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Session 2pAAb: Dah-You Maa: His Contributions and Life in Acoustics

2pAAb12. A hybrid modal analysis for enclosed sound fields and its applications

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In 1939, Dr. Dah-You Maa presented a paper entitled "The Distribution of Eigentones in a Rectangular Chamber at Lower Frequency Ranges" [D. Y. Maa, J. Acoust. Soc. Am. 10, 258 (1939)]. Since then, his interest in room acoustics has not diminished. More than six decades later, Maa proposed an idea of adding a monopole solution to the normal modal expansion to improve the accuracy in simulating the near-field sound field in rooms [D. Y. Maa, Acta Acust. (Beijing) 27, 385-388 (2002)]. Following this idea, a hybrid model that combines the free field Green's function and a modal expansion has been proposed by the authors based on a rigorous mathematical derivation [B. Xu et al, J. Acoust. Soc. Am.128, 2857-2867 (2010)]. The hybrid modal expansion can be further extended for complex sound sources by introducing the multipole expansion to the solution. In this paper, the theoretical derivation of the hybrid modal expansion will be reviewed, followed by examples demonstrating the use of the hybrid modal expansion in real-world applications.

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INTRODUCTION

To ordinary Chinese people, Dr. Dah-You Maa is most well known for his leading role in designing the acoustics and the audio system of the Great Hall of the People in 1959. Maa's interest in architectural acoustics can be traced back to the 1930's. In one of his earliest JASA publications, he studied the distribution of normal modes in rooms, which is an important contribution to the general framework of modal analysis (MA).

Modal analysis has been widely used to study the low-frequency response in enclosed sound fields. In the literature related to the sound field computation based on MA, distributed sources on boundaries are often considered, e.g., a piston source mounted on the wall in a room. Small sources (e.g., point sources) inside an enclosure are not sufficiently studied, partially due to the very slow convergence rate of MA in the near field. Maa recognized this deficiency in a study on active noise control (ANC) in rooms (Maa, 1994). Later, in a 2002 paper (Maa, 2002), he proposed adding the free field monopole solution in the near-field to the classical modal expansion solution (Morse and Ingard, 1968), which essentially divides the sound field into a direct field and a reverberant field (Beranek, 1954). Even though this proposed solution generally works well in many applications, it is not quite the exact solution to the wave equation.

Inspired by Maa's work, the authors proposed a hybrid model that combines the free-field Green's function with a modal expansion based on a rigorous mathematical derivation (Xu and Sommerfeldt, 2010). The hybrid model can be further extended for complex sound sources by introducing the multipole expansion to the solution. It has been shown that this hybrid method not only greatly improves the convergence rate, but also provides an elegant way to study the physical properties of enclosed sound fields. In this paper, the theoretical derivation of the hybrid modal expansion will be reviewed, followed by examples demonstrating the use of the expansion in real-world applications.

SUMMARY OF THE THEORETICAL DERIVATION

The sound field in an enclosure that is generated by a point source satisfies the nonhomogeneous Helmholtz equation (Pierce, 1989):

$$\nabla^2 \hat{p} + k_0^2 \hat{p} = -i\rho_0\omega_0 \hat{Q}_s(\omega_0) \delta(\mathbf{r} - \mathbf{r}_o), \quad (1)$$

where p is the sound pressure, ω_0 is the excitation frequency, k_0 is the acoustic wave number, ρ_0 is the air density, and Q_s represents the source strength of the point source. Here, $\hat{\cdot}$ indicates that the quantity is in the Fourier transform domain.

The boundary condition is usually assumed to be locally reacting and given as follows:

$$\left. \frac{\partial \hat{p}}{\partial \mathbf{n}} \right|_S = \beta = -ik_0 \frac{\rho_0 c}{z}, \quad (2)$$

where z is the specific acoustic impedance of the boundary, and β represents the *normalized* specific acoustic admittance.

