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Why are resonant frequencies sometimes defined in terms of zero reactance?

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The resonant frequencies of a sinusoidally driven system are usually defined as those for which some physical response is a relative maximum. There is also a tendency to define them as frequencies for which the reactance of the system vanishes, since zero-reactance frequencies are often approximately equal to maximum-response frequencies and are sometimes easier to calculate. However, there are many systems for which the two types of frequencies are significantly different. There are also systems for which the reactance does not vanish in certain frequency ranges, though maximum responses still occur. It is concluded that vanishing reactance is not valid as a general criterion for resonance, and students should be warned against its use.

I. INTRODUCTION

In studying the properties of sinusoidally driven systems, one often finds it important to determine their resonant frequencies. These are usually defined to be those frequencies for which some type of physical response of the system is a relative maximum. However there seems to be a tendency, often expressed in current physics and electrical engineering textbooks, to define the resonant frequencies as those for which the reactance of the system vanishes. This latter point of view raises some questions, since for some systems the zero-reactance frequencies can be quite different from those at which quantities other than reactance achieve maximum response. The intent of this paper, therefore, is to examine the condition of zero reactance and evaluate its validity as a criterion for resonance. We begin with a consideration of what is really meant by resonance in a sinusoidally excited system.

II. THE MAXIMUM-RESPONSE CONCEPT OF RESONANCE

The *AIP Handbook* defines resonance for a system in forced oscillation as a condition where any change, however small, in the frequency of excitation causes a decrease in the response of the system.[1] In other words, a curve representing some physical response as a function of frequency exhibits a relative maximum at resonance.

There are many familiar illustrations of this concept. A mechanical oscillator, for example, exhibits a maximum in its displacement amplitude at one resonant frequency, and a maximum velocity amplitude at another. Good examples of electrical circuits that exhibit resonance are the simple series *RLC* circuit in which the current amplitude is maximum at the resonant frequency, and the simple parallel circuit with its maximum voltage at resonance. There are also many acoustical systems, such as the air column in a musical wind instrument, that exhibit maximum-response resonances. For example, the resonant frequencies of a clarinet or a trumpet are almost always defined in terms of maximum pressure amplitude at the reed, whereas resonance in a flute corresponds to maximum volume velocity.

It is interesting to note that in all the above cases the maximum-response criterion for resonance has a real physical validity, expressed in terms of properties that are involved in the perception of resonance or interaction with other systems. The zero-reactance criterion, to be discussed next, seems by contrast to be somewhat arbitrary and artificial.

III. THE ZERO-REACTANCE CRITERION

Reasons are not usually stated for identifying zero reactance[2] with the concept of resonance, but it is often done in engineering applications, and is commonly found in physics and engineering textbooks. Some texts simply list zero reactance as one of several alternative conditions for resonance,[3] while others actually define resonance in this way.[4,5]

The practice of identifying zero reactance with resonance probably grew out of the use of complex analysis in the treatment of resonant electrical circuits. The usual procedure for determining a maximum-response frequency is to derive an expression for the complex circuit impedance **Z**, and then to find the driving frequency for which $|\mathbf{Z}|$ becomes maximum (or minimum). The zero-reactance frequency, of course, is found by setting the imaginary part of **Z** equal to zero. When this method is applied to the simple series *RLC* circuit, it is found that the frequency of maximum current response is identical to the zero-reactance frequency, and this is also the case for the maximum voltage response in a simple parallel circuit. Hence for these two cases, which are the ones most often used in textbook illustrations of resonance, the zero-reactance criterion appears to be valid. Furthermore, it is usually much easier to solve for the frequency at which the reactance of a circuit vanishes than it is to maximize (or minimize) the expression for $|\mathbf{Z}|$. And finally, for the high-Q resonant circuits that are of greatest interest to electrical engineers, the zero-reactance frequency often turns out to be so close to the frequency of maximum response that the difference can be neglected for practical purposes. It is apparently for these three major reasons (successful application to simple circuits, ease of calculation, and negligible error in many practical situations) that zero reactance has come to be so widely used as a criterion for resonance.

From an analytical point of view, however, the criterion is suspect. Note that the impedance of a circuit can be written as $\mathbf{Z} = r + jx$, where the "resistance" r and the reactance x are generally both functions of the frequency ω. To determine the maximum response we set $d|\mathbf{Z}|/d\omega = 0$, which leads to the condition that

$$r\frac{dr}{d\omega} + x\frac{dx}{d\omega} = 0.$$

It is clear that vanishing reactance does not in itself guarantee a maximum response; the term $r(dr/d\omega)$ must vanish as well. The term does vanish in many cases, such as those discussed above, but there are important exceptions. We now wish to examine some of these, and to evaluate the effectiveness and validity of the zero-reactance criterion. We shall consider its application first to electrical circuits, and then to acoustical air columns.

