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Recording Lissajous Figures

WAYNE B. HALES*
Brigham Young University, Provo, Utah

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THE following article is not intended to contribute anything new on the theory of vibrating bodies or of Lissajous figures but it is a description of a refined method of using a double pendulum for the production of these figures and the making of them into a permanent record. The beauty and design of the many patterns produced should be of interest to the scientist and teacher, as well as to the artist, and all interested in unique design.

Lissajous curves or figures could be just as properly called Bowditch curves or figures since a Yankee, Nathaniel Bowditch of navigation fame, as early as 1815 produced similar curves by a pendulum suspended from two points. The *American Academy of Arts Science Memoirs*[1] carries an article by him and one by James Dean, Professor of Mathematics and Natural Philosophy, University of Vermont, in which were announced the invention and description of the double pendulum. It consisted of a Y-shaped affair having different oscillating lengths at right angles to each other and, therefore, different periods. It is not known just how Bowditch recorded the path of his pendulum, if at all, but

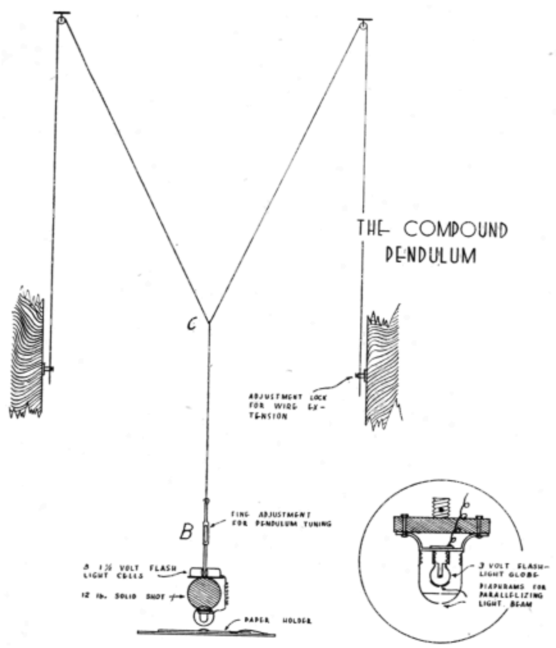

FIG. 1. Diagram of double pendulum and light housing.

* Now at Rutgers University, New Brunswick, New Jersey.
[1] *American Academy of Arts Science Memoirs* (1815), Vol. 3.

