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Stimulated Thomson scattering

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The theory of stimulated Thomson scattering is investigated both quantum mechanically and classically. In the interaction of a collisionless plasma with two electromagnetic waves, both with frequencies well above the plasma frequency, energy is transferred from the high-frequency wave to the low-frequency wave via stimulated Thomson scattering. This process is mediated by the nonlinear interaction of the plasma electrons with a beat wave at the difference frequency between the two waves; this beat frequency must be well above the plasma frequency. The gain coefficient for stimulated Thomson scattering is calculated both quantum mechanically and classically, and identical results are obtained. The classical calculation also yields the first nonlinear term in the index of refraction due to stimulated Thomson scattering, as well as the details of the saturation of the gain and the index of refraction. The authors present explicit formulas for the gain coefficient and the index of refraction, in the unsaturated limit, for both very cold and very hot plasmas. The calculations indicate that it should be possible to detect stimulated Thomson scattering experimentally by means of polarization enhancement. In an appendix, the theory of stimulated Thomson scattering is used to treat the free-electron amplifier.

I. INTRODUCTION

This paper discusses the theory of stimulated Thomson scattering. We show that when a collisionless plasma is illuminated by two oppositely directed electromagnetic waves of different frequencies, energy is transferred from the high-frequency pump wave to the low-frequency probe wave via stimulated Thomson scattering. The small signal gain of the plasma due to stimulated Thomson scattering is calculated both quantum mechanically and classically; the results are identical.

The classical calculation is very fruitful, yielding not only the gain but also the first nonlinear correction to the index of refraction. Furthermore, these quantities are calculated in both the saturated and unsaturated limits. Explicit expressions for the unsaturated gain are given in two limits; (i) the homogeneously broadened limit, where the electron gas is very cold, so that the spectral width of the radiation from spontaneous Thomson scattering is determined by interruption of the phase of the high frequency pump wave, and (ii) the inhomogeneously broadened limit, where the electron gas is very hot, so that the spectral width of the radiation from spontaneous Thomson scattering is determined by Doppler broadening. When saturation is important, the gain is calculated numerically; the results of these calculations are given for the same two limits. The calculation of the index of refraction is given in the same limits, both for the saturated and unsaturated cases. The gain and the nonlinear index of refraction due to stimulated Thomson scattering are shown to be large enough that they can be measured experimentally using polarization enhancement of the sensitivity as introduced by Wieman and Hänsch.[1] From such an experiment, both the electron density and the electron temperature of the plasma can be determined.

Finally, it is shown in Appendix A that if the unsaturated homogenously broadened gain formula is transformed to a frame in which the electrons are highly relativistic, the gain of the free-electron amplifier is obtained.

II. QUANTUM-MECHANICAL THEORY OF STIMULATED THOMSON SCATTERING

Here we present the quantum-mechanical theory of stimulated Thomson scattering following the approach of Madey.[2] Consider two oppositely directed electromagnetic waves, of frequencies ω_0 and ω, incident on a plasma of electron density n_e. The probe wave, at frequency ω, is to stimulate Thomson scattering of the pump wave, at frequency ω_0. The total rate of Thomson scattering of photons into a mode, including both spontaneous and stimulated photons, is $\gamma(n+1)$, where γ is the rate of spontaneous Thomson scattering into the mode, and n is the number of photons in the mode. The net gain coefficient for stimulated scattering into the mode is γ/c minus the total loss rate from the mode divided by c, where c is the speed of light. The primary loss mechanism is the inverse process, in which the wave at frequency ω_0 stimulates Thomson scattering of the wave at frequency ω. We ignore other loss mechanisms.

We take the pump wave, of intensity I_0, to be polarized such that the polarization of the Thomson backscattered radiation matches the polarization of the probe wave. The spontaneous rate per unit volume and per unit solid angle at which photons are Thomson backscattered from the pump wave is given by the equation

$$R=(n_e r_e^2 I_0)/(\hbar\omega_0)\,, \tag{1}$$

where r_e is the classical electron radius. In Eq. (1) we have made use of the fact that the differential Thomson scattering cross section in the backward direction is r_e^2. The polarized photons are scattered into $(\omega_0^2\Delta\omega_0)/(2\pi c)^3$ modes per unit volume and per unit solid angle, where $\Delta\omega_0$ is the bandwidth of the scattered radiation. Thus, for light of proper polarization,

$$\begin{aligned}\frac{\gamma}{c}&=\frac{(2\pi c)^3 R}{c\omega_0^2\Delta\omega_0}\\&=\frac{8\pi^3c^2n_e r_e^2 I_0 g[\omega-(\omega_0-2\hbar\omega_0^2/mc^2)]}{\hbar\omega_0^3}\,,\end{aligned}\tag{2}$$

where $1/\Delta\omega_0$ is replaced by g, the normalized line shape centered about the recoil shifted frequency. The net gain coefficient, including loss from the inverse process, is given by

$$\begin{aligned}\alpha&=\frac{8\pi^3c^2n_e r_e^2 I_0}{\hbar\omega_0^3}\\&\quad\times\left[g\left(\omega-\omega_0+\frac{2\hbar\omega_0^2}{mc^2}\right)-g\left(\omega-\omega_0-\frac{2\hbar\omega_0^2}{mc^2}\right)\right]\\&=\frac{32\pi^3 n_e r_e^2 I_0}{\omega_0 m}\left(\frac{d}{d\omega}\,g(\omega-\omega_0)\right).\end{aligned}\tag{3}$$

In the homogeneously broadened limit, where the electron gas is very cold and the spectral broadening of the Thomson scattering is the result of the phase of the pump wave being interrupted after a time τ, the normalized line shape is given by

$$g(\omega-\omega_0)=\tau\sin^2\{\tfrac{1}{2}[(\omega-\omega_0)\tau]\}/2\pi\{\tfrac{1}{2}[(\omega-\omega_0)\tau]\}^2\,,\tag{4}$$

and the net gain is

$$\alpha=\frac{8\pi^2\tau^2 n_e r_e^2 I_0}{m\omega_0}\,\frac{d}{d\eta}\left(\frac{\sin^2\eta}{\eta^2}\right)\bigg|_{\eta=(\omega-\omega_0)\tau/2}.\tag{5}$$

In an inhomogeneously broadened system, where the spectral broadening of the Thomson scattering is the result of Doppler shifts from the distribution of velocities in the electron gas, and where $f(v_z)$ is a normalized distribution function for electron velocities parallel to the light beam, the net gain is given by

$$\begin{aligned}\alpha&=\frac{32\pi^3 n_e r_e^2 I_0}{m\omega_0}\int_{-\infty}^{\infty}\frac{d}{d\omega}\,g\left(\omega-\omega_0+\frac{2\omega_0 v_z}{c}\right)f(v_z)\,dv_z\\&=-\frac{16\pi^3 n_e r_e^2 I_0 c}{m\omega_0^2}\int_{-\infty}^{\infty}g\left(\omega-\omega_0+\frac{2\omega_0 v_z}{c}\right)\frac{df(v_z)}{dv_z}\,dv_z.\end{aligned}\tag{6}$$

In the limit that inhomogeneous broadening dominates over homogeneous broadening, the net gain at, or near, the line center is given by

$$\alpha=-\frac{8\pi^3 n_e c^2 r_e^2 I_0}{m\omega_0^3}\,\frac{df(v_z)}{dv_z}\bigg|_{v_z=(\omega_0-\omega)c/2\omega_0}.\tag{7}$$

For a Maxwell-Boltzmann distribution function, the net gain is given by

$$\alpha=(8\pi^{5/2}n_e c^2 r_e^2 I_0/\omega_0^3 k_B T)\,y e^{-y^2}\tag{8}$$

where

$$y=(mc^2/8k_BT)^{1/2}[(\omega_0-\omega)/\omega_0]\,,\tag{9}$$

T is the temperature of the electron gas, and k_B is Boltzmann's constant. Dreicer has discussed stimulated Thomson scattering.[3] His results can be used to derive Eq. (8) except for a difference of a factor of 2π.