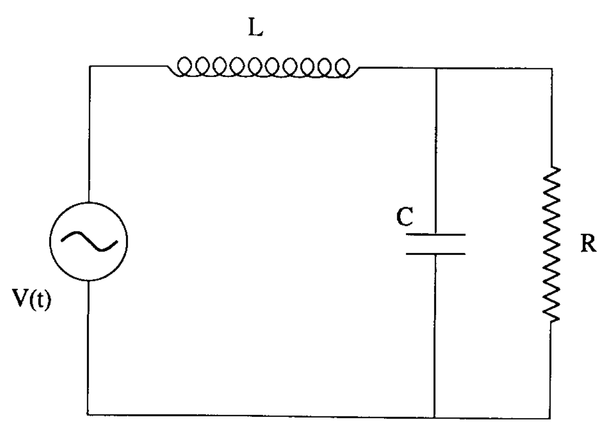

Fig. 1. *RLC* circuit, driven by a sinusoidal voltage source $V(t) = V_0 \sin \omega t$.

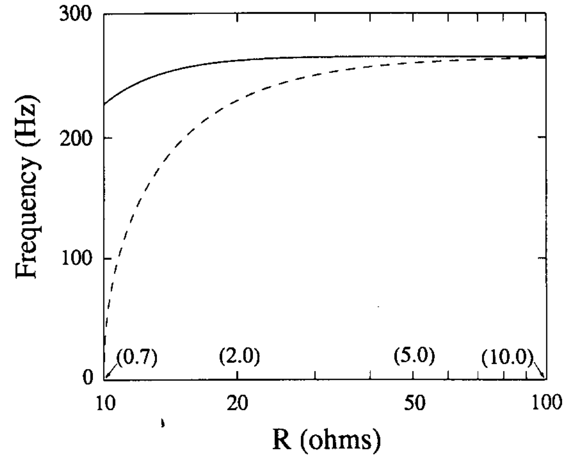

Fig. 2. Maximum current frequency (solid line) and zero-reactance frequency (dashed line) as functions of R, for the *RLC* circuit in Fig. 1. Corresponding values of Q are shown in parentheses.

IV. HOW VALID IS THE ZERO-REACTANCE CRITERION FOR ELECTRICAL CIRCUITS?

While vanishing reactance appears to be a useful criterion for resonance in many electrical circuits, its usage also raises some questions. An electrical circuit is likely to have a number of maximum-response frequencies, corresponding for example to maxima in the total current, voltage, power, and current through or voltage across individual components. These frequencies may differ from each other, and the zero-reactance frequency is usually different from any of them.[6] In a high-Q circuit it often turns out that the various frequencies for maximum response are nearly equal, and that the zero-reactance frequency is approximately the same. However it is quite possible to have a circuit of lower Q for which the use of zero reactance to approximate the corresponding peak frequency could result in significant error.

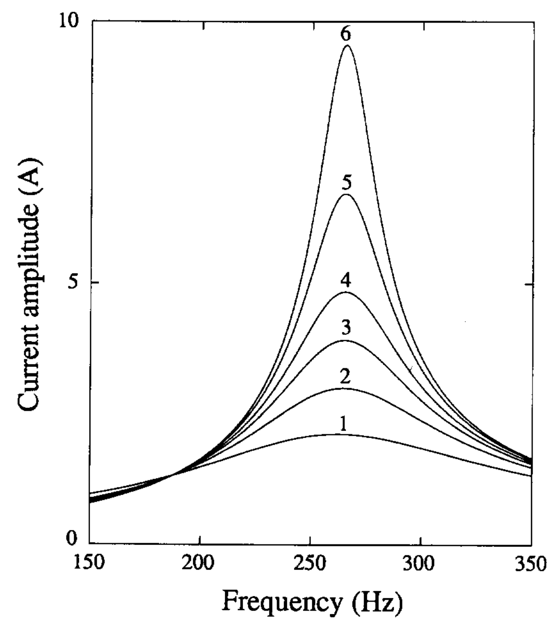

Fig. 3. Resonance curves for the *RLC* circuit in Fig. 1. See Table I for corresponding values of the parameters.

Table I. Parameters corresponding to resonance curves in Fig. 3.