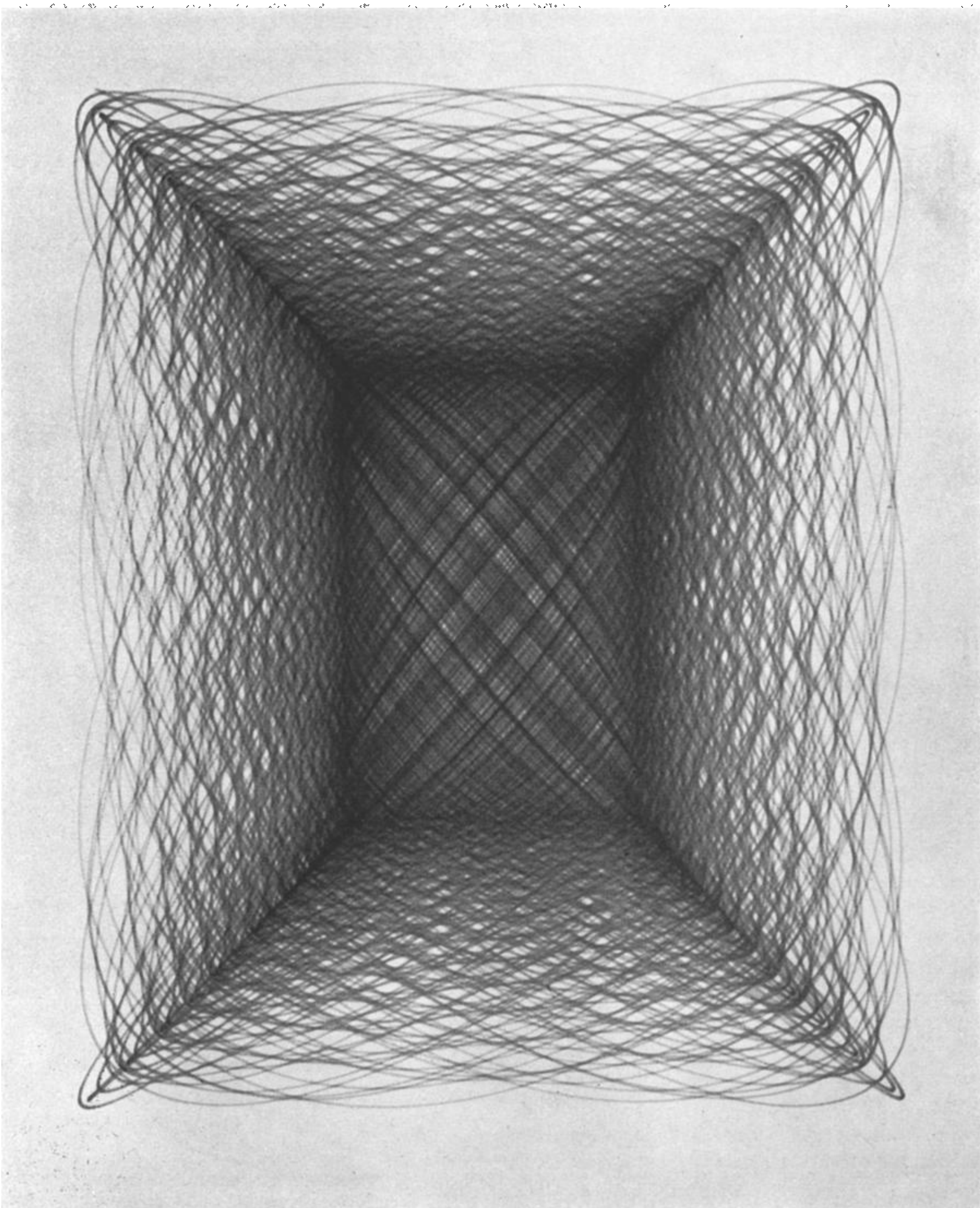

FIG. 2A. $r=7/6$. $T=3.66$ sec. $T_1=3.13$ sec. Total time of exposure = 25 min. Number of cycles = 68.2.