The gain mechanism of the free-electron amplifier is essentially stimulated Thomson scattering.[4] In the free-electron amplifier, a relativistic electron beam, is incident on a spatially periodic circularly polarized magnetic field. When transformed to a frame of reference moving with the electrons, the spatially periodic magnetic field appears as a plane electromagnetic wave; it plays the role of the pump wave. In Appendix A, we show that Eq. (5), when transformed to the highly relativistic electron frame, is the same as the gain formula for the free electron amplifier calculated by others.[5-7]

Since Planck's constant does not appear in the gain coefficient of the free-electron laser, a classical approach might be expected to give the same result. This was first suggested by Madey *et al.*[8] Since that suggestion, many papers have appeared exploring the classical interpretation of both the free-electron laser and stimulated Thomson scattering.[5-7,9-16] In Sec. III, we present a classical calculation of the gain and dispersion of a plasma in the presence of two counterpropagating electromagnetic waves. The gain formula obtained agrees, in the unsaturated limit, with that obtained in this section. In addition, the classical calculation points up the transient nature of stimulated Thomson scattering, and yields the saturation of the effect due to nonlinear Landau damping.

III. CLASSICAL THEORY OF STIMULATED THOMSON SCATTERING

A. Gain and dispersion in the plasma

Here we introduce sufficient formalism to determine the gain and dispersion of a plasma, due to stimulated Thomson scattering, in terms of the motion of the electrons in the plasma. As in Sec. II, we are interested in studying two oppositely directed electromagnetic waves, of angular frequencies ω and ω_0, propagating in a plasma. It is convenient to work in the Coulomb gauge, where $\vec{\nabla}\cdot\vec{A}=0$, so that the vector potential satisfies the inhomogeneous wave equation

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi}{c}\vec{J}+\frac{1}{c}\vec{\nabla}\frac{\partial\Phi}{\partial t}, \tag{10}$$

where $\vec{J}$ is the current density and where Φ is the scalar potential. The current density is given by

$$\vec{J}=-e\int d\vec{r}_0\int d\vec{v}_0\,\vec{v}'(\vec{r}_0,\vec{v}_0,t)\times\delta[\vec{r}-\vec{r}'(\vec{r}_0,\vec{v}_0,t)]F(\vec{r}_0,\vec{v}_0). \tag{11}$$

The primed coordinates $\vec{r}'$ and $\vec{v}'$ represent the instantaneous position and velocity at time t of an electron that at time $t=0$ was at position $\vec{r}_0$ with velocity $\vec{v}_0$. The electron distribution at time $t=0$ is $F(\vec{r}_0,\vec{v}_0)$, normalized so that its integral over all velocities and over the plasma volume gives the number of electrons in the plasma. In order to study the two oppositely directed waves, we seek a solution to Eq. (10) of the form

$$\vec{A}_T=\vec{A}_0+\vec{A}=[(E_0c/\omega_0)e^{-\alpha_0 z/2}\sin(k_0z-\omega_0t)+(Ec/\omega)e^{-\alpha z/2}\sin(kz+\omega t-\phi)]\hat{x}, \tag{12}$$

where the absorption coefficient for the pump wave α_0, and the gain coefficient for the probe wave α, are both expected to be extremely small compared to either k_0 or k. We take the plasma to be optically thin so that $\alpha_0 l$ and αl are both much less than one; l is the length of the plasma in the z direction. Note that $\alpha>0$ corresponds to gain for a wave propagating in the negative z direction.

To determine α, we substitute Eq. (12) into Eq. (10), multiply by $\cos(kz+\omega t-\phi)$, and integrate over the plasma volume V, and the coherence time τ, to obtain

$$\begin{aligned}\alpha&=\frac{8\pi\omega}{c^2kEV\tau}\int_0^\tau dt\int_V d\vec{r}\cos(kz+\omega t-\phi)\hat{x}\cdot\vec{J}\\&\simeq-\frac{8\pi e}{V\tau Ec}\int_V d\vec{r}_0\int d\vec{v}_0\int_0^\tau dt\,\cos(kz'+\omega t-\phi)\\&\quad\times v_x'F(\vec{r}_0,\vec{v}_0).\end{aligned} \tag{13}$$

In obtaining this expression we have assumed that $\tau\gg 2\pi/\omega$ or $2\pi/\omega_0$, $e^{\alpha z/2}\simeq 1$, and $ck/\omega\simeq 1$, and we have used Eq. (11) for $\vec{J}$. In addition we have ignored surface contributions to the integral, and we have taken the plasma to be homogeneous in the x direction, so that $\hat{x}\cdot\vec{\nabla}(\partial\phi/\partial t)=0$.

Equation (13) can be given a simple physical interpretation by writing it in the form

$$-\alpha\frac{cE^2}{8\pi}=\frac{1}{V}\int_V d\vec{r}\left(\frac{1}{\tau}\int_0^\tau dt\,\vec{E}\cdot\vec{J}\right), \tag{14}$$

where $\vec{E}=-Ee^{-\alpha z/2}\cos(kz+\omega t-\phi)\hat{x}$ and $e^{\alpha z/2}\approx 1$. The left-hand side of Eq. (14) is the product of $-\alpha$ and the intensity I of the probe wave. The right-hand side of Eq. (14) is the average of $\vec{E}\cdot\vec{J}$ for the probe wave. Hence, Eq. (14) can be written in the form

$$\frac{dI}{dz}=-\alpha I=\frac{1}{V}\int_V d\vec{r}\left(\frac{1}{\tau}\int_0^\tau dt\,\vec{E}\cdot\vec{J}\right). \tag{15}$$

In a similar manner we obtain an expression for the nonlinear index of refraction n

$$\begin{aligned}n^2=\left(\frac{kc}{\omega}\right)^2&\simeq 1+\frac{8\pi}{V\tau\omega E}\int_0^\tau dt\int_V d\vec{r}\sin(kz+\omega t-\phi)\hat{x}\cdot\vec{J}\\&\simeq 1-\frac{8\pi e}{V\tau\omega E}\int_V d\vec{r}_0\int d\vec{v}_0\int_0^\tau dt\sin(kz'+\omega t-\phi)\\&\quad\times v_x'F(\vec{r}_0,\vec{v}_0).\end{aligned} \tag{16}$$

Since α and n^2 depend on the value of z' and v_x' along the particle orbits, useful expressions for α and n^2 can be obtained only if the details of the electron motion are known. This is the subject of Sec. III B.

Henceforth, we drop the prime notation, but we remind the reader that these integrals are evaluated along particle orbits.

B. Particle orbits

Here we calculate the orbits of plasma electrons under the influence of the two electromagnetic waves. It is assumed that the gain and absorption coefficients α and α_0 are so small that the wave fields may be taken to be constant over the plasma volume. The scalar potential Φ is neglected in solving for the particle orbits; the criteria for the validity of this procedure are discussed in Sec. III F.

