{l}}^2) - i \sin\theta \cos\psi (u_{\vec{l}} v_{\vec{l}})] \mathcal{S}_{\vec{l}}^{-} , \\
\mathcal{S}_{\vec{l}}^{-} \to\ & [-i \sin\theta \cos\psi (u_{\vec{l}}^2 - v_{\vec{l}}^2) + 2i u_{\vec{l}} v_{\vec{l}} (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)] \mathcal{S}_{\vec{l}}^{(z)} \\
& + [\tfrac{1}{2} (\sin\phi \sin\psi - \cos\theta \cos\phi \cos\psi) + \tfrac{1}{2} i (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)(u_{\vec{l}}^2 - v_{\vec{l}}^2) + i \sin\theta \cos\psi (u_{\vec{l}} v_{\vec{l}})] \mathcal{S}_{\vec{l}}^{-} \\
& + [\tfrac{1}{2} (\cos\theta \cos\phi \cos\psi - \sin\phi \sin\psi) + \tfrac{1}{2} i (\sin\psi \cos\phi + \cos\theta \sin\phi \cos\psi)(u_{\vec{l}}^2 - v_{\vec{l}}^2) + i \sin\theta \cos\psi (u_{\vec{l}} v_{\vec{l}})] \mathcal{S}_{\vec{l}}^{+} .
\end{aligned} \tag{4.8}$$

Therefore, from Eq. (4.8) our Hamiltonian, written in physical variables [Eq. (3.43) of I], which is

$$H = -\sum_{\vec{l}} B_{\vec{l}} \mathcal{S}_{\vec{l}}^{(z)} , \tag{4.9}$$

is invariant in form under Eq. (4.2), written in physical variables, as was our original Hamiltonian under Eq. (4.2). However, our physical ground state (all spins aligned parallel) [Eq. (3.44) of I], now is not an eigenstate of the symmetry generator $\vec{S}_{\vec{l}}$, written in physical variables, as was the original ground state under $\vec{S}_{\vec{l}}$:

$$\begin{aligned}
&\vec{S}_{\vec{l}} |0\rangle_{\text{orig}} = (S_{\vec{l}}^{(x)} \hat{i} + S_{\vec{l}}^{(y)} \hat{j} + S_{\vec{l}}^{(z)} \hat{k}) |0\rangle_{\text{orig}} = 0 , \\
&(\vec{S}_{\vec{l}})_{\text{phys}} |0\rangle_{\text{phys}} = \{[\hat{i} u_{\vec{l}} v_{\vec{l}} + \hat{k} \tfrac{1}{2} (u_{\vec{l}}^2 - v_{\vec{l}}^2)] |0\rangle_{\text{phys}} \\
&\quad + [\hat{i} \tfrac{1}{2} (u_{\vec{l}}^2 - v_{\vec{l}}^2) - \hat{j} \tfrac{1}{2} (i)^{-1} - \hat{k} u_{\vec{l}} v_{\vec{l}}] |\uparrow\uparrow \cdots \downarrow \cdots \uparrow \cdots\rangle\} ,
\end{aligned} \tag{4.10}$$

where $\downarrow$ is in the $\vec{l}$th position, and $|0\rangle_{\text{orig}}$ is taken as a linear combination of all possible ground-state spin orientations of H_{original}, since there is nothing *a priori* in the original theory to tell us which particular one to pick. Therefore, for every state with a spin-up specification, there is a corresponding specification with that spin in the opposite direction, so that $\vec{S}_{\vec{l}} |0\rangle_{\text{orig}} = 0$. [This is the typical relationship taken between a symmetry generator and a ground state (or vacuum).[17]] Consequently, our physical Hamiltonian [Eq. (4.9)], is still symmetric under the original symmetry transformation, written in physical variables, but from Eq. (4.10) the physical ground state $|0\rangle_{\text{phys}}$ is not. We have a broken symmetry and, as we would expect, our ferromagnetic representation is built up on an asymmetric ground state. This, along with the fact that our interaction is short ranged (nearest neighbor), means that the conditions of Goldstone's theorem are met. Therefore, there must be "massless" particles present in this physical representation. This can be seen directly, for if we calculate $\dot{\mathcal{S}}_{\vec{l}}$ using the Hamiltonian in Eq. (3.12) of I [(Eq. (1.1), when transformed using Eqs. (4.4) and (4.3)] before it is normal-ordered, we obtain

$$i \frac{d\mathcal{S}_{\vec{l}}^{-}}{dt} = i \dot{\mathcal{S}}_{\vec{l}}^{-} = 2 \sum_{\vec{l}'} J_{\vec{l}-\vec{l}'} (\mathcal{S}_{\vec{l}}^{(z)} \mathcal{S}_{\vec{l}'}^{-} - \mathcal{S}_{\vec{l}}^{-} \mathcal{S}_{\vec{l}'}^{(z)}) . \tag{4.11}$$