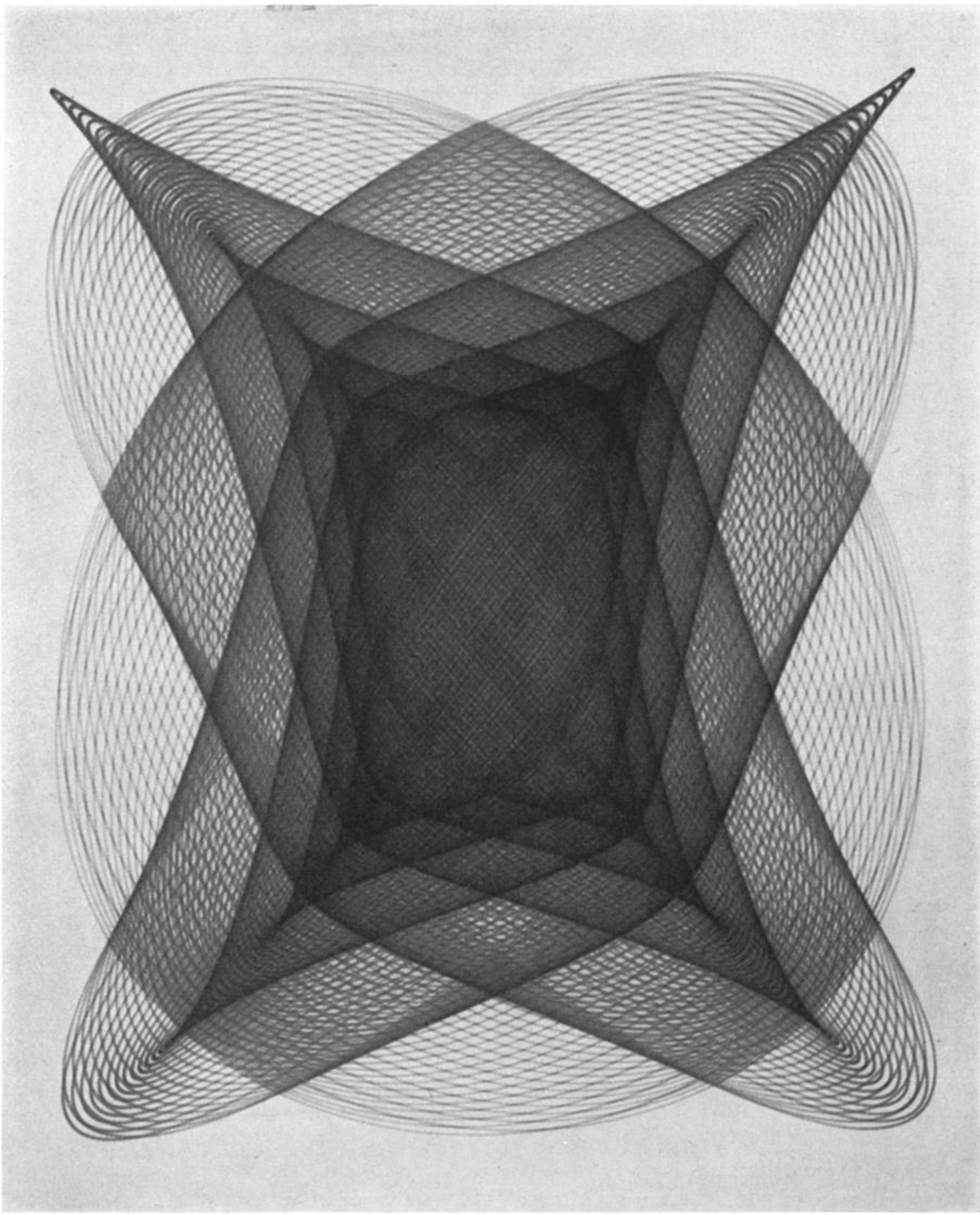

FIG. 2B. $r=7/6$. $T=3.648$ sec. $T_1=3.128$ sec. Total time of exposure $=32$ min. Number of cycles $=87.0$.

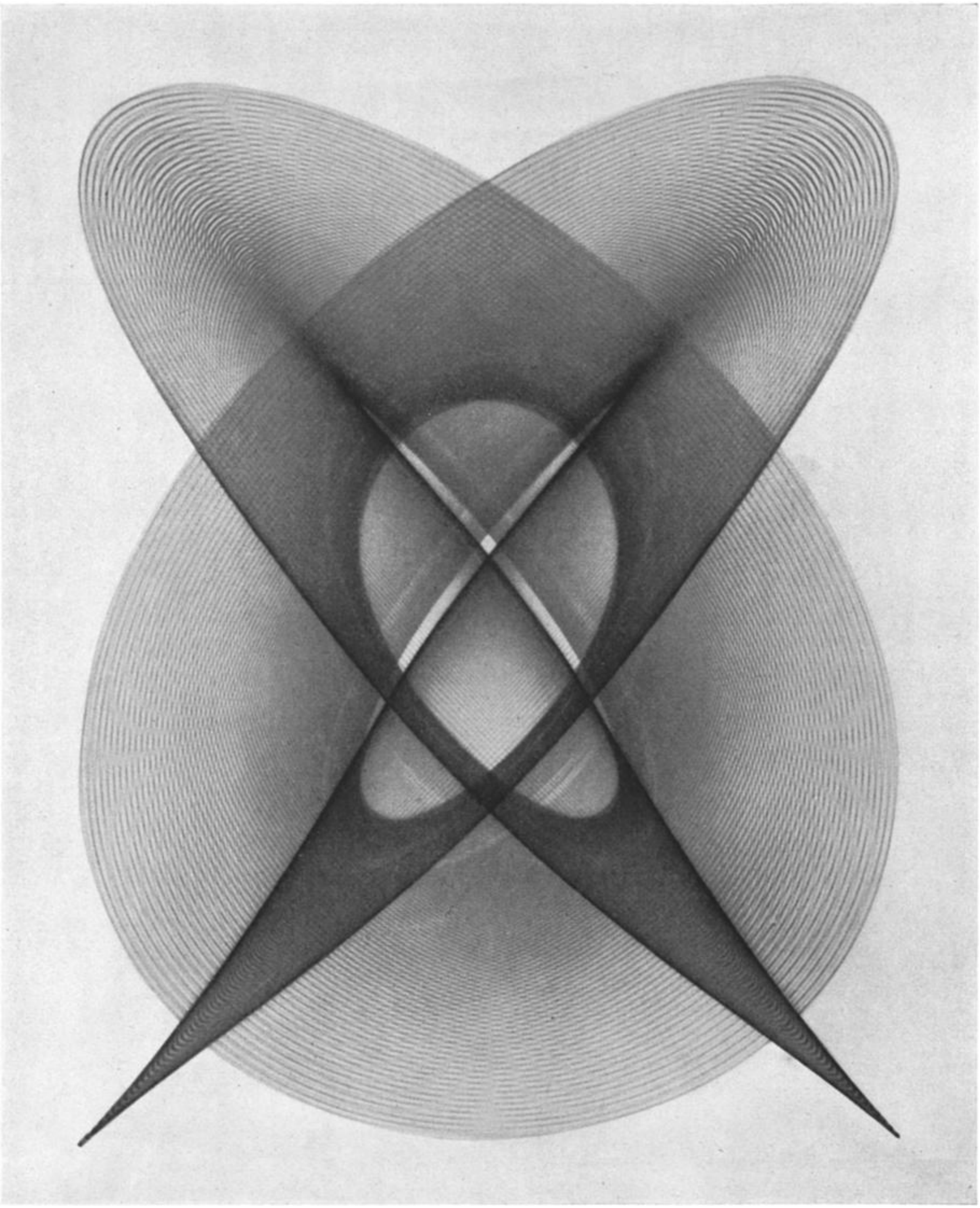

FIG. 2C. $r=5/4$. $T=3.650$ sec. $T_1=2.919$ sec. Total time of exposure = 24.5 min. Number of cycles = 100.

