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Generalized emissivity inverse problem

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Inverse problems have recently drawn considerable attention from the physics community due to of potential widespread applications [K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer Verlag, Berlin, 1989)]. An inverse emissivity problem that determines the emissivity $g(\nu)$ from measurements of only the total radiated power $J(T)$ has recently been studied [Tao Wen, DengMing Ming, Xianxi Dai, Jixin Dai, and William E. Evenson, Phys. Rev. E **63**, 045601(R) (2001)]. In this paper, a new type of generalized emissivity and transmissivity inverse (GETI) problem is proposed. The present problem differs from our previous work on inverse problems by allowing the unknown (emissivity) function $g(\nu)$ to be temperature dependent as well as frequency dependent. Based on published experimental information, we have developed an exact solution formula for this GETI problem. A universal function set suggested for numerical calculation is shown to be robust, making this inversion method practical and convenient for realistic calculations.

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I. INTRODUCTION

It is well known that every logical proposition has an inverse. Some inverse propositions have drawn great attention in past decades. Inverse problems having unique solutions have been proposed and studied already, such as, the inverse black body radiation problem [1], the specific heat-phonon spectrum inversion [2,3], and inverse scattering problems in acoustics and electromagnetics. Most recently we proposed a new type of inverse problem: emissivity and transmissivity inversion (ETI) [4], which is of potential importance and interest for antiremote sensing.

However, it is well known that not all inverse problems have unique solutions. For some problems, it is necessary to add physical constraints in order to get a meaningful solution. We call these "generalized inverse problems," and propose one such problem here: a generalized emissivity inverse problem. While the solution to this problem might not be unique, it has significant practical importance. We show that by considering a physical model that introduces constraints, one gets interesting results for this class of generalized inverse problem.

II. A GENERALIZED EMISSIVITY INVERSE PROBLEM

The emissivity $g(\nu)$ is defined as a measure of the radiation capability of a body relative to a black body. If the emissivity $g(\nu)$ is known, then the total radiated power $J(T)$ can be written as function of emissivity $g(\nu)$ in an integral expression. In the emissivity and transmissivity inverse problem (ETI) [4] one can obtain the emissivity $g(\nu)$ from the total radiated power $J(T)$ as a function of temperature by solving an integral equation. In ETI the emissivity is *frequency dependent* only, but, in practice the emissivities of many materials can also depend on the temperature of the radiators in a complicated way [6]. In order to include the temperature dependence of the emissivity, we present here a generalized emissivity and transmissivity inversion (GETI) problem.

The key integral equation is

$$J(T)=\frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3 g(\nu,T)d\nu}{e^{h\nu/k_BT}-1}. \tag{1}$$

Here the emissivity $g(\nu,T)$ is both *frequency* and *temperature* dependent. Hence this formulation is much closer to reality, as well as having practical importance. However, the above integral equation is very difficult to solve, and furthermore the key equation is not adequate to determine $g(\nu,T)$. As a first step, one must find additional information that, added to Eq. (1), will complete the problem. Such additional conditions will be shown to be essential in solving these new inverse problems.

Empirical models have been developed for many physical quantities, based on experimental results, to describe complicated real systems. In past decades, many experiments have been performed investigating the emissivity of various radiators. From those studies, the temperature dependence of $g(\nu,T)$ for many materials is known to be described by the following relation [7,8]:

$$\ln g(\nu,T)=a_0+b_0\lambda+c_0\lambda^2+\cdots, \tag{2}$$

where $\lambda=c/\nu$ (c is the speed of light in vacuum), and $a_0,b_0,c_0,\ldots$ are independent of wavelength, but can vary with temperature. These constants have the dimensions $1,\lambda^{-1},\lambda^{-2},\ldots$. The formula (2) accurately describes the emissivity behaviors of many materials within a relatively

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wide wavelength region and temperature interval, such as, W [9], Mg [10], K, Ta, Ir, Re, NbB_2, etc. [11,12], and some highly emitting nonmetals, such as ceramic matrix composites [13].