Then normal ordering and taking the limit as $V \to \infty$, in the same manner as we obtained the bilinear Hamiltonian in Eq. (3.41) of I in terms of the b's, or the linear Hamiltonian in Eq. (3.43) of I [same as Eq. (4.9) above], in terms of the $\mathcal{S}$'s, we obtain the linearized equation of motion for $\mathcal{S}_{\vec{l}}^{-}$,

$$i \dot{\mathcal{S}}_{\vec{l}}^{-} = 2s \sum_{\vec{l}'} J_{\vec{l}-\vec{l}'} (\mathcal{S}_{\vec{l}'}^{-} - \mathcal{S}_{\vec{l}}^{-}) . \tag{4.12}$$

Fourier-transforming Eq. (4.12), using for the time dependence $e^{-i\omega_{\vec{k}} t}$, gives

$$\sum_{\vec{k}} \omega_{\vec{k}} \mathcal{S}_{\vec{k}}^{-} e^{i \vec{k} \cdot \vec{l}} = 2s \sum_{\vec{k}} e^{i \vec{k} \cdot \vec{l}} \mathcal{S}_{\vec{k}}^{-} (J_{\vec{k}} - J_0) , \tag{4.13}$$

and so

$$\omega_{\vec{k}} = 2s (J_{\vec{k}} - J_0) = 2s \sum_{\vec{l}-\vec{l}'} J_{\vec{l}-\vec{l}'} (e^{-i \vec{k} \cdot (\vec{l}-\vec{l}')} - 1) , \tag{4.14}$$

where s is the eigenvalue of $\mathcal{S}_{\vec{l}}^{(z)}$. Therefore,

$$\lim_{\vec{k} \to 0} (\omega_{\vec{k}}) = 0 . \tag{4.15}$$

A similar result is found for $\mathcal{S}_{\vec{l}}^{+}$ with Eq. (4.15) again being satisfied. Thus, $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$ correspond to the spin-wave fields.[18]

Now from Umezawa's work discussed in the section on the dynamical rearrangement of symmetries, since we have an asymmetric ground state (all spins aligned parallel) [Eq. (3.44) of I], we expect a dynamical rearrangement of symmetry to have taken place in obtaining our ferromagnetic representation. This is the case, as is seen in Eq. (4.8). The spin-rotation symmetry transformation in physical variables has been taken over by $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$, leaving $\mathcal{S}_{\vec{l}}^{(z)}$, the field accounting for the physical fermion property of spin for the system, frozen. That is, $\mathcal{S}_{\vec{l}}^{(z)}$ does not respond at all to the transformation. We also get a good idea of this from the physical Hamiltonian in Eq.

(4.9). Since for the physical ground state all spins are up ($B_{\vec{l}} > 0$), then with respect to this state there is an energy difference of $2B_{\vec{l}}\, s$ between a spin-up and a spin-down fermion. In other words, going to physical variables has separated the (↑↓) doublet in energy and isolated it into two singlets.[19]

Since $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$ take up the spin-symmetry transformation in physical variables, we expect to find the symmetry quantum numbers of spin s associated with $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$ in some way. Let us look at the ground-state expectation values of Eq. (4.8):

$$\begin{aligned}({}_{\rm phys}\langle 0|\, e^{i\vec{\theta}\cdot\Sigma_{\vec{l}}(\vec{S}_{\vec{l}})_{\rm phys}}\, \mathcal{S}_{\vec{l}}^{+}\, e^{-i\vec{\theta}\cdot\Sigma_{\vec{l}}(\vec{S}_{\vec{l}})_{\rm phys}}\, |0\rangle_{\rm phys}) &= \{s[-v_{\vec{l}}\, u_{\vec{l}}\, i(2\cos\phi\,\sin\psi + 2\cos\theta\,\sin\phi\,\cos\psi) \\ &\quad + (u_{\vec{l}}^2 - v_{\vec{l}}^2) i\cos\psi\,\sin\theta]\} \equiv s\eta \;, \qquad (4.16)\end{aligned}$$

