Computer evaluation of the complementary error function

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$$= 2\pi i \left(\frac{dB}{dK}\right)_{K=0} \exp(-i\Omega t)$$

$$+ 2\pi B(0) \left(\frac{d\Omega}{dK}\right)_{K=0} t \exp(-i\Omega t). \qquad (7)$$

From Eqs. (5)–(7),

$$\langle z(t)\rangle = i \left(\frac{dB}{dK}\right)_{K=0} [B(0)]^{-1} + \left(\frac{d\Omega}{dK}\right)_{K=0} t. \qquad (8)$$

The first term is simply the initial centroid of the modulating function, $\langle z(0)\rangle$. On transforming back to the original variables, Eq. (8) becomes

$$\langle z(t)\rangle = \langle z(0)\rangle + \left(\frac{d\omega}{dk}\right)_{k=k_0} t. \qquad (9)$$

Equation (9) can be looked upon as the equation of motion of the centroid of the wave packet.

[1]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 208.
[2]G. C. Johnson, *Field and Wave Electrodynamics* (McGraw-Hill, New York, 1965), p. 112.
[3]R. N. Bracewell, *The Fourier Transform and Its Applications* (McGraw-Hill, New York, 1965), p. 136.

Computer evaluation of the complementary error function

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Over the past two years, several notes[1–3] have been published on approximations to the error function. In general, these articles concerned themselves with various time-saving approaches which yield erf$x$ to accuracies of 4.5%,[1] 0.04%,[2] and 2.5×10^{-5}.[3] These approaches, which are not as accurate as the power series approach given below, in reality do not save time because for small x the power series expansion converges so rapidly and for larger x, where time saving might be important, the error function is so close to unity that these less accurate values are not very meaningful. This note is to point out that the power series expansion for erf$x$ coupled with the asymptotic expansion for erfc$x$ = 1 − erf$x$ can be programmed to run as rapidly as the less accurate techniques suggested, with a precision limited only by the number of figures carried by the computer. To be specific, with a computer that has a word equivalent of ten decimal digits, one can calculate erf$x = 2\pi^{-1/2}\int_0^x \exp(-y^2)\,dy$ to nine significant figures and erfc$x$ to six significant figures. The expansions are[4]

$$\mathrm{erf}x = 2x\pi^{-1/2}\exp(-x^2)\sum_{n=0}^{\infty}\frac{(2x^2)^n}{1\cdot 3\cdots(2n+1)}, \qquad x<a \qquad (1)$$

and

$$\mathrm{erfc}x = \frac{\exp(-x^2)}{x\pi^{1/2}}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdots(2n-1)}{(2x^2)^n} + 1, \qquad x>a. \qquad (2)$$

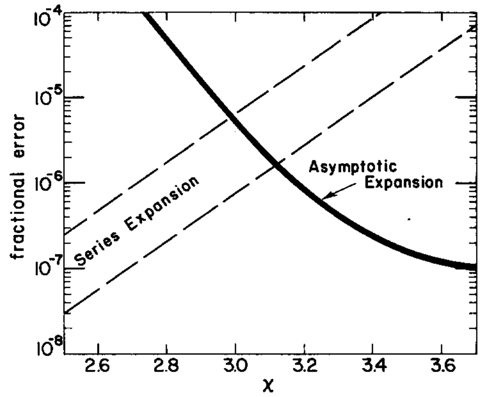

Fig. 1. The fractional error in the calculated erfc$x$ for the series expansion and asymptotic expansion in the region where it is necessary to change from one to the other.

The series expansion [Eq. (1)] is rigorously derived from the definition of the error function and involves no approximations. It converges for all values of x, but as x becomes large the convergence is slower and round-off error is more significant. The crossover point a is chosen as the point where the average error in erfc$x$ calculated by the series expansion exceeds the error in the asymptotic expansion as shown in Fig. 1 for a computer with a 12-decimal digit word length. The appropriate crossover point is near 2.8 for a 10-decimal digit word.

Note that the power series, Eq. (1), converges very rapidly with a maximum of less than 34 terms being used near $x = 3.1$. A great savings in computer time and accuracy is accomplished by not calculating each term individually but by calculating the $(n+1)$-st term from the nth term. The accuracy in the asymptotic region $(x > a)$ is improved by removing half the last term in the sum since the correct result must lie between any two consecutive partial sums of the alternating series. This series should be truncated at the ninth term for $a = 3$ because this is the smallest term in the asymptotic expansion near $x = 3$.