The Lagrangian for an electron with these approximations is

$$\begin{aligned}L&=\tfrac{1}{2}mv^2-(e/c)\vec{v}\cdot\vec{A}=\tfrac{1}{2}m(v_x^2+v_y^2+v_z^2)\\&\quad-(e/c)[(E_0c/\omega_0)\sin(k_0z-\omega_0t)\\&\quad+(Ec/\omega)\sin(kz+\omega t-\phi)]v_x.\end{aligned} \tag{17}$$

Since x and y are ignorable coordinates, there are two constants of the motion

$$p_x=\frac{\partial L}{\partial v_x}=mv_x-\frac{e}{c}\left(\frac{E_0c}{\omega_0}\sin(k_0z-\omega_0t)+\frac{Ec}{\omega}\sin(kz+\omega t-\phi)\right) \tag{18}$$

and

$$p_y = \frac{\partial L}{\partial v_y} = m v_y \,. \tag{19}$$

Lagrange's equation for z yields

$$m\dot{v}_z + \frac{e}{c}\left(\frac{E_0 c k_0}{\omega_0}\cos(k_0 z - \omega_0 t) + \frac{Eck}{\omega}\cos(kz + \omega t - \phi)\right) v_x = 0 \,. \tag{20}$$

Combining Eqs. (18) and (20), and setting $ck_0/\omega_0 \approx 1$ and $ck/\omega \approx 1$ yields

$$\begin{aligned} m\dot{v}_z = &-(ep_x/mc)[E_0\cos(k_0 z - \omega_0 t) + E\cos(kz + \omega t - \phi)] \\ &- (e^2/mc)[(E_0^2/2\omega_0)\sin(2k_0 z - 2\omega_0 t) + (E^2/2\omega)\sin(2kz + 2\omega t - 2\phi) \\ &+ (E_0 E/\omega)\cos(k_0 z - \omega_0 t)\sin(kz + \omega t - \phi) + (E_0 E/\omega_0)\sin(k_0 z - \omega_0 t)\cos(kz + \omega t - \phi)] \,. \end{aligned} \tag{21}$$

The first four terms on the right-hand side of Eq. (21) represent forces that oscillate at optical frequencies. Assuming electron temperatures of a few kilovolts, or less, and laser intensities of about 10^{13} W/cm^2, we find that these terms contribute z displacements far less than the wavelength of either electromagnetic wave. The first four terms are therefore neglected. The last two terms on the right-hand side of Eq. (21) are combined to yield a low-frequency equation of motion

$$\frac{d^2 z}{dt^2} \simeq -\frac{e^2 E_0 E}{m^2 c\bar{\omega}}\sin[(k_0 + k)z - \delta\omega t - \phi] \,, \tag{22}$$

where $\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$ and $\delta\omega = \omega_0 - \omega$. This acceleration is a ponderomotive force produced by the beating of the two electromagnetic waves. It is the classical recoil force that causes stimulated Thomson scattering. The oscillation of a particle produced by the electric field of one wave combines with the magnetic field of the other wave to produce an acceleration in the z direction. This low frequency acceleration can contribute displacements comparable to the wavelengths of the two electromagnetic waves for $\delta\omega \lesssim 10^{-3}\bar{\omega}$.

The physical significance of Eq. (22) can be seen by making a Galilean transformation to a frame of reference where the two waves are Doppler shifted so that they have the same frequency:

$$\bar{z} = z - [\delta\omega/(k_0 + k)]t - \phi/(k_0 + k) \,. \tag{23}$$

In the $\bar{z}$ coordinate system, Eq. (22) becomes

$$\frac{d^2\bar{z}}{dt^2} = -\frac{e^2 E_0 E}{m^2 c\bar{\omega}}\sin(k_0 + k)\bar{z} \,. \tag{24}$$

Equation (24) shows that in this frame the particles move in a time-independent potential. Particles with small $\bar{z}$ velocities are trapped in the minima of this potential and execute harmonic motion. Particles with $\bar{z}$ velocities on the order of $\bar{v}_z = [4e^2E_0E/m^2\bar{\omega}c(k_0 + k)]^{1/2}$ are "barely trapped" in the potential minima and tend to "stick" near the peaks of the potential. Particles with $\bar{z}$ velocities well above $\bar{v}_z$ are not trapped; the potential only causes a slight ripple in their $\bar{z}$ velocities.

It is convenient to make the change of variable

$$\xi = (k_0 + k)\bar{z} = (k_0 + k)z - \delta\omega t - \phi \,, \tag{25}$$

so that Eq. (24) becomes the simple pendulum equation

$$\frac{d^2\xi}{dt^2} = -\Omega^2 \sin\xi \,, \tag{26}$$

where

$$\Omega^2 = (k_0 + k)e^2E_0E/m^2c\bar{\omega} \approx 2E_0Ee^2/m^2c^2 \,.$$

The swinging, "sticking," and rotating solutions of the simple pendulum equation correspond, respectively, to trapped, "barely trapped," and untrapped particles in the reference frame described by Eq. (23). An analysis of the free-electron laser in terms of a pendulum equation has been given by Colson.[7] The solutions of the simple pendulum equation are given in terms of Jacobian elliptic functions. We use Milne-Thomson's notation for these functions.[17] To describe the solutions, we define two quantities ϵ and p by the equations

$$\epsilon^2 = \dot{\xi}^2 + 4\Omega^2\sin^2(\tfrac{1}{2}\xi) \tag{27}$$

and

$$p = (\epsilon/2\Omega)^2 \,. \tag{28}$$

Note that ϵ^2 is a constant of the motion, analogous to the energy of a pendulum. The two types of solutions of Eq. (26) are now described.

1. Swinging solutions

These solutions are characterized by the condition $0 \leq p \leq 1$, and satisfy the equation

$$\sin(\tfrac{1}{2}\xi) = (\epsilon/2\Omega)\,\mathrm{sn}[\Omega(t - t_0)\,|p] \,. \tag{29}$$

From Eq. (29) it follows that

$$\dot{\xi} = \epsilon\,\mathrm{cn}[\Omega(t - t_0)\,|p] \,. \tag{30}$$

The constants ϵ and t_0 are determined by the initial values of ξ and $\dot{\xi}$ as follows:

$$\sin(\tfrac{1}{2}\xi_0) = \sin\{\tfrac{1}{2}[\xi(t=0)]\} = (\epsilon/2\Omega)\,\mathrm{sn}(-\Omega t_0|p)\,, \tag{31}$$

$$\dot{\xi}_0 = \dot{\xi}(t=0) = \epsilon\,\mathrm{cn}(-\Omega t_0|p)\,. \tag{32}$$

2. Rotating solutions

These solutions are characterized by the condition $p \geq 1$ and satisfy

$$\sin(\tfrac{1}{2}\xi) = \mathrm{sn}\{\tfrac{1}{2}[\epsilon(t-t_0)]|p^{-1}\} \tag{33}$$

and

$$\dot{\xi} = \epsilon\,\mathrm{dn}\{\tfrac{1}{2}[\epsilon(t-t_0)]|p^{-1}\}\,. \tag{34}$$

The constants ϵ and t_0 are determined by

$$\sin(\tfrac{1}{2}\xi_0) = \mathrm{sn}[-\tfrac{1}{2}(\epsilon t_0)|p^{-1}] \tag{35}$$

$$\dot{\xi}_0 = \epsilon\,\mathrm{dn}[-\tfrac{1}{2}(\epsilon t_0)|p^{-1}]\,. \tag{36}$$

Note that since $\mathrm{dn}(x|p^{-1})$ is always positive, Eq. (36) requires that ϵ have the same sign as $\dot{\xi}_0$.

The integrals on the right-hand side of Eqs. (13) and (16) can now be evaluated by using v_x obtained from Eq. (18) and the simple pendulum solutions for the z motion.