Curve	R (Ω)	Q	f_r (Hz)	f_x (Hz)	Δf (%)
1	20	2.0	261.89	229.72	− 12.3
2	30	3.0	264.52	250.09	− 5.5
3	40	4.0	265.01	256.84	− 3.1
4	50	5.0	265.16	259.90	− 2.0
5	70	7.0	265.23	262.54	− 1.0
6	100	10.0	265.25	263.93	− 0.5

For example, consider the RLC circuit in Fig. 1. If we let L and C have the nominal values of 6 mH and 60 μF, respectively, and plot the maximum-current and zero-reactance frequencies as functions of R, we obtain the curves of Fig. 2. Note that the two frequencies are approximately equal when Q is high, but become widely divergent as Q decreases. We can check the sharpness of the maximum-current resonance by examining the family of curves in Fig. 3. Table I lists the values of R, Q, the resonant frequency f_r (for maximum current amplitude), the zero-reactance frequency f_x, and the percentage Δf by which f_x differs from f_r, for each curve. It is clear that a definite resonance effect continues to exist (as shown, for example, by curves 2–4 of Fig. 3) well into the region where the divergence between the zero-reactance and maximum-current frequencies becomes significant. Such a region exists for most electrical circuits that exhibit resonance, and certainly makes the validity of the zero-reactance criterion questionable.

V. HOW VALID IS THE ZERO-REACTANCE CRITERION FOR ACOUSTICAL SYSTEMS?

We will consider just two acoustical systems, a cylindrical tube and a truncated cone, both of which contain air columns that are useful in musical wind instruments. Both are taken to be essentially closed at the driven end (the small end, for the cone) with a sinusoidal input volume velocity. Under these conditions the resonant frequencies, corresponding to relative maxima of the input pressure amplitude, are those for which the magnitude of the input acoustical impedance is maximized. Hence, we can determine the resonant frequencies by plotting $|\mathbf{Z}|$ as a function of frequency and identifying the positions of the peaks.

Such a curve for a cylindrical air column of radius 0.5 cm and length 50 cm, with typical viscous, thermal, and radiation losses, is shown in Fig. 4(a). Note the sharp, well-defined resonance peaks, with frequencies that are close to the odd-harmonic series one expects for the idealized case. Figure 4(b) shows the input reactance curve for the same air column. Note that the zero-reactance frequencies, where the reactance changes steeply from positive to negative, correspond very closely to the peak frequencies of the impedance curve. (Indeed, for an idealized cylinder with

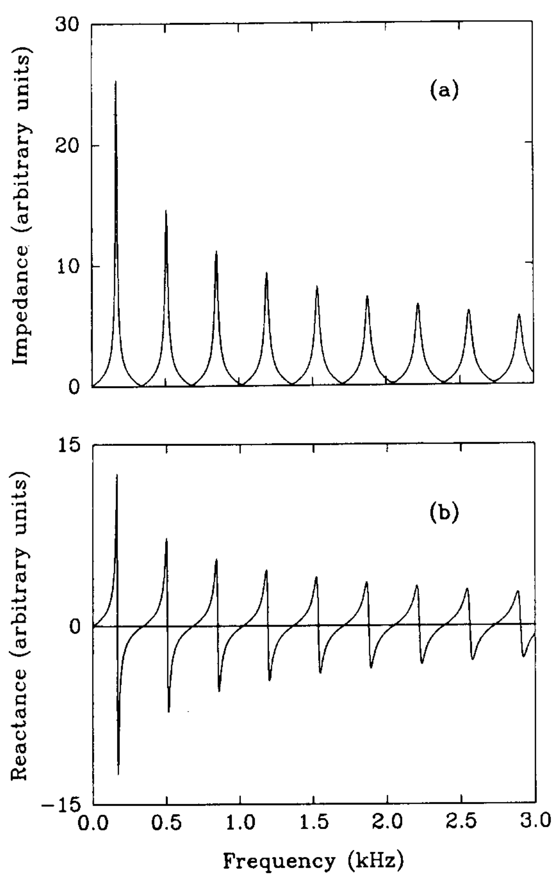

Fig. 4. Impedance and reactance curves for a cylindrical air column. (a) Magnitude of the input impedance versus frequency. (b) Input reactance versus frequency.