To solve for \hat{p} , it can be written as the sum of the free-field Green's function, $G(\mathbf{r}|\mathbf{r}_o)$, and a homogeneous solution [$F(\mathbf{r})$]:

$$\hat{p}(\mathbf{r}) = \frac{i\rho_0\omega_0 \hat{Q}_s}{4\pi} G(\mathbf{r}|\mathbf{r}_o) + F(\mathbf{r}), \quad (3)$$

where

$$G(\mathbf{r}|\mathbf{r}_o) = \frac{1}{|\mathbf{r} - \mathbf{r}_o|} e^{-ik_0(\mathbf{r} - \mathbf{r}_o)}, \quad (4)$$

and $F(\mathbf{r})$ is a solution to the homogeneous Helmholtz equation. Neither $G(\mathbf{r}|\mathbf{r}_0)$ nor $F(\mathbf{r})$ alone would satisfy the boundary condition represented by Eq. (2), but together they can be constructed to do so. With the form of Eq. (3), the sound field is actually divided into a direct field (for a point source), $\frac{i\rho_0\omega_0\hat{Q}_s}{4\pi}G(\mathbf{r}|\mathbf{r}_0)$, and a reverberant field, $F(\mathbf{r})$.

$F(\mathbf{r})$ can be solved using a modal expansion,

$$F = \sum_{n=1}^N q_n \psi_n, \tag{5}$$

where q_n is called the modal amplitude, ψ_n is a modal function, and N is the total number of modal functions. Given a complete set of modal functions, q_n can be solved from the following equation, which is obtained by applying the Green's theorem and the boundary condition [Eq. (2)]:

$$\sum_n [(k_0^2 - k_m^2)C_{mn} + D_{mn}] q_n = -\frac{i\rho_0\omega_0\hat{Q}_s}{4\pi} \oint_S \psi_m^* \left(\beta G - \frac{\partial G}{\partial \mathbf{n}} \right) da, \tag{6}$$

where $C_{mn} = \iiint_V \psi_m^* \psi_n d^3x$ and $D_{mn} = \oint_S (\beta - \beta') \psi_m^* \psi_n da$. Here, β' represents the **normalized** specific acoustic admittance on the boundary that is satisfied by the modal function. The volume integral covers the entire volume inside the enclosure, and the surface integral is evaluated on the entire inside surface of the enclosure. Assuming the modal functions are orthogonal, the matrix form of Eq. (6) reads

$$\begin{bmatrix} (k_0^2 - k_1^2)C_{11} + D_{11} & D_{12} & \cdots \\ D_{21} & (k_0^2 - k_2^2)C_{22} + D_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} = -\frac{i\rho_0\omega_0\hat{Q}_s}{4\pi} \begin{bmatrix} \oint_S \psi_1^* \left(\beta G - \frac{\partial G}{\partial \mathbf{n}} \right) da \\ \oint_S \psi_2^* \left(\beta G - \frac{\partial G}{\partial \mathbf{n}} \right) da \\ \vdots \end{bmatrix},$$

or

$$A \cdot \mathbf{Q} = B, \tag{7}$$

where $A_{mn} = (k_0^2 - k_m^2)C_{mn} + D_{mn}$, $Q_n = q_n$, and $B_m = -\frac{i\rho_0\omega_0\hat{Q}_s}{4\pi} \oint_S \psi_m^* \left(\beta G - \frac{\partial G}{\partial \mathbf{n}} \right) da$.

Matrix A is nondiagonal, though often sparse and Hermitian. For an exact solution, N is often required to be infinity, which makes it impossible to solve for the q_n 's. In practice, however, with a set of properly chosen modal functions, ψ_n , q_n often diminishes quickly as n goes to a large number. It is therefore possible to keep only a finite number of ψ_n 's as well as q_n 's to obtain a desired accuracy (Xu and Sommerfeldt, 2010).

B_m in Eq. (7) contains a surface integral and may not be easy to evaluate analytically, but a numerical evaluation should be generally straightforward. In addition, recognizing the spherical spreading nature of G and $\partial G/\partial \mathbf{n}$, one can mesh the surface S accordingly to improve the efficiency of the computation.