C. Gain coefficient in the homogeneous limit

Here we evaluate the classical expression for the gain of a plasma due to stimulated Thomson scattering [Eq. (13)] and obtain a simple formula for it in the homogeneously broadened limit. The distribution function $F(\vec{r}_0, \vec{v}_0)$ is assumed to be spatially homogeneous and factorable in velocity space, making it useful to define a reduced distribution function by

$$f(v_{z0}) = \frac{1}{n_e}\int_{-\infty}^{\infty} dv_{x0}\int_{-\infty}^{\infty} dv_{y0}\,F(\vec{r}_0,\vec{v}_0)\,, \tag{37}$$

where $\int_{-\infty}^{\infty} dv_{z0} f(v_{z0}) = 1$ and where n_e is the electron density. The expression for v_x obtained from Eq. (18) is substituted into Eq. (13). In the resulting expression, all terms with frequencies on the order of $\bar{\omega}$ or $2\bar{\omega}$ nearly time average to zero; they are neglected. If a cylindrical plasma volume of area A in the x-y plane and length l in the z direction is used, Eq. (13) becomes

$$\alpha = -\frac{4\pi e^2 n_e E_0}{l\tau E m c\omega_0} \times\int_{-\infty}^{\infty} dv_{z0}\int_{-l/2}^{l/2} dz_0\int_0^{\tau} dt\,\sin[(k_0+k)z - \delta\omega t - \phi] \times f(v_{z0})\,. \tag{38}$$

We now change variables from z, z_0, and v_{z0} to $\xi = (k_0+k)z - \delta\omega t - \phi$, $\xi_0 = (k_0+k)z_0 - \phi$, and $\dot{\xi}_0 = (k_0+k)v_{z0} - \delta\omega$. We use Eq. (26) to perform the time integration, and we note that the integrand is periodic in ξ_0. Ignoring end effects, we obtain

$$\alpha = \frac{2e^2 n_e E_0}{\tau E m c\omega_0(k_0+k)\Omega^2}\int_{-\infty}^{\infty} d\dot{\xi}_0\int_{-\pi}^{\pi} d\xi_0[\dot{\xi}(\tau) - \dot{\xi}_0] \times f\left(\frac{\dot{\xi}_0 + \delta\omega}{k_0+k}\right)\,. \tag{39}$$

This is the general expression for the gain coefficient. A similar calculation yields α_0. The relationship between α and α_0 is discussed in Appendix B.

In the homogeneously broadened limit, Doppler broadening is unimportant; this limit is obtained by setting

$$f((\dot{\xi}_0 + \delta\omega)/(k_0+k)) = (k_0+k)\delta(\dot{\xi}_0 + \delta\omega)\,. \tag{40}$$

Using this distribution function, Eq. (39) becomes

$$\alpha = (8\pi^2 r_e^2 n_e \tau^2 I_0/m\omega_0)J\,, \tag{41}$$

where

$$I_0 = cE_0^2/8\pi \tag{42}$$

and

$$J = \frac{4}{\pi\Omega^4\tau^3}\int_{-\pi}^{\pi} d\xi_0[\dot{\xi}(\tau) + \delta\omega]\,. \tag{43}$$

It can be shown that J is a function only of $\delta\omega\tau$ and $\Omega\tau$. For general values of $\delta\omega\tau$ and $\Omega\tau$, J is difficult to evaluate analytically. In the limit that $\beta = (2\Omega/\delta\omega)^2 \ll 1$ and $\Omega\tau \lesssim 1$, however, it is possible to obtain an approximate expression for J by expanding the integrand of Eq. (43) in powers of β. Note that in this limit, only rotating solutions are used, so the Eq. (43) can be written

$$J = \frac{4}{\pi\Omega^4\tau^3}\int_{-\pi}^{\pi} d\xi_0\left\{\epsilon\,\mathrm{dn}\left[\frac{\epsilon(t-t_0)}{2}\,\middle|\left(\frac{2\Omega}{\epsilon}\right)^2\right] + \delta\omega\right\}. \tag{44}$$

The expansion of the integrand of Eq. (44) is tedious, but straightforward; with the help of Ref. 17 we obtain

$$J(\delta\omega\tau, \Omega\tau) = \frac{d}{d\eta}\left(\frac{\sin^2\eta}{\eta^2}\right)\bigg|_{\eta=-\delta\omega\tau/2}\,. \tag{45}$$

As stated, the expansion for J in Eq. (43) is valid when $\beta \ll 1$ and when $\Omega\tau \lesssim 1$. This implies that $\eta \gg \Omega\tau$. Equation (45) is also valid for values of $\eta \leq \Omega\tau$ with the added restriction that $\Omega\tau \ll 1$. This follows from an expansion of the integrand of Eq. (43) in powers of $\Omega\tau$ with $\delta\omega \lesssim 2\Omega$. Hence, in the unsaturated limit ($\Omega\tau \ll 1$) the homogeneously broadened gain coefficient is given by

$$\alpha = \frac{8\pi^2 r_e^2 n_e \tau^2 I_0}{m\omega_0}\frac{d}{d\eta}\left(\frac{\sin^2\eta}{\eta^2}\right)\bigg|_{\eta=-\delta\omega\tau/2} \tag{46}$$

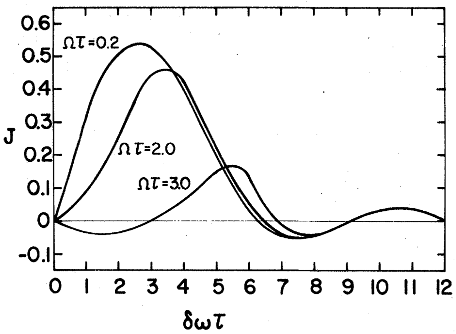

FIG. 1. J as a function of $\delta\omega\tau$ for selected values of $\Omega\tau$.

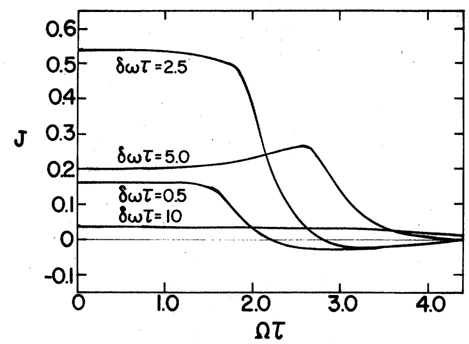

FIG. 2. J as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$.

in exact agreement with the quantum-mechanical expression, Eq. (5).

When $\Omega\tau$ is no longer small, the gain saturates. The gain saturates because of the phase mixing of the trapped and nearly trapped electrons as in nonlinear Landau damping of plasma waves. To calculate the saturation behavior, the integral in Eq. (43) is performed numerically, using Eqs. (27) through (36). The homogeneously broadened gain saturates more easily near $\eta = 0$ than in the wings of the line shape. Figure 1 shows $J(\delta\omega\tau, \Omega\tau)$ as a function of $\delta\omega\tau$ for selected values of $\Omega\tau$. Figure 2 shows $J(\delta\omega\tau, \Omega\tau)$ as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$.

D. Index of refraction in the homogeneously broadened limit

Here we calculate the index of refraction of the plasma, using the same assumptions about the distribution function and the plasma volume as in Sec. IIIC. The expression for v_x obtained from Eq. (18) is substituted into Eq. (16). In the resulting expression, the high-frequency terms are time averaged, and the variables z, z_0, and v_{x0} are changed to ξ, ξ_0, and $\dot{\xi}_0$ to obtain

$$n^2 = \left(\frac{kc}{\omega}\right)^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 + \frac{2e^2 n_e E_0}{\tau m \omega \omega_0 E(k_0 + k)} \times \int_{-\infty}^{\infty} d\dot{\xi}_0 \int_{-\pi}^{\pi} d\xi_0 \int_0^{\tau} dt \cos\xi f\left(\frac{\dot{\xi}_0 + \delta\omega}{k_0 + k}\right), \quad (47)$$

the general expression for the index of refraction.