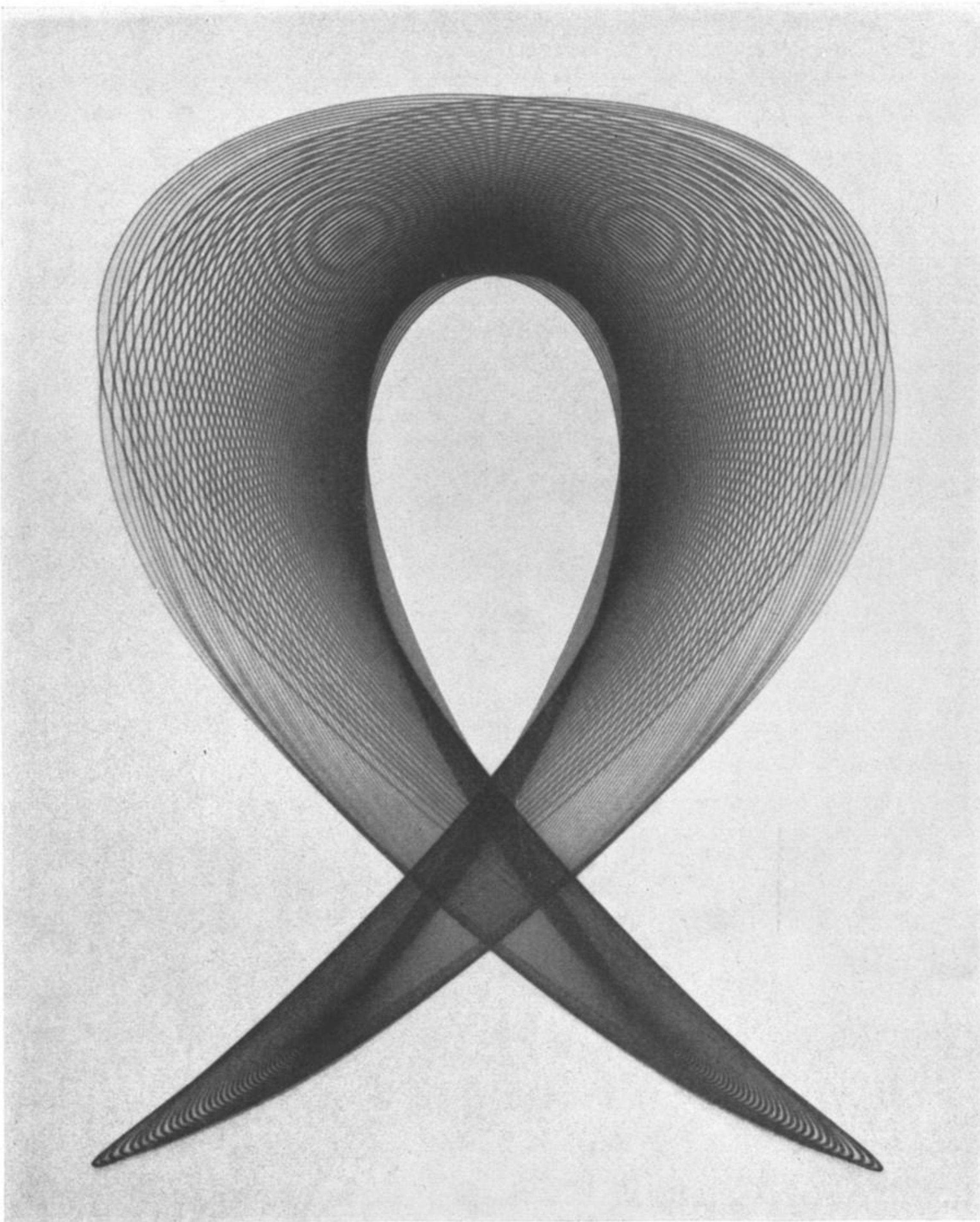

FIG. 2D. $r=3/2$. $T=3.650$ sec. $T_1=2.433$ sec. Total time of exposure = 7.3 min. Number of cycles = 60.