In general, a linear model can be used to describe the temperature dependence of the coefficients $a_0, b_0, \ldots,$ so that in a suitable temperature and frequency region $g(\nu,T)$ may be written as follows, using only the first two terms of Eq. (2):

$$g(\nu,T)=g_\beta(\nu)\exp(\mu T-\beta T/\nu), \tag{3}$$

where $\beta>0$, and μ are constants, and $g_\beta(\nu)$ denotes the temperature-independent part of $g(\nu,T)$. In many cases, μ is very small, and Eq. (3) can be simplified further to

$$g(\nu,T)=g_\beta(\nu)\exp(-\beta T/\nu). \tag{4}$$

Materials to which Eq. (4) applies may be called β-type materials. We focus on β-type materials in what follows. By inserting this relation into Eq. (1), the GETI problem can be rewritten as follows:

$$J(T)=\frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3\exp(-\beta T/\nu)}{e^{h\nu/k_BT}-1}g_\beta(\nu)d\nu. \tag{5}$$

Here the parameter β manifests the temperature dependence of the emissivity, and the above new equation is quite different from the previous one we studied [4].

In some cases, the variation of emissivity with temperature is accurately represented by expanding Eq. (4) in powers of temperature [14–16]

$$g(\nu,T)=g_\beta(\nu)\left(1-\frac{\beta T}{\nu}+\frac{1}{2}\left(\frac{\beta T}{\nu}\right)^2+\cdots\right). \tag{6}$$

In order to solve this new integral equation, we follow our previous work [4], beginning with the asymptotic behavior analysis of $J(T)$. In general, $0\leqslant g(\nu,T)\leqslant 1$, so $0\leqslant J(T)\leqslant\sigma T^4$. In the limiting case of ideal black-body radiation, $g(\nu,T)\equiv 1$, and $J(T)=\sigma T^4$. Furthermore, for many materials, it is only in some finite interval of ν that the value of $g(\nu,T)$ is significantly greater than zero, while for $\nu\to 0$ or $\nu\to\infty$, $g(\nu,T)\to 0$ as, for example, $g(\nu,T)\sim\nu^s\exp(-\alpha\nu)$, $(0<a<\nu<b<\infty)$. In those cases, $J(T)$ has the following asymptotic behavior:

$$J(T)\sim\begin{cases}T^{s_1} & \text{when } T\to\infty\\ T^{s_2} & \text{when } T\to 0,\end{cases} \tag{7}$$

where $0<s_1<s_2$, and generally $1\leqslant s_1\leqslant 4, 4\leqslant s_2\leqslant\infty$.

By choosing s $(s_1<s<s_2)$, we have

$$J(T)/T^s=\frac{2\pi h}{c^2}\int_0^\infty \frac{(\nu^3/T^s)\exp(-\beta T/\nu)}{e^{h\nu/k_BT}-1}g_\beta(\nu)d\nu. \tag{8}$$

Then $\lim_{T\to 0}J(T)/T^s=0$, and $\lim_{T\to\infty}J(T)/T^s=0$. Introducing a logarithmic transformation of the variables (ν,T) by defining

$$x=\ln(T/T_0),\quad y=\ln(h\nu/k_BT_0), \tag{9}$$

Eq. (5) becomes

$$Q_0(x)=\int_{-\infty}^{\infty}K_{\beta_1}(y-x)F_0(y)dy, \tag{10}$$

where $\beta_1=(h/k_B)\beta$ (for simplicity we later use the symbol β, without subscript for this β_1), and

$$Q_0(x)=\frac{1}{\sigma T_0^4}J(T_0e^x)e^{-sx}, \tag{11}$$

$$F_0(y)=e^{(4-s)y}g_\beta\left(\frac{k_BT_0}{h}e^y\right), \tag{12}$$

$$K_\beta(y-x)=\frac{e^{s(y-x)}\exp(-\beta e^{x-y})}{\exp(e^{y-x})-1}. \tag{13}$$

The Fourier transform of the kernel $K_\beta(x)$ is

$$\hat{K}_\beta(-k)=\int_{-\infty}^{\infty}\frac{e^{ikx+sx}e^{-\beta/e^x}}{e^{e^x}-1}dx$$
$$=\int_0^\infty\frac{\xi^{s+ik-1}e^{-\beta/\xi}}{e^\xi-1}d\xi. \tag{14}$$

By a convolution theorem similar to that for the Fourier transform, one has

$$\hat{F}_0(k)=\frac{\hat{Q}_0(k)}{\hat{K}_\beta(-k)}. \tag{15}$$

Then, taking the inverse Fourier transform, one obtains the following exact solution formula:

$$g_\beta(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{Q}_0(k)\left(\dfrac{h\nu}{k_BT_0}\right)^{s+ik-4}}{\hat{K}_\beta(-k)}dk, \tag{16}$$

Notice that introducing the parameter s is critical for the derivation [2].