In the homogeneously broadened limit, Eq. (40) is used to obtain

$$\tilde{n}^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 + \frac{8\pi^2 n_e r_e^2 c \tau^2 I_0}{\omega_0 \omega m} K, \quad (48)$$

where

$$K = \frac{4}{\pi \Omega^2 \tau^3} \int_0^{\tau} dt \int_{-\pi}^{\pi} d\xi_0 \cos\xi. \quad (49)$$

Like the function J defined in Sec. IIIC, K can be shown to be only a function of $\delta\omega\tau$ and $\Omega\tau$. In the limit that $\beta < 1$, we obtain

$$K = \frac{4}{\pi \Omega^2 \tau^3} \int_0^{\tau} dt \int_{-\pi}^{\pi} d\xi_0 \times \left\{ 1 - 2\,\mathrm{sn}^2\left[\frac{\epsilon(t - t_0)}{2} \middle| \left(\frac{2\Omega}{\epsilon}\right)^2\right]\right\}. \quad (50)$$

In the limit $\beta \ll 1$ and when $\Omega\tau \lesssim 1$, a first-order expansion in β yields

$$K(\delta\omega\tau, \Omega\tau) = \frac{d}{d\eta}\left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2}\right) \Bigg|_{\eta = -\delta\omega\tau/2} \quad (51)$$

for $\eta \gg \Omega\tau$. This expression is also valid for $\eta \lesssim \Omega\tau$ if $\Omega\tau \ll 1$. Hence, in the unsaturated limit ($\Omega\tau \ll 1$) we obtain

$$n^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 + \frac{8\pi^2 n_e r_e^2 c \tau^2 I_0}{m \omega \omega_0} \times \frac{d}{d\eta}\left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2}\right) \Bigg|_{\eta = -\delta\omega\tau/2}, \quad (52)$$

where the third term on the right-hand side may also be obtained by applying the Kramer-Kronig relations, with respect to $\delta\omega$, to the gain formula (46).

If $\Omega\tau$ is no longer small, the index of refraction

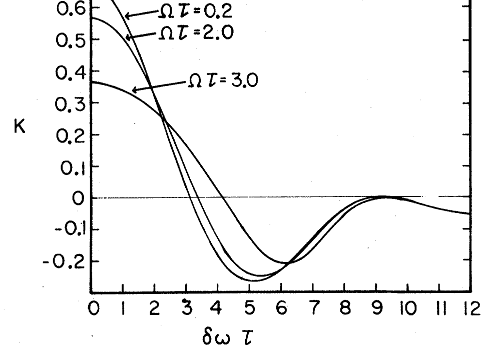

FIG. 3. K as a function of $\delta\omega\tau$ for selected values of $\Omega\tau$.

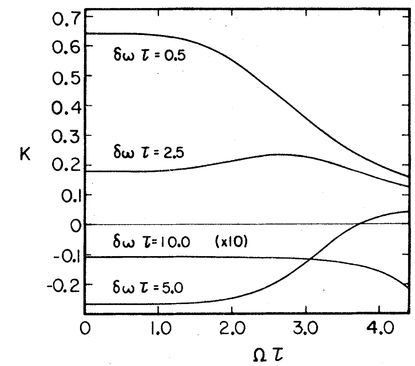

FIG. 4. K as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$.

must be calculated numerically. Figure 3 shows $K(\delta\omega\tau, \Omega\tau)$ as a function of $\delta\omega\tau$ for selected values of $\Omega\tau$. Figure 4 shows $K(\delta\omega\tau, \Omega\tau)$ as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$. Note that like the homogeneous gain, the homogeneous dispersion saturates more easily near $\eta = 0$ than in the wings of the line shape.

E. Gain and the index of refraction in the inhomogeneously broadened limit

When the electrons in a plasma have a distribution of velocities, the gain can be obtained as a simple convolution of the homogeneous gain formula with the electron distribution function. Equations (39)–(41), and the expression for the distribution function

$$f\left(\frac{\dot{\xi}_0 + \delta\omega}{k_0 + k}\right) = \int_{-\infty}^{\infty} d\sigma\, \delta(\sigma - \dot{\xi}_0 - \delta\omega) f\left(\frac{\sigma}{k_0 + k}\right) \quad (53)$$

can be combined to yield the equation

$$\alpha = \frac{8\pi^2 r_e^2 n_e \tau^2 I_0}{m\omega_0(k_0 + k)} \times \int_{-\infty}^{\infty} d\sigma f\left(\frac{\sigma}{k_0 + k}\right) J((\delta\omega - \sigma)\tau, \Omega\tau)\,. \quad (54)$$

In the unsaturated limit, Eq. (45) gives the expression for $J(\delta\omega\tau, \Omega\tau)$; substituting it into Eq. (67) and integrating by parts yields

$$\alpha = -\frac{32\pi^2 r_e^2 n_e I_0}{m\omega_0(k_0 + k)^2} \times \int_{-\infty}^{\infty} d\lambda f'\left(\frac{\delta\omega}{k_0 + k} + \frac{2\lambda}{(k_0 + k)\tau}\right)\frac{\sin^2\lambda}{\lambda^2} \quad (55)$$

where $\lambda = \frac{1}{2}(\sigma - \delta\omega)\tau$. In the inhomogeneous limit, the width, in λ, of the distribution function is much larger than the width of the homogeneous gain curve. This limit is described by the condition

$$\bar{v}_z/c \gg (\bar{\omega}\tau)^{-1} \quad (56)$$

where $\bar{v}_z$ is the width, in velocity, of the distribution function $f(v_z)$. In this limit, the slope of the electron distribution function is nearly constant over the width of the homogeneous gain curve. Setting

$$f'\left(\frac{\delta\omega}{(k_0 + k)} + \frac{\lambda}{(k_0 + k)\tau}\right) \simeq f'\left(\frac{\delta\omega}{(k_0 + k)}\right)$$

in the integrand of Eq. (55) and using the result that $\int_{-\infty}^{\infty} \sin^2\lambda/\lambda^2 = \pi$ yields for the inhomogeneously broadened gain in the unsaturated limit

$$\alpha = -\frac{32\pi^3 r_e^2 n_e I_0}{m\omega_0(k_0 + k)^2}\frac{df}{dv_z}\bigg|_{v_z = (\omega_0 - \omega)c/2\omega_0} \simeq -\frac{8\pi^3 r_e^2 c^2 n_e I_0}{m\omega_0^3}\frac{df}{dv_z}\bigg|_{v_z = (\omega_0 - \omega)c/2\omega_0}, \quad (57)$$

in agreement with the quantum-mechanical result, Eq. (7).

If the gain is allowed to saturate, then the homogeneous gain curve is no longer given by Eq. (45). The saturated homogeneous gain curve is flattened near the center, but is still an odd function of $\delta\omega\tau$ as shown in Fig. 2. We define a function $M(\lambda, \Omega\tau)$ by

$$M(\lambda, \Omega\tau) = \int_{-\infty}^{\lambda} J(-2\lambda', \Omega\tau)\, d\lambda'\,, \quad (58)$$

and integrate Eq. (54) by parts to obtain an equation for the saturated inhomogeneous gain coefficient

$$\alpha = -\frac{32\pi^2 r_e^2 n_e I_0}{m\omega_0(k_0 + k)^2} \times \int_{-\infty}^{\infty} d\lambda f'\left(\frac{\delta\omega}{k_0 + k} + \frac{2\lambda}{(k_0 + k)\tau}\right) M(\lambda, \Omega\tau)\,. \quad (59)$$

If the distribution function is much wider than M, Eq. (59) may be written

$$\alpha = -\frac{32\pi^2 r_e^2 n_e I_0}{m\omega_0(k_0 + k)^2} \times \left[\left(\frac{df}{dv_z}\right)\bigg|_{v_z = (\omega_0 - \omega)c/2\omega_0}\right]\Delta(\Omega\tau)\,, \quad (60)$$

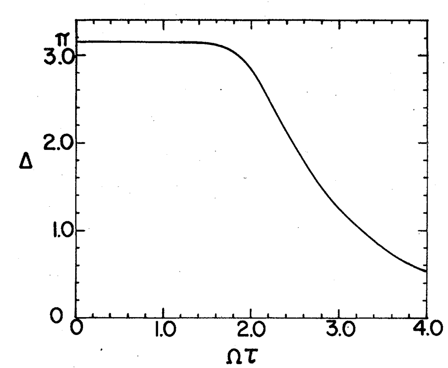

FIG. 5. Δ as a function of $\Omega\tau$.

where

$$\Delta(\Omega\tau)=\int_{-\infty}^{\infty} M(\lambda,\Omega\tau)\,d\lambda=\int_{-\infty}^{\infty}\lambda J(-2\lambda,\Omega\tau)\,d\lambda\,. \tag{61}$$

The quantity $\Delta(\Omega\tau)$ is shown in Fig. 5.