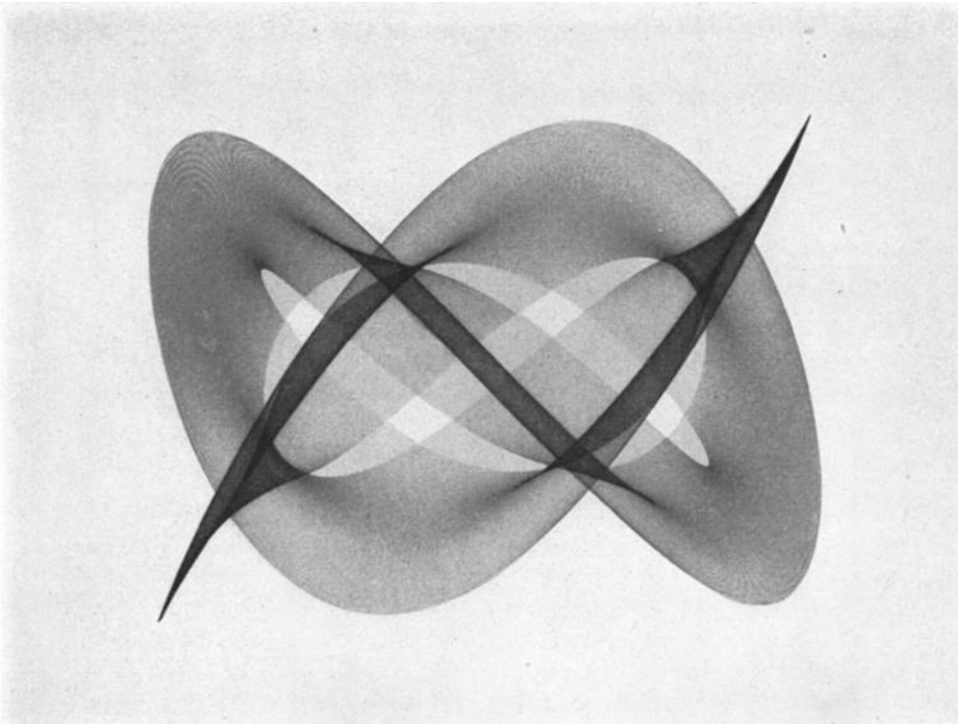

FIG. 3A. $r=5/3$. $T=3.700$ sec. $T_1=2.22$ sec. Total time of exposure$=30.0$ min. Number of cycles$=162$.

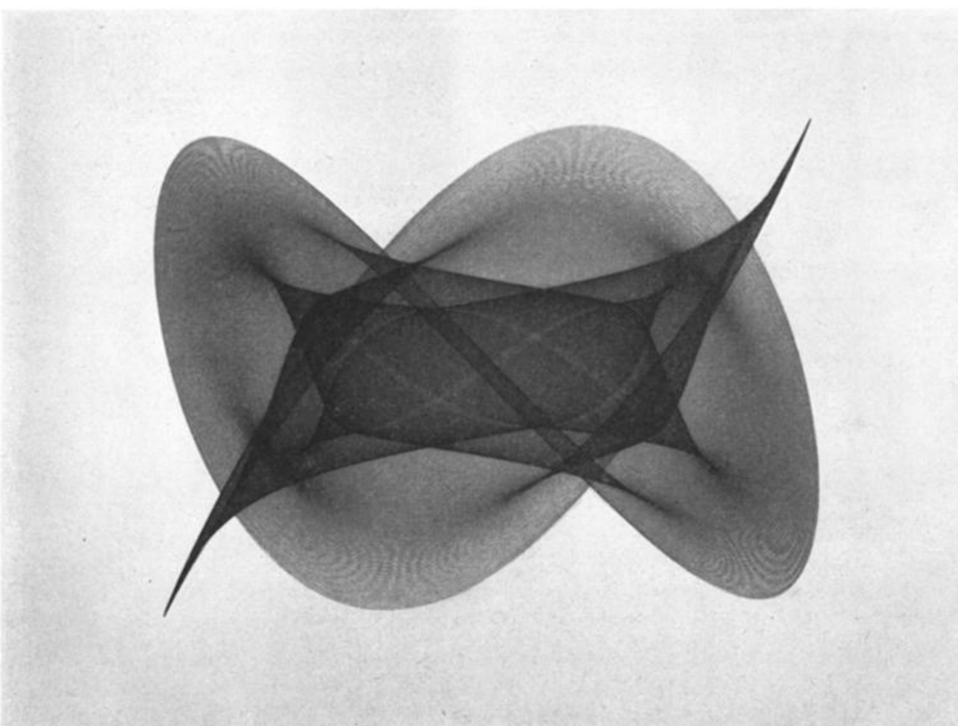

FIG. 3B. $r=5/3$. $T=3.700$ sec. $T_1=2.22$ sec. Total time of exposure$=60.0$ min. Number of cycles$=324$.