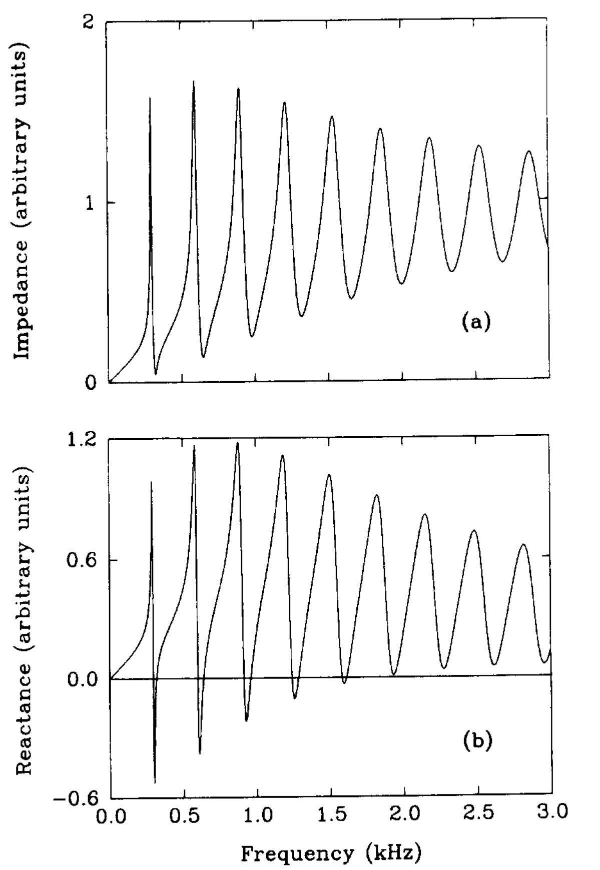

Fig. 5. Impedance and reactance curves for the air column in a truncated cone. (a) Magnitude of the input impedance (at the small end) versus frequency. (b) Input reactance versus frequency.

no losses the two sets of frequencies are exactly the same.)

From these results one might jump to the conclusion that it would be advantageous to use the zero-reactance criterion with air columns, but there are at least two reasons why this is not true. For one thing, when losses are taken into account the analytical calculations of maximum-impedance frequencies and zero-reactance frequencies alike become rather complicated, even for a cylindrical air column. For precise results, such calculations are most easily done numerically with a computer. That being the case, it becomes just as easy to determine the maximum-impedance frequencies as it is to determine those for zero reactance, and the "ease of calculation" advantage no longer exists. Furthermore, it turns out that for many tube shapes other than cylindrical the zero-reactance frequencies are significantly different from the maximum-impedance frequencies. This is well illustrated by our next example, the air column in a truncated cone.

Figures 5(a) and 5(b) show the impedance and reactance curves for a truncated conical air column with inlet radius 0.5 cm, outlet radius 6.0 cm, and length 50 cm, corresponding roughly to the dimensions of a trumpet bell. Note the well-defined resonance peaks in the impedance curve. These are fairly sharp (high-Q), so one might expect from experience with electrical circuits that the zero-reactance criterion would yield good approximations. However, it turns out that the zero-reactance frequencies are consistently too high, by significant amounts that increase successively with each peak. Furthermore, Fig. 5(b) reveals that *the reactance doesn't even vanish after the fifth mode!* There are plenty of reasonably sharp impedance peaks above the fifth-mode frequency, but no reactance zeroes at all. The criterion totally breaks down here, and we must again raise serious doubts about the validity of zero reactance as a condition for resonance.

VI. CONCLUSIONS

Zero-reactance frequencies are sometimes easier to determine than frequencies of maximum response (particularly in the complex analysis of electrical circuits), and may be approximately equal to some of the maximum-response frequencies for certain high-Q systems, but there are at least three good reasons for not using zero reactance as a general criterion for resonance:

(1) It can lead to significant error in the determination of resonant frequencies, as it does with many acoustical air columns and lower-Q electrical circuits.

(2) Reactance zeroes do not even exist for some systems in certain frequency ranges, though maximum responses may occur in those same ranges.

(3) The criterion itself is rather artificial, and has no conceptual relationship to the maximum physical responses of a system.

As we teach the concept of resonance we should explain why it is sometimes defined in terms of zero reactance, point out the fallacy of this, and warn students of the possibility of error. Indeed, we should probably recommend that the definition not be used at all.

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[1] *American Institute of Physics Handbook*, edited by D. E. Gray (McGraw-Hill, New York, 1972), 3rd ed., p. 3-10.

[2] Electrical engineers may also refer to this condition as real impedance, a purely resistive load, zero phase angle, or unity power factor.

[3] For example, see J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1979), 3rd ed., pp. 276–278.

[4] See J. W. Nilsson, *Electric Circuits* (Addison-Wesley, Reading, MA, 1983), pp. 456, 471, and 488, for example. Note that several types of maximum response are examined, but the word "resonance" is used only for the condition of zero reactance.

[5] L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics* (Wiley, New York, 1982), 3rd ed. Resonance in acoustical systems is characterized by zero reactance throughout the book.

[6] A good discussion of these various resonances was given 25 years ago by E. J. Burge, Am. J. Phys. **29**, 19 (1961), who then recognized a need that we see now.

The momentum of a transverse wave

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The mechanism by which a traveling transverse wave in a string stores momentum and transports energy is studied. The existence of longitudinal waves with much higher propagation speed is shown to be essential.

It is commonplace that the quantum treatment of traveling waves possess a momentum density. Recoil is associated with the emission and absorption of photons, phonons, and other kinds of quanta as in the Compton effect. The relation between momentum density and energy flux in a traveling wave does not involve Planck's constant, and in fact, has been long known in the case of electromagnetic waves where the Poynting vector specifies both. This con-