For a complex source that cannot be treated as a monopole, but that is small compared to the dimensions of the enclosure, the hybrid modal expansion method can be modified to take into account the free-field radiation \hat{p}_0 of the source. Equation (3) becomes

$$\hat{p}(\mathbf{r}) = \hat{p}_0 + F(\mathbf{r}), \tag{8}$$

and Eq. (6) becomes

$$\sum_n [(k_0^2 - k_m^2)C_{mn} + D_{mn}] q_n = -\oint_S \psi_m^* \left(\beta \hat{p}_0 - \frac{\partial \hat{p}_0}{\partial \mathbf{n}} \right) da. \tag{9}$$

If the multipole expansion is known for a source in the free-field, e.g., $\hat{p}_0 = \sum_{m=1}^M A_m \Phi_m$, Eq. (9) can be further modified to the form

$$\sum_n [(k_0^2 - k_m^2)C_{mn} + D_{mn}] q_n = -\sum_{m=1}^M A_m \oint_S \psi_m^* \left(\beta \Phi_m - \frac{\partial \Phi_m}{\partial \mathbf{n}} \right) da, \tag{10}$$

where Φ_m represents the m th multipole and A_m stands for the multipole amplitude.

APPLICATIONS

Effective Absorption Coefficient

The calculation of the effective absorption coefficient α_e in rooms has been a long-time topic in room acoustics, and a variety of formulas have been given (Sabine, 1964; Eyring, 1930; Joyce, 1978, 1980; Hodgson, 1993; Beranek, 2006; Jing and Xiang, 2008). In this example, α_e will be calculated based on the critical-distance results simulated from the hybrid modal expansion. The effective absorption coefficient is directly tied to the critical distance, the distance from the source to where the direct sound pressure, \hat{p}_0 , equals the reverberant sound pressure, $F(\mathbf{r})$:

$$R_c = \sqrt{\frac{S\alpha_e}{16\pi(1-\alpha_e)}}, \quad (11)$$

where R_c stands for the critical distance, S is the inner surface area of the room, and α_e is the effective absorption coefficient.

In this example, the critical distances of a set of rooms will be computed based on the hybrid modal analysis, and thus α_e can be solved from Eq. (11). The rooms all have the same dimensions ($\sqrt{15}m \times \pi m \times 5m$) but different inner surfaces. The Sabine absorption coefficient varies from 0.05 to 0.8. The sound fields excited by a randomly placed point source in each room are computed at five frequencies in the 630 Hz one-third octave band. The computed critical distances are averaged over ten source locations and then inserted into Eq. (11) to solve for α_e . Figure 1 compares the R_c -based results with Sabine's formula and Eyring's formula. The former

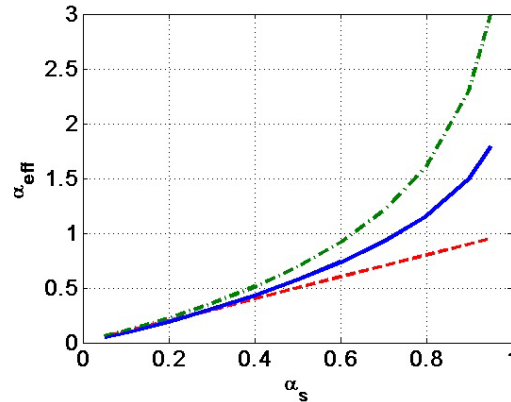


FIGURE 1: (Color online) Effective absorption coefficient as a function of Sabine's absorption coefficient calculated in a rectangular room for the 630 Hz one-third octave band. "—": prediction based on the hybrid modal expansion and the critical distance; "--": Sabine's formula; "- · -": Eyring's formula.

generally fall in the middle of them, which is very similar to the results of Joyce (Joyce, 1978) (see curve "s = 7/9" in his Fig. 4) and Jing *et al.* (Jing and Xiang, 2008).

Complex Sources in an Enclosure

In this example, the hybrid modal expansion is applied to compute the enclosed sound fields excited by two different sources: a dipole source [$D(r, \phi, \theta) = \cos(\theta)$] and a complex source [$D(r, \phi, \theta) = \sqrt{2}\cos(\theta/2)$]. Both sources are located at the center of an enclosure (dimensions: $em \times \pi m \times 2m$). The effects of the source directivity are clearly represented by the hybrid modal expansion (Fig. 2).