The index of refraction can also be expressed, in the inhomogeneously broadened limit, as a convolution of the homogeneous expression with the distribution function. Using Eqs. (47) and (48) we obtain

$$n^2=1-\left(\frac{\omega_p}{\omega}\right)^2+\frac{8\pi^2 r_e^2 c n_e \tau^2 I_0}{\omega\omega_0 m(k_0+k)}\times\int_{-\infty}^{\infty} f\left(\frac{\sigma}{k_0+k}\right)K((\delta\omega-\sigma)\tau,\Omega\tau)d\sigma\,. \tag{62}$$

In the unsaturated limit, Eq. (51) is used to obtain

$$n^2=1-\left(\frac{\omega_p}{\omega}\right)^2-\frac{32\pi^2 r_e^2 c n_e \tau I_0}{\omega\omega_0 m(k_0+k)}\frac{d}{d(\delta\omega)}\times\int_{-\infty}^{\infty} d\sigma f\left(\frac{\sigma}{k_0+k}\right)\left(\frac{1}{(\sigma-\delta\omega)\tau}-\frac{\sin(\sigma-\delta\omega)\tau}{(\sigma-\delta\omega)^2\tau^2}\right). \tag{63}$$

In the inhomogeneous limit, the distribution function is nearly constant, in σ, over the center of the integral of the homogeneous dispersion curve. Since this curve is odd, the center of the integral of the homogeneous dispersion curve makes only a small contribution to the index of refraction. There is, however, a significant contribution from the wings of the integral of the homogeneous dispersion curve; in fact, since the integral of $1/(\sigma-\delta\omega)\tau$ diverges, it samples the entire width of the distribution function. It is convenient to replace the fairly complicated integral of the homogeneous dispersion curve by a simpler function that also contributes nothing for σ near $\delta\omega$, and that has the same behavior for $|\sigma|\gg\delta\omega$. Such a function is $\mathrm{P}[1/(\sigma-\delta\omega)\tau]$, where P stands for principal value. In this approximation, Eq. (63) becomes

$$n^2=1-\left(\frac{\omega_p}{\omega}\right)^2-\frac{32\pi^2 r_e^2 c n_e I_0}{\omega\omega_0 m(k_0+k)}\times\left(\frac{d}{d(\delta\omega)}\mathrm{P}\int_{-\infty}^{\infty} d\sigma\frac{f(\sigma/(k_0+k))}{\sigma-\delta\omega}\right). \tag{64}$$

If the distribution function is Maxwellian, we have

$$f(v_z)=\left(\frac{m}{2\pi k_B T}\right)^{1/2} e^{-mv_z^2/2k_B T}\,, \tag{65}$$

and Eq. (64) becomes

$$n^2=1-\left(\frac{\omega_p}{\omega}\right)^2-\frac{8\pi^2 r_e^2 c^3 n_e I_0}{k_B T\omega_0^4}Z'(y)\,, \tag{66}$$

where y is given by Eq. (9) and where

$$Z'(y)=\frac{d}{dy}\left[\mathrm{Re}\left(\frac{1}{\sqrt{\pi}}\mathrm{P}\int_{-\infty}^{\infty} dx\,\frac{e^{-x^2}}{x-y}\right)\right] \tag{67}$$

is the derivative of the real part of the plasma dispersion function for a real argument.[18]

If the index of refraction is allowed to saturate, then $K(\delta\omega\tau,\Omega c)$ changes near $\delta\omega=0$, but remains nearly the same as in the unsaturated limit for $\delta\omega\gg 1/\tau$; since nearly all of the contribution to the inhomogeneous dispersion comes from the wings of the integral of the homogeneous dispersion curve, the inhomogeneous index of refraction essentially does not saturate.

It is interesting to note that the way the folding together of the distribution function with the homogeneously broadened line shape, $(d/d\eta)(\sin^2\eta/\eta^2)$, gives rise to the first derivative of the distribution function in the inhomogeneously broadened limit [Eq. (57)] is completely analogous to the way the first derivative of the distribution function appears in the Landau-damping formula. In the physical derivation of Landau damping by Dawson,[19] the same function $(d/dx)(\sin^2 x/x^2)$ appears in exactly the same way. Indeed, the unsaturated gain due to stimulated Thomson scattering arises from a process analogous to linear Landau damping, while the saturation mechanism discussed in Sec. III C–III E is analogous to nonlinear Landau damping.

F. Effect of finite electron density on stimulated Thomson scattering

The discussion of stimulated Thomson scattering in Secs. III A–III E applies when the electron density is very low. In those calculations the gain and index of refraction are obtained by adding up the contributions to the current from each electron in the plasma, assuming that the electrons are independent, i.e., the motion of the electrons is calculated ignoring the electrostatic fields produced by the changing positions of the electrons. In this section we discuss the gain due to stimulated Thomson scattering from a cold plasma (the homogeneously broadened limit) with a finite plasma density.

The gain is obtained using a perturbation expansion of the fluid equations for the plasma electrons. The electrons are neutralized by a background of ions whose motion is neglected. We assume that at time $t=0$, two counterpropagating electromagnetic waves are suddenly present in a cold, stationary, neutralized electron fluid. A perturbation expansion of the fluid quantities is carried out as follows:

$$n = n_e + \delta n(z, t)\,, \tag{68}$$

$$\vec{v} = \vec{v}_1 + \delta v(z, t)\hat{z}\,, \tag{69}$$

where $\delta v \ll v_1$,

$$\vec{E} = [E_0 e^{-\alpha_0 z/2}\cos(k_0 z - \omega_0 t) - E e^{-\alpha z/2}\cos(kz + \omega t - \phi)]\hat{x} + \delta E(z, t)\hat{z}\,, \tag{70}$$

where $\delta E \ll E, E_0$ and

$$\vec{B} = [E_0 e^{-\alpha_0 z/2}\cos(k_0 z - \omega_0 t) + E e^{-\alpha z/2}\cos(kz + \omega t - \phi)]\hat{y}\,. \tag{71}$$

The velocity $\vec{v}_1$ represents the response of the fluid to the electric fields of the two counterpropagating waves, and δn, δv, and δE describe the effects of the nonlinear ponderomotive force discussed in Sec. III B [see Eq. (22)]. The fluid equations are linearized to obtain

$$\frac{\partial \delta v}{\partial t}\hat{z} \cong -\frac{e}{mc}\vec{v}_1 \times \vec{B} - \frac{e}{m}\delta E\hat{z}\,, \tag{72}$$

$$\vec{v}_1 = (e/mc)\vec{A}\,, \tag{73}$$

$$\frac{\partial \delta E}{\partial z} = -4\pi e\delta n\,, \tag{74}$$

$$\frac{\partial \delta n}{\partial t} + n_e\frac{\partial \delta v}{\partial z} = 0\,, \tag{75}$$

subject to the initial conditions

$$\delta n|_{t=0} = 0, \quad \delta v|_{t=0} = 0, \quad \delta E|_{t=0} = 0\,. \tag{76}$$