When $\beta=0$, the denominator $\hat{K}_0(-k)$ in the solution formula (16) becomes [4]

$$\hat{K}_0(-k)=\Gamma(s+ik)\zeta(s+ik), \tag{17}$$

where $\Gamma(z),\zeta(z)$ are the Euler gamma function and the Riemann zeta function, respectively. The following asymptotic behavior control condition is necessary and sufficient to guarantee the existence and uniqueness of the solution [5,4]:

$$\hat{Q}_0(k)=o\left[k^{s-1/2}\exp\left\{-k\tan^{-1}\left(\frac{k}{s}\right)\right\}\right],\quad k\to\pm\infty,\ s>1. \tag{18}$$

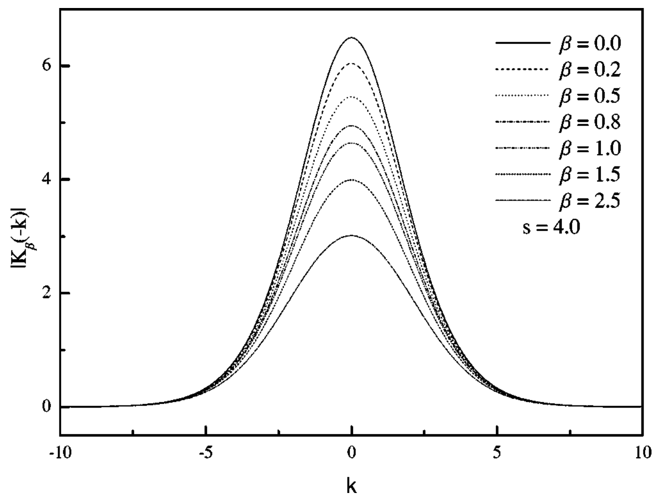

FIG. 1. $\hat{K}_\beta(-k)$ versus β.

In the general case, there is a weight factor $e^{-\beta/\xi}$ appearing in the definition of $\hat{K}_\beta(-k)$ in Eq. (14), with $\beta>0$. This monotone increasing function confines contributions to the integrand of $\hat{K}_\beta(-k)$ primarily to large ξ, compared to $\hat{K}_0(-k)$. On the other hand the factor $\xi^{ik}=\exp(ik\ln\xi)$ oscillates more strongly with increasing ξ. These two properties imply that $|\hat{K}_\beta(-k)|$ decreases with β. Figure 1 shows the variation of $|\hat{K}_\beta(-k)|$ versus β. In order to guarantee the existence of the solution to Eq. (5) of GETI, one needs at least condition (18) or some stronger one.

In practice, experimental data could hardly satisfy the controlling condition (18) for large k, which goes to zero exponentially [i.e., faster than $\Gamma(s+ik)$]. Here we propose a universal function set (UFS) method to overcome this difficulty. The key point is to choose a complete orthogonal function set to guarantee the correct asymptotic behavior in advance. The Hermite functions [17] are one such basis,

$$u_n(x)=\sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\exp[-(1/2)\alpha^2x^2]H_n(\alpha x), \quad (19)$$

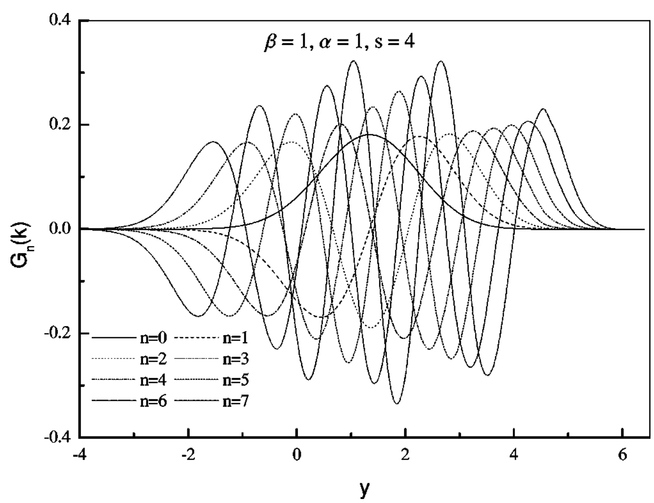

FIG. 2. Universal function set $G_n(y)$:$n=0-7$.

