Simplified Equations for Relativistic Rotating Perfect Fluids with Axial Symmetry*

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The general relativistic equations for a stationary axially symmetric rotating perfect fluid, in a comoving coordinate system, are presented in an elegant general form. The coordinates can then be chosen in any of several ways to further simplify the equations. One choice reduces the number of independent metric functions to three.

FOR work on the general relativistic problem of the stationary, rotating, axially symmetric perfect fluid, one uses equations that are as simple as possible. In this paper we find equations which have an elegant vectorial form, and which can be adapted to any of several approaches to the problem by proper choice of coordinates. One of these approaches reduces the number of independent metric coefficients to three. This simplification has not been noted before in the literature, although it was found independently by Wahlquist in parallel work.[1] The derivation proceeds from the usual tensor form of Einstein's equations, but with occasional insights from the dyadic analysis of Wahlquist and Estabrook.[2]

We consider the timelike congruence of the world lines of a perfect fluid. The absolute angular velocity of this congruence, locally defined, will be denoted by $\mathbf{\Omega}$. Stationarity requires that the congruence be rigid, so that the rate of strain dyadic S and all time derivatives vanish. The triad determining the spacelike coordinates will be fixed in the rigid body, so that its angular velocity $\boldsymbol{\omega}=\mathbf{\Omega}$. The absolute acceleration of the congruence is denoted by $\mathbf{a}$.

These choices enable one to choose a metric of the form[3]

$$-ds^2=-e^{2U}(dx^0+f_\alpha dx^\alpha)^2+e^{-2U}\eta_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{1}$$

where time derivatives vanish and Greek letters run from 1 to 3. The fluid velocity four-vector is

$$u^i=(e^{-U},0,0,0)\,. \tag{2}$$

We specialize immediately to the case of diagonal space metric:

$$\eta_{\alpha\beta}=\delta_{\alpha\beta}h_\alpha{}^2 \text{ (no summation)}\,, \tag{3}$$

and define the usual vector operations for such a metric:

$$\nabla F=\sum_\alpha h_\alpha{}^{-1}F_{,\alpha}\mathbf{u}^\alpha\,, \tag{4}$$

$$\nabla\cdot\mathbf{V}=\eta^{-1/2}\sum_\alpha(\eta^{1/2}h_\alpha{}^{-1}V^{(\alpha)})_{,\alpha}\,, \tag{5}$$

$$\nabla\times\mathbf{V}=\eta^{-1/2}\sum_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma}h_\alpha(h_\gamma V^{(\gamma)})_{,\beta}\mathbf{u}^\alpha\,, \tag{6}$$

$$\nabla^2F=\eta^{-1/2}\sum_\alpha(\eta^{1/2}h_\alpha{}^{-2}F_{,\alpha})_{,\alpha}\,, \tag{7}$$

where the $\mathbf{u}^\alpha$ are the orthonormal triad of vectors along the space axes, $\mathbf{V}=\sum_\alpha V^{(\alpha)}\mathbf{u}^\alpha$, $\eta=\det(\eta_{\alpha\beta})$, and commas denote ordinary differentiation.

We can now easily write the Einstein equations, using previously derived expressions for the Ricci tensor for the metric (1).[4] After some simplification, they are

$$\nabla\times\mathbf{f}=e^{-4U}\nabla\psi\,, \tag{8}$$

$$\nabla^2\psi-4\nabla\psi\cdot\nabla U=0\,, \tag{9}$$

$$\nabla^2U+\tfrac{1}{2}e^{-4U}(\nabla\psi)^2=e^{-2U}(3P+\rho)\,, \tag{10}$$

$$P_{\alpha\beta}=2U_{,\alpha}U_{,\beta}+\tfrac{1}{2}e^{-4U}\psi_{,\alpha}\psi_{,\beta}-4Pe^{-2U}\eta_{\alpha\beta}\,. \tag{11}$$

Here ψ is a potential, the use of which satisfies the $R_{0\alpha}$ equation,[4] $P_{\alpha\beta}$ is the three-dimensional Ricci tensor formed from the $\eta_{\alpha\beta}$, and the components $f^{(\alpha)}$ of the vector $\mathbf{f}$ are given by

$$f^{(\alpha)}=h_\alpha{}^{-1}f_\alpha \text{ (no sum)}\,. \tag{12}$$

(Thus $\mathbf{f}=\sum_\alpha f_\alpha\nabla x^\alpha$; for details on the various bases, see Ref. 2.) Units are chosen so that $c=1$, $G=(4\pi)^{-1}$; P and ρ are the perfect fluid pressure and density, respectively.