Note that we have assumed that the plasma oscillations contained in these equations are undamped. A discussion of the effects of damping is found in Ref. 20. From these equations, the perturbed transverse current $\delta\vec{J} = -e\delta n\vec{v}_1$ is obtained and substituted into Maxwell's equations yielding

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = -\frac{4\pi e}{c}(n_e + \delta n)\vec{v}_1\,. \tag{77}$$

As in Sec. IIIC, the gain coefficient is obtained by averaging Eq. (77) over space and fast time scales. The result of this calculation is

$$\alpha = \frac{8\pi^2 r_e^2 n_e \tau^2 I_0}{m\omega_0} P(\eta, \omega_p\tau)\Big|_{\eta = -\delta\omega\tau/2}\,, \tag{78}$$

where

$$P(\eta, \omega_p\tau) = \frac{1}{\omega_p\tau}\left(\frac{\sin^2(\eta + \frac{1}{2}\omega_p\tau)}{(\eta + \frac{1}{2}\omega_p\tau)^2} - \frac{\sin^2(\eta - \frac{1}{2}\omega_p\tau)}{(\eta - \frac{1}{2}\omega_p\tau)^2}\right). \tag{79}$$

As $\omega_p\tau$ approaches zero, P approaches $(d/d\eta)$ $\times(\sin^2\eta/\eta^2)$ in agreement with the results obtained in Secs. II and IIIC.

The fluid calculation describes the interaction of three waves, the pump wave at frequency ω_0, the probe wave at frequency ω, and a plasma wave at frequency ω_p. When $\delta\omega \gg \omega_p$ the interaction is far off resonance; in this limit the three-wave process gives the same gain as stimulated Thomson scattering. When $\delta\omega = \omega_p$ the interaction is on resonance, and the plasma wave grows as a result of a three-wave parametric instability driven by the ponderomotive force of Eq. (22). When $\delta\omega = \omega_p$ stimulated Raman scattering occurs; the gain coefficient obtained from Eq. (78) is

$$\alpha = \frac{8\pi^2 r_e^2 n_e \tau I_0}{m\omega_0\omega_p}\left(1 - \frac{\sin^2\omega_p\tau}{(\omega_p\tau)^2}\right). \tag{80}$$

Three-wave parametric instabilities have been extensively studied,[21,22] especially by Cohen *et al.*,[23,24] and are reviewed in Ref. 25. Our results look different from those usually obtained because we have solved the initial value problem rather than Fourier transforming in the time at the outset.

The calculation presented is valid as long as the electrons can be described by the electron fluid equations; when crossing of charge sheets occurs, the fluid equations are no longer adequate. It is then necessary to follow each electron, as was done in Sec. III. Charge sheet crossing becomes important when $\Omega\tau$ is no longer small. It is, in fact, this phase mixing of the electrons that produces the saturation of the gain as described in Sec. IIIC. Thus, the perturbation expansion of the fluid equations of motion cannot describe saturation; but in the limit that $\Omega\tau \ll 1$, the homogeneously broadened gain formulas of Sec. IIIC agree with the gain calculated from the fluid equations for the transient coupling of the two counterpropagating electromagnetic waves and a plasma wave.

One last comment is in order. For a hot inhomogeneously broadened plasma in which $\omega v_{\text{thermal}}/c \gg \omega_p$, the independent electron model of Sec. III is correct, i.e., electrostatic effects may be neglected.

G. Proposed experimental detection of stimulated Thomson scattering

Here we discuss a proposed experiment to detect stimulated Thomson scattering. Figure 6 is a schematic diagram of the apparatus of the proposed experiment. In order to evaluate the possibility of detecting stimulated Thomson scattering, we must assume reasonable values for certain experimental parameters. We assume that the plasma is produced by a pinch discharge, and that it has an electron density of $n_e = 10^{15}$ cm^{-3} and an electron temperature of $k_B T_e = 10$ eV. These parameters are reasonable for a pinch discharge.[26] We assume that the pump laser has a pulse duration of 3 nsec and an energy per pulse of 1.0 J at a

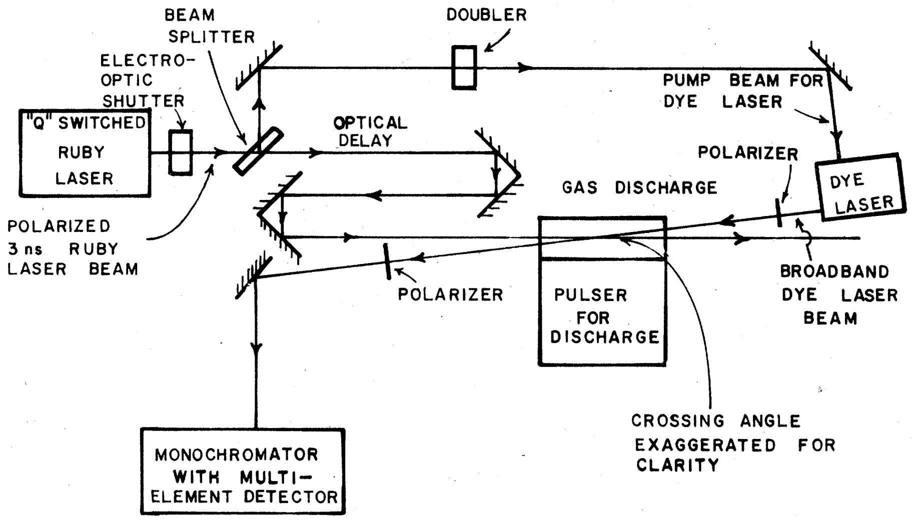

FIG. 6. Schematic diagram of the proposed experiment to detect stimulated Thomson scattering.

wavelength of 694.3 nm. We also assume the pump laser has a waist diameter at focus of 0.01 cm and hence a Rayleigh range of $Z_R = \pi w_0^2/\lambda = 4.5$ cm. The pump intensity at the focus is $I_0 = 2.1 \times 10^{12}$ W/cm^2. If the plasma is completely ionized (i.e., the neutral gas pressure is negligible) then the pump laser will heat the plasma primarily by inverse bremsstrahlung. The percentage increase in the electron temperature due to heating by the focused laser beam is on the order of 5%, a tolerable perturbation to the plasma.[27] A broad-band dye laser centered at wavelength 694.3 nm is used as the probe laser. The dye laser has an energy per pulse of 10^{-2} J and a pulse duration of 3 nsec. The dye laser has the same waist and Rayleigh range as the pump laser. Plasmas and lasers with these properties are available.

The peak gain coefficient is $\alpha = 2.8 \times 10^{-4}$ cm^{-1} at a wavelength of 700.4 nm with the assumed experimental parameters. We note that $v_{\text{thermal}}\, \omega/c$ is more than five times as large as the plasma frequency. The single-pass gain product for the stimulated Thomson scattering is $\alpha l \cong \alpha Z_R = 1.3 \times 10^{-3}$. Although the gain product is small, it is possible to detect by the use of polarization enhancement as introduced by Wieman and Hänsch.[1] Polarization enhancement may provide a three-orders-of-magnitude increase in signal to background. In this situation the light transmitted by the blocking polarizer is increased by 130% above the light transmitted without stimulated Thomson scattering. We believe that with existing plasmas and lasers it will be possible to detect stimulated Thomson scattering.