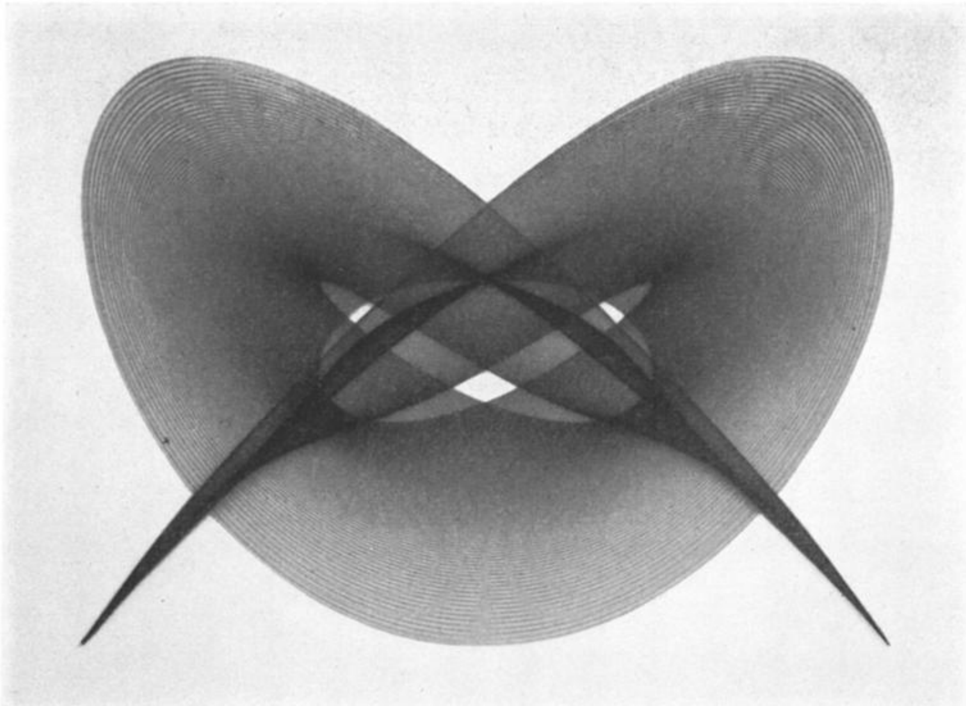

FIG. 4A. $r=4/3$. $T=3.650$ sec. $T_1=2.738$ sec. Time of exposure = 22.0 min.
Number of cycles = 120.0.

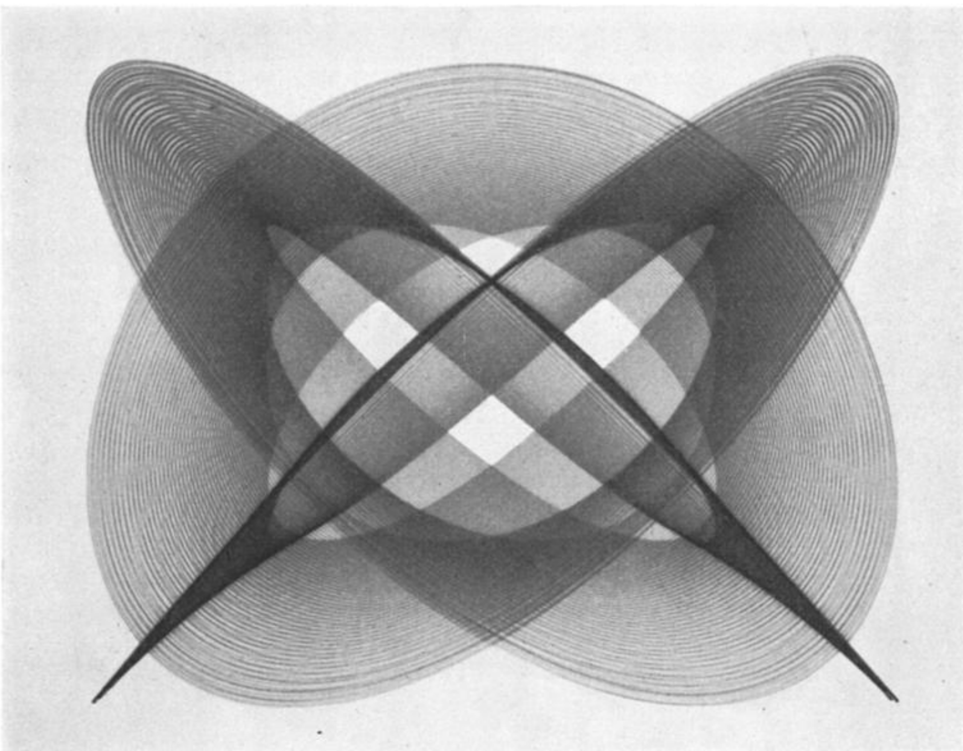

FIG. 4B. $r=6/5$. $T=3.650$ sec. $T_1=3.04$ sec. Time of exposure = 15.2 min.
Number of cycles = 50.0.