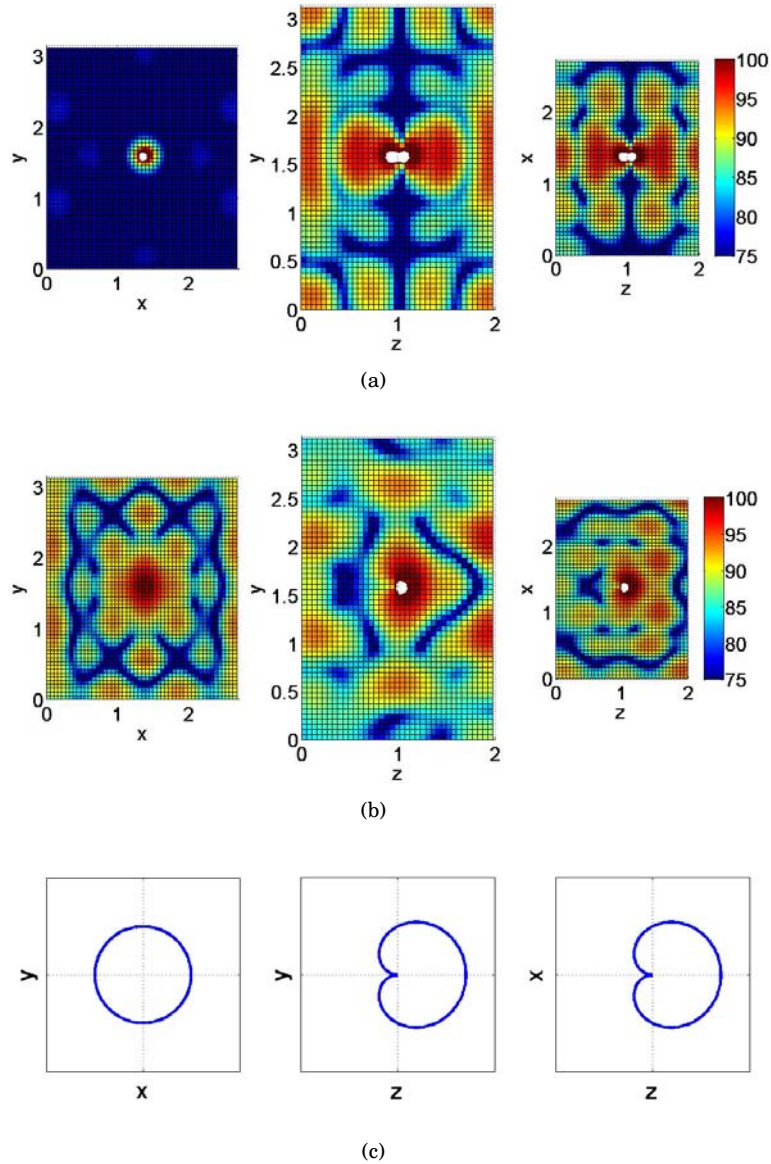


FIGURE 2: Sound pressure level computed by the hybrid modal expansion for (a) a dipole source and (b) a small complex source in a rectangular room at 400 Hz. Both sources are placed at the center of the room. Pressure fields on the x-y, y-z and x-z planes that include the source are plotted with the white dots representing the location of the sources. The directivity pattern of the small complex source is shown in (c).

Insertion Loss of an Open Rectangular Enclosure

An accurate estimation of the insertion loss (IL) for enclosures with one or more openings is very important in many noise control applications. A numerical model based on the hybrid modal analysis was established and used to study the IL below the Schroeder frequency. In this example, an enclosure with dimensions $1.2m \times 2m \times 0.8m$ and a single opening ($0.5m \times 0.5m$) is studied. The opening is located at the center of the ceiling ($1.2m \times 2m$) and a point source is placed on the floor (in a corner). The IL predictions from the numerical model are compared with the experimental results. As shown in Fig. 3, the prediction is very good.

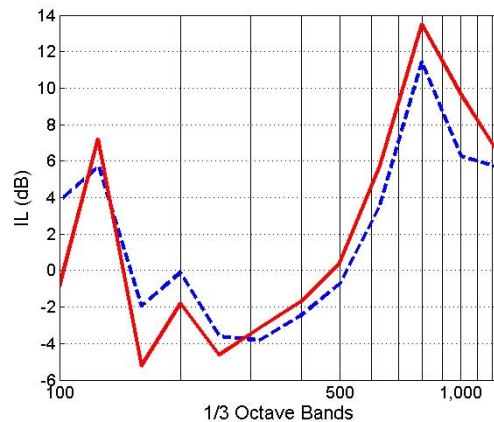


FIGURE 3: Insertion loss of a partially opened enclosure. The sound source was placed in the corner furthest from the aperture. “—” the model prediction; “--”: the experimental result.

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