The probe beam can be analyzed with a low-resolution grating monochrometer and detected with a multielement detector array. The measurement of the gain for $\lambda > 694.3$ nm or absorption for $\lambda < 694.3$ nm as a function of λ will yield both n_e and T_e for the plasma. If a 30-element array with 33% quantum efficiency were used, then each of the 30 elements would detect about 3.5×10^7 dye-laser photons. This number assumes the dye laser has an energy of 10^{-2} J per pulse and assumes an extinction ratio of 10^{-7} for the crossed polarizers. If the signal-to-noise ratio for the detector array were limited by photon statistics, then an intensity modulation of 1.7×10^{-4} would just be detectable. This corresponds to the gain from a plasma with $n_e \simeq 1.3 \times 10^{11}$ cm^{-3} and $T_e = 10$ eV. Stimulated Thomson scattering may be useful as a diagnostic for laboratory plasmas and arcs. One additional advantage of using stimulated Thomson scattering as a diagnostic is that spontaneous emission from the plasma would not be a major source of noise. A small gain or absorption is detected with a near diffraction limited probe laser beam. The probe laser beam is spatially filtered after traversing the plasma. Since all the stimulated photons are emitted into the same small solid angle as the probe beam, only the spontaneous emission into this small solid angle acts as a background noise source.

ACKNOWLEDGMENTS

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APPENDIX A: FREE-ELECTRON AMPLIFIER

The free-electron amplifier is an interesting example of the stimulated Thomson scattering process. Madey[2] presented a quantum-mechanical theory of this effect, and Elias, Fairbank, Madey, Schwettman, and Smith[4] performed the first experiment demonstrating the effect. Deacon, Elias, Madey, Ramian, Schwettman, and Smith[28] subsequently built a free-electron laser. Hopf, Meystre, Scully, and Louisell[5] showed that the free-electron amplifier could be explained using a classical theory. Their classical theory of the free-electron amplifier agrees with the quantum-mechanical theory. Ichimaru and Iwamoto,[6] Colson,[7] Dawson, Kwan, and Lin,[10,11] Sprangle,[14] Cocke,[15] and Kroll and McMullin[16] have also presented classical calculations of the performance of free-electron amplifiers and lasers. Hopf *et al.*[29] have discussed some aspects of the saturation of the free-electron amplifier.

Here the free-electron amplifier experiment is analyzed briefly in terms of the theory of stimulated Thomson scattering. In the free-electron amplifier, highly relativistic electrons are shot down the axis of a helical magnetic field in the presence of an electromagnetic wave propagating in the direction of the electron velocity. The electromagnetic wave was observed to increase, or decrease, in intensity as the electron energy was varied. It is useful to view this experiment from the rest frame of the electrons. In this frame, the transverse part of the helical field is approximately a left-circularly-polarized electromagnetic pump wave; the electromagnetic probe wave is Doppler shifted to a lower frequency. Thus, except for the difference in polarization, the free-electron amplifier corresponds to the physical situation discussed in this paper. Assuming that the electron beam is monoenergetic, we expect Eq. (46), properly Lorentz transformed and modified for circularly polarized light, to give the small signal gain of the free-electron amplifier.

It is a simple matter to carry out the calculations in Sec. III for circularly polarized light to obtain the following results. If the two oppositely directed waves have the same circular polarization, the gain coefficient is zero. If they are of opposite circular polarization, the gain coefficient is twice as large as for plane polarized light. In the free electron amplifier, these effects are observed.[4]

The quantities in Eq. (46) are related to quantities in the free-electron amplifier by the Lorentz transformation. With the assumption that $v \approx c$, these quantities are the following:

(i) If L is the length of the helical field and γ is the Lorentz factor for the electrons, then the coherence time τ is given by

$$\tau = L/\gamma v \cong L/\gamma c . \tag{A1}$$

(ii) If ω^* is the probe-wave frequency in the laboratory frame, then the probe-wave frequency in the electron frame is given by

$$\omega = \left(\frac{1-v/c}{1+v/c}\right)^{1/2} \omega^* \simeq \frac{\omega^*}{2\gamma} . \tag{A2}$$

(iii) If λ_q is the period length of the helical field, then the pump-wave frequency in the electron frame is given by

$$\omega_0 = 2\pi\gamma v/\lambda_q \cong 2\pi\gamma c/\lambda_q . \tag{A3}$$

(iv) If B is the magnitude of the transverse part of the helical magnetic field in the laboratory, then the intensity of each linearly polarized component of the circularly polarized probe wave is given by

$$I_0 = c\gamma^2 B^2/8\pi . \tag{A4}$$

(v) If n^* is the electron density in the laboratory, then the electron density in the electron frame is given by

$$n = n^*/\gamma . \tag{A5}$$

(vi) If $q = 2\pi/\lambda_q$, then the homogeneous line-shape variable η is given by

$$\eta = -\tfrac{1}{2}\delta\omega\tau = \tfrac{1}{2}L(\omega^*/2\gamma^2 c - q) . \tag{A6}$$

The gain per pass of the amplifier can be calculated by finding the spatial separation L' in the electron frame between the following two events: (a) the front of the probe wave enters the helical field, and (b) the front of the probe wave leaves the helical field. Application of the Lorentz transformation with $v \simeq c$ shows that $L' \cong L/2\gamma$, so that the gain per pass in the electron frame is given by

$$G = L\alpha/2\gamma . \tag{A7}$$

Since this is just the logarithm of the ratio of the intensity at event (a) to that at event (b), Eq. (A7) also gives the gain per pass in the laboratory. Hence, the small signal gain coefficient in the laboratory, α^*, is given by

$$\begin{aligned}\alpha^* &= \frac{G}{L} = \frac{r_e^2 n^* B^2 \lambda_q L^2 F}{2mc^2\gamma^3}\frac{d}{d\eta}\left(\frac{\sin^2\eta}{\eta^2}\right)\\ &= \tfrac{3}{2}\,(n^*\sigma_T)\frac{B^2}{8\pi mc^2}\frac{\lambda_q L^2}{\gamma^3}\\ &\quad\times F\frac{d}{d\eta}\left(\frac{\sin^2\eta}{\eta^2}\right) ,\end{aligned} \tag{A8}$$

where F is a filling factor that gives the fraction of the electromagnetic-wave cross section that interacts with the electron beam (assuming that the wave cross section is larger than the electron-beam cross section), and where η is given by Eq. (A6). Equation (A8) agrees with the results obtained by others.[5-7]

In evaluating Eq. (A9) for the free-electron amplifier, care must be taken in evaluating γ; it is reduced from its value when the electrons are outside of the helical field since the electrons spiral down the helix. The factor γ is given by

$$\gamma = \frac{E}{mc^2[1+(\lambda_q^2 r_e B^2)/4\pi^2 mc^2]^{1/2}}, \tag{A9}$$

where E is the electron energy.

When the parameters of the free-electron-amplifier experiment are used to calculate the peak gain per pass G from Eq. (A8), it is found to be 5.4%. The experiment measured a peak gain per pass of 7%.

APPENDIX B: RELATIONSHIP BETWEEN α_0 AND α

In Secs. III A and III C, the gain coefficient of the probe wave α is calculated. The same procedure may be used to calculate the absorption coefficient of the pump wave α_0; the result obtained is that α_0 and α are connected by the relation

$$\alpha I \cong \alpha_0 I_0 , \tag{B1}$$

where I is the probe-wave intensity, and I_0 is the pump-wave intensity. Since $I \propto e^{-\alpha z}$, and $I_0 \propto e^{-\alpha_0 z}$, Eq. (B1) can be written in the form

$$\frac{dI}{dz} \cong \frac{dI_0}{dz}, \tag{B2}$$

i.e., at all values of z, $I - I_0$ is a constant. Hence, to the degree of approximation employed in this paper, the electrons act as intermediates, transferring energy from one wave to the other.

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