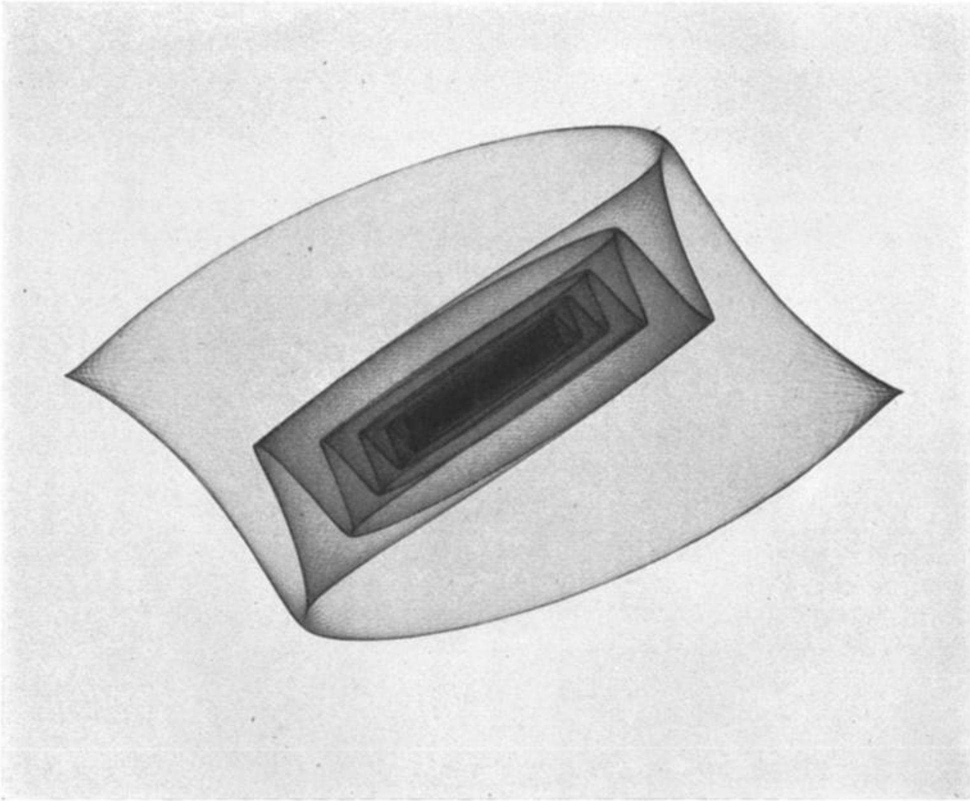

FIG. 5. This unusual pattern was obtained by holding the pendulum line to one side in a semirigid way and allowing the short length arm of the pendulum to oscillate for one hour as a Foucault pendulum.

about twenty-nine years later Hugh Blackburn of Glasgow invented a double pendulum having a conical bob, the apex of which had a small hole through which sand or salt could filter and leave a record of the track being described.

During the years 1856–58, Jules Antoine Lissajous, a teacher at the Lycée St. Louis in Paris, made extensive studies of these compounded harmonic motions at right angles to each other. He published two papers on this subject. This publication together with his more extensive lecturing and demonstrating of this phenomenon has resulted in his name being more effectively attached thereto. Lissajous used an optical method of producing these figures and this permitted them to be projected on a screen and observed by an audience of any size. His apparatus consisted of two tuning forks each carrying a small mirror and vibrating at right angles to each other. A spot of light is reflected successively from the two mirrors onto a screen where the combined motion of the two forks is clearly shown. If the vibrations are fast enough, the peristence of vision enables the observer to see a continuous curve.

Many methods have been designed for reproducing these figures—such as the Harmonograph of A. E. Donkin and the Kaleidophone of Sir Charles Wheatstone.

The late Professor Wallace Clement Sabine of Harvard drew some very beautiful curves by mechanical means. Eight of them are reproduced in Wm. F. Osgood's *Mechanics*, and according to Professor Osgood, this is the only reproduction that has been made of Professor Sabine's drawings. The details of his mechanism are not known by this author. Numerous other reproductions are found in *Harmonic Vibrations and Vibration Figures* by Goold, Benham, and others, and in *Harmonic Curves* by William F. Rigge.

The most modern way of producing these figures is by means of the cathode-ray oscillograph. In this instrument the vibrations at right angles to each other are electrical and produce proportionate displacements of a beam of electrons which subsequently falls on a photographic plate or phosphorescent screen where the resultant Lissajous figure is recorded. The formation of such figures with frequencies up to at least 10 megacycles has been produced.