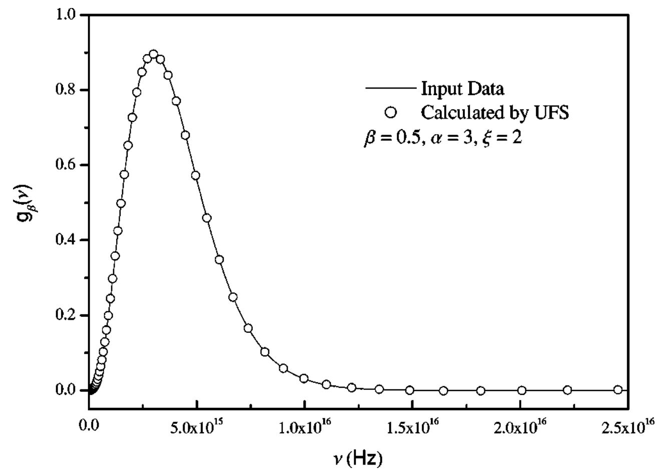

FIG. 3. Comparison of $g(\nu)$ calculated by the UFS (open circles) with the known input function (solid curve).

where α is a parameter. One can expand $Q_0(x)$ in terms of $u_n(x)$

$$Q_0(x)=\sum_{n=0}^{\infty} C_n u_n(x), \quad (20)$$

then

$$g_\beta(\nu)=\sum_{n=0}^{\infty} C_n G_{\beta,n}(\nu), \quad (21)$$

where

$$G_{\beta,n}(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{u}_n(k)\left(\frac{h\nu}{k_BT_0}\right)^{s+ik-4}}{\hat{K}_\beta(-k)}dk \quad (22)$$

and

$$\tilde{u}_n(k)=(-i)^n\sqrt{\frac{2\sqrt{\pi}}{\alpha 2^n n!}}e^{-(k^2/2\alpha^2)}H_n(k/\alpha). \quad (23)$$

One notices that the Fourier transform of $\{u_n(x)\}$ is a function set that is orthogonal in k space. The universal function set $\{G_{\beta,n}(\nu)\}$ can be calculated to high precision. Figure 2 shows some of these functions [where the variable ν is scaled into y through the transformation in Eq. (9)]. The temperature-dependent emissivity $g(\nu,T)$ can be obtained from Eqs. (21) and (4).

The solution formulas, Eqs. (16) and (22), depend on the preestablished parameter β. The key point is that one cannot determine β from the solution formula itself. One possible and practical method is to measure the emissivity at two different temperatures but at the same frequency, $g(\nu_0,T_1)$, $g(\nu_0,T_2)$. Then according to Eq. (4)

$$\beta=\frac{\nu_0}{T_2-T_1}\ln\left(\frac{g(\nu_0,T_1)}{g(\nu_0,T_2)}\right). \quad (24)$$

Another method is by fitting $J(T)$ to obtain β.

In order to check the exact solution formula (16), we have chosen a known function $g_0(\nu)$ and a definite parameter β, and have obtained the corresponding $J(T)$ to use as input in GETI, as shown in Fig. 3 by the solid curve. By comparing the calculated $g_\beta(\nu)$ with the known function $g_0(\nu)$, one finds excellent agreement for this test function.

III. CONCLUDING REMARKS

In this paper a generalized inversion problem is presented, in which the defining integral equation does not determine the solution uniquely. There are many such inverse problems in practice, including noisy data cases, which are of potential importance. In order to solve such problems, one must add additional conditions from physical considerations. In mathematics this procedure is analogous to adding suitable boundary conditions to a differential equation, but the main difference here is that the key equation is an integral equation instead of a differential equation. Fortunately, β-type materials are found experimentally, and the GETI problem can be solved exactly and uniquely by including the condition (4) for the emissivity. The exact solution formula to the GETI is given by considering the asymptotic behavior control conditions and introducing the parameter s to eliminate divergences. A modified UFS was developed to make practical calculations possible.

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