We programmed this problem as a subroutine on an HP9100 desk calculator, using seven words of memory for the program and four words for storage. This calculator has a 12-decimal digit word and yields erfc$x$ accurate to seven significant figures for $-5 < x < 15$, with a computation time less than 2 sec. The accuracy approaches ten significant figures for values of x well away from $x = a$. Also, we programmed this problem for the HP65 hand calculator[5] with an accuracy in erfc$x$ to at least five significant figures and a computation time less than 17 sec. The maximum time is for x just below the crossover point between the expansions. The expansion approach has the advantage over the rational fraction approximation[3] for small computers with small memories in that a large number of constants need not be stored in the calculator's memory.

Another paper on this subject appeared[6] after this paper was written, which gives an approximation of erf$x$ in closed form. This method is considerably faster in calculation time than any of the others but yields erf$x$ to only 0.7% and could not be used to calculate erfc$x$ for $x > 1.9$, where the error is $> 100\%$.

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[2]R. M. Hashway, Am. J. Phys. **42**, 75 (1974).
[3]R. K. Hobbie, Am. J. Phys. **42**, 913 (1974).
[4]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1969), p. 297.
[5]HP65 User's Library, Program No. 01293A.
[6]R. Manzel, Am. J. Phys. **43**, 366 (1974).

Asymmetrical conclusions drawn by twins performing identical measurements in special relativity

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It is well known that there is a basic asymmetry between the two twins in the famous twin problem of special relativity. One way to demonstrate this asymmetry is to analyze a situation wherein both twins perform measurements in an identical manner and yet obtain different results. Usually such a demonstration involves some additional assumptions about the measuring apparatus of the traveling twin during his acceleration period (behavior of his "rigid" rods, variations of his lines of simultaneity, etc.), and as a result leaves room for criticism.

There exists, however, a rather simple measurement procedure, the so-called Fock "radar station"[1] that lends itself to analysis completely within the framework of special relativity with no additional assumptions necessary. In this note we describe some of the results obtained when the accelerated twin makes space–time measurements utilizing radar techniques. The transformations between the two twins' radar space–time measurements are determined, and Minkowski diagrams are drawn for both twins. One result is that the traveling twin determines that during a considerable part of his trip the twin who stayed at home remains motionless.

With the radar method an observer reflects a light signal from the event of interest and notes the times on his clock at which the signal is emitted (t_e) and received back (t_r). Considering the motions only in one spatial direction, the distance x of the event from the observer and the time t assigned to the event by the observer are then defined by the expressions

$$x = \tfrac{1}{2} c (t_r - t_e), \quad t = \tfrac{1}{2} (t_r + t_e). \tag{1}$$

This procedure provides a well-defined, unambiguous method for an observer to assign space–time coordinates to any event. Moreover, the process is operational and no additional assumption needs to be made because the behavior of light signals is describable completely within the framework of special relativity.

When the radar method (together with the principle of relativity) is applied to two unaccelerated inertial observers moving relative to each other with constant velocity, the results are the same as those obtained by the usual procedure of using rigid rods and stationary clocks. The space–time measurements of the observers are related by the Lorentz transformations, and each observer appears as an inclined straight line on the other's Minkowski diagram. In fact, the radar method provides an interesting way of deriving the Lorentz transformations between two inertial observers.[2]

There is, however, no need to restrict the radar definition (1) to observers who remain unaccelerated throughout their entire histories. The definition (1) is completely general and can be applied to accelerated as well as unaccelerated observers, as has been demonstrated previously for hyperbolically accelerated observers.[3]

In particular, we now describe the observations of the traveling twin, whom we will call observer O′, of the twin problem of special relativity, who uses the radar

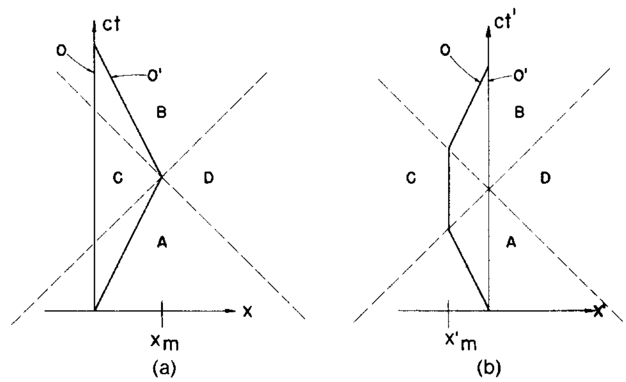

Fig. 1. (a) Minkowski diagram for twin O showing the four regions A, B, C, and D. (b) Minkowski diagram for twin O′ showing the four regions A, B, C, and D.