The equations of motion for these curves are given by

$$X=a\sin\left(r\frac{2\pi t}{T}+\alpha\right),$$

$$Y=b\sin\frac{2\pi t}{T},$$

where a and b are the respective amplitudes of vibration, T the fundamental period of vibration, α the phase difference in the two motions, and r a constant representing the ratios of the periods of the two vibrations.

$$r=T/T_1=(L/l)^{\frac{1}{2}},$$

where

$$T=2\pi(L/g)^{\frac{1}{2}} \quad \text{and} \quad T_1=2\pi(l/g)^{\frac{1}{2}}.$$

When $r=1$ and $\alpha=0$ the motion is a straight line which may be changed to a circular or elliptical path by introducing a phase angle α of the desired value.

As r increases in simple ratios, the complexity of the motion becomes much greater and an analytical treatment of the problem by eliminating t and obtaining X in terms of Y likewise becomes greater. When $r=2$, the motion describes a quadratic path; when $r=3$, the path will be a cubic. Beyond this degree the analytical analyses of these curves become very involved and they are usually studied by graphical methods. A complete discussion of the theory connected with these curves will be found in any good treatise on sound or mechanics.

When the frequencies are very near whole numbers, the differences have the effect of introducing a small or large phase angle between the vibrating components and, as a result, there is a migration of the cusps of the figure being described. This migration will be rapid or slow depending on the accuracy of the tuning or the integral relationship of the ratio. If, for example, the frequency ratio is 4/3, the short length pendulum will make four complete vibrations while the longer one makes three complete vibrations, and they return to the point of origin together to begin another cycle. If, however, the ratio is 4.01/3, then the shorter one will arrive at the point of origin just a little ahead of the longer one and they will be out of step. This lead will increase as time goes on until the short pendulum is a half-vibration ahead of the longer one and the cusps will appear on the opposite side of the sheet and the vibrations appear to be in step again. This continues through the period of the exposure. Figure 3A is a splendid example of sharp or accurate tuning. This exposure continued for 30.0 minutes and the two vibrations were in step even at the end. During this time 162 complete cycles were executed and the bob came back to the initial point at the end of each cycle. Figure 2B is an example of near tuning in which the cusps migrated from corner to corner during the 32.0 minutes of its exposure. The basket weave of Fig. 2A is a result of very poor tuning. The frequencies were so incommensurate that they were never together after the release of the bob at the beginning of the exposure.

The present method of recording Lissajous figures was begun several years ago in the Physics Laboratory of Brigham Young University. A number of students have worked on the project, each contributing refinement which added to the precision of timing and recording and to the beauty of the resulting pattern. Professor Charles A. Slichter of Wisconsin used a similar method (unknown to us during the development of our method) of making stereoptican photographs of single cycles of harmonic curves of three frequencies.[2]

The apparatus in its final form consisted of an ordinary Blackburn pendulum, the principal length L of which was 325 centimeters. The bob consisted of a heavy brass ball—weight twelve pounds—on which was constructed a rack to carry two dry cells and under which was constructed a housing for the flashlight globe, and

[2] His work is described in *Transactions, Wisconsin Academy of Science, Arts, and Letters* (1896–1897), Vol. 11.

a series of three pinholes in line to insure parallel light. A piece of 11×14 inch double weight Kodabromide F5 paper was held in place by a printing board of convenient size. The bob was tied back to one corner of the paper and released simultaneously with the turning on of the light and starting the chronometer. All this was done in the red light of an Eastman safe light for bromide paper. The exposure was allowed to continue for a few seconds for one cycle or for an hour and a half for 450 cycles or more.

The pencil of light is fine enough and the damping of the bob sufficiently large to record every path as a fine distinct line, so that areas of light and dark are visible to produce the fine structure of the pattern. Even the black areas where the damping is small near the end of the exposure show a unique and fine structure detail and symmetry in the originals.

One sees, therefore, that an infinite variety of designs or patterns could be produced by varying any one of a number of factors which include frequency ratio, phase, amplitude, damping, and time of exposure. Much time was always consumed before each exposure to insure perfect timing for each ratio of periods recorded. It was found that ratios had to be synchronized by the method of timing because direct measurements of respective lengths proved quite inaccurate.

Rough adjustments to change the period or frequency ratios were made by means of the clamp *C*, Fig. 1, until the ratio of lengths was the desired value. Finer adjustments were later made by timing the periods in each phase of motion and making adjustments at the turn buckel *B*.

Since we began recording Lissajous figures by this method, nearly 200 exposures have been made and 150 different and attractive patterns have been produced—no two of which were alike. Sixty of these were exhibited in the foyer of the Hotel Pennsylvania during the May, 1944 meeting of the Acoustical Society of America; nine of them are reproduced with this article.