$$\begin{aligned}({}_{\rm phys}\langle 0|\, e^{i\vec{\theta}\cdot\Sigma_{\vec{l}}(\vec{S}_{\vec{l}})_{\rm phys}}\, \mathcal{S}_{\vec{l}}^{-}\, e^{-i\vec{\theta}\cdot\Sigma_{\vec{l}}(\vec{S}_{\vec{l}})_{\rm phys}}\, |0\rangle_{\rm phys}) &= \{s[2u_{\vec{l}}\, v_{\vec{l}}\, i(\cos\phi\,\sin\psi + \cos\psi\,\cos\theta\,\sin\phi) \\ &\quad - (u_{\vec{l}}^2 - v_{\vec{l}}^2) i\sin\theta\,\cos\psi]\} \equiv s\eta^* \;, \qquad (4.17)\end{aligned}$$

where phys means written in physical variables, and η and η^* are defined by comparison in Eqs. (4.16) and (4.17). Using the equations from I,

$$\begin{aligned}&\mathcal{S}_{\vec{l}}^{(z)}|0\rangle_{\rm phys} = s|0\rangle_{\rm phys} \;, \\ &\mathcal{S}_{\vec{l}}^{+}|0\rangle_{\rm phys} = 0 \;, \qquad (4.18) \\ &\mathcal{S}_{\vec{l}}^{-}|0\rangle_{\rm phys} = |\uparrow\uparrow\cdots\downarrow\cdots\uparrow\uparrow\cdots\rangle \;,\end{aligned}$$

where ↓ occurs in the $\vec{l}$th position, we can also write Eq. (4.17), for example, as

$${}_{\rm phys}\langle 0'|\mathcal{S}_{\vec{l}}^{-}|0'\rangle_{\rm phys} = {}_{\rm phys}\langle 0|\mathcal{S}_{\vec{l}}^{-} + s\eta^*|0\rangle_{\rm phys} = s\eta^* \;, \qquad (4.19)$$

where

$$|0'\rangle_{\rm phys} = e^{-i\vec{\theta}\cdot\Sigma_{\vec{l}}(\vec{S}_{\vec{l}})_{\rm phys}}|0\rangle_{\rm phys} \;.$$

Likewise, for Eq. (4.16), we have

$${}_{\rm phys}\langle 0'|\mathcal{S}_{\vec{l}}^{+}|0'\rangle_{\rm phys} = {}_{\rm phys}\langle 0|\mathcal{S}_{\vec{l}}^{+} + s\eta|0\rangle_{\rm phys} = s\eta \;. \qquad (4.20)$$

Thus, the c numbers η and η^* exhibit the effect of the Bose-Einstein condensation of the massless fields $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$. [See Eq. (3.1).] As can be seen above, the Bose-Einstein condensations of $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$ are directly proportional to the spin s and we can interpret this by saying that the condensation of the massless fields "prints" the quantum-number spin onto the physical ground state. This requires no energy or momentum because the fields are gapless. This rearrangement of spin in the ground state is certainly a remarkable feature of the theory.

Of course, probably the best known technique for obtaining a gapless spectrum for the Heisenberg exchange model is that of Holstein and Primakoff.[20] Their approach is based on the expression of spin operators in terms of the creation and annihilation operators of the harmonic oscillator. The basic assumption is that the original operator $S_{\vec{l}}^{(z)}$ can be replaced by the quantum number s so as to insure the correct commutation relations for the oscillator operators. However, as shown by our calculations, this assumption is only valid in our physical representation where, since all spins are aligned in the same direction, $\mathcal{S}_{\vec{l}}^{(z)}$ always has the same value at any site and then could justifiably be replaced by s. Then our spin-commutation relations become for $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$,

$$[(2s)^{-1/2}\mathcal{S}_{\vec{l}}^{+},\; (2s)^{-1/2}\mathcal{S}_{\vec{l}}^{-}] = 1 \;, \qquad (4.21)$$

which means that we can interpret $\mathcal{S}_{\vec{l}}^{+}$ and $\mathcal{S}_{\vec{l}}^{-}$ as boson fields, so that above we have Bose-Einstein condensation of massless Bose fields with respect to the physical ground state.

CONCLUSION

Thus, in I we have found that ferromagnetism can be described as a self-consistent choice by the system of a state which does not carry the full symmetry of the original Hamiltonian. It is produced by a self-consistent magnetic field which depends upon the exchange interaction. Now, in this paper we have found the physical implications producing this asymmetric representation. The appearance of "massless" particles which are Bose-Einstein-condensed with respect to the ferromagnetic ground state were explicitly found and by further analysis shown to restore the full consequences of the original symmetry. This then provides a very good explanation for the microscopic mechanism producing the macroscopic asymmetry of ferromagnetism.

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