From the contracted Bianchi identities, we have

$$P_{,\alpha}+(P+\rho)U_{,\alpha}=0\,, \tag{13}$$

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[1] H. D. Wahlquist (private communication).

[2] H. D. Wahlquist and F. B. Estabrook, Jet Propulsion Laboratory Technical Report No. 32-868, 1966 (unpublished); see also F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. **5**, 1629 (1964); H. D. Wahlquist and F. B. Estabrook, *ibid.* **7**, 894 (1966).

[3] Sign conventions are those found in Landau and Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Addison-Wesley, Reading, Mass., 1962).

[4] B. K. Harrison, J. Math. Phys. **9**, 1744 (1968).

which yields

$$P=P(U), \quad \rho=\rho(U), \quad dP/dU=-P-\rho\,, \tag{14}$$

relations noted before.[5] P and ρ may be obtained as explicit functions of U if an equation of state $P=P(\rho)$ is known.

Introduction of a complex Ernst potential[6]

$$\xi=e^{2U}+i\psi \tag{15}$$

simplifies Eqs. (9)–(11) to

$$\nabla^2\xi=e^{-2U}(\nabla\xi)^2+2(3P+\rho)\,, \tag{16}$$

$$P_{\alpha\beta}=\tfrac{1}{4}e^{-4U}(\xi_{,\alpha}\xi_{,\beta}{}^*+\xi_{,\beta}\xi_{,\alpha}{}^*)-4Pe^{-2U}\eta_{\alpha\beta}\,. \tag{17}$$

Comparison of metric (1) with Eq. (24) of Ref. 2 shows that the quantities $\phi=e^{-U}$, $h_{\alpha\beta}=e^{-2U}\eta_{\alpha\beta}$, and $A_\alpha=-f_\alpha$. These identifications enable us to evaluate the vectors $\mathbf{a}$ and $\mathbf{\Omega}$. They are given in Ref. 2 as $\mathbf{a}=-\phi^{-1}\nabla'\phi$ and $\mathbf{\Omega}=\frac{1}{2}\phi^{-1}\nabla'\times\mathbf{A}'$, where ∇' is the del operator for the metric $h_{\alpha\beta}$, and $\mathbf{A}'$ is defined relative to $h_{\alpha\beta}$. The following relations hold between arbitrary quantities defined for $h_{\alpha\beta}$ (primed) and for $\eta_{\alpha\beta}$ (unprimed):

$$\nabla'F=e^U\nabla F\,, \quad \nabla'\times\mathbf{v}=e^{2U}\nabla\times(e^{-U}\mathbf{v}')\,, \quad \mathbf{v}'=e^U\mathbf{v}\,.$$

Thus,

$$\mathbf{f}=-\mathbf{A}=-e^{-U}\mathbf{A}'\,,$$

and we find

$$\mathbf{a}=e^U\nabla U\,, \tag{18}$$

$$\mathbf{\Omega}=-\tfrac{1}{2}e^{3U}\nabla\times\mathbf{f}=-\tfrac{1}{2}e^{-U}\nabla\psi\,, \tag{19}$$

by Eq. (8). We note that $\nabla\xi=2e^U(\mathbf{a}-i\mathbf{\Omega})$.

We now specialize to the axial symmetry case and put $\partial/\partial x^3=0$. We also write $h_1=e^A$, $h_2=e^B$, $h_3=e^C$, and $f_3=Q$. It is easily shown from Eq. (8) that $f_1=\zeta_{,1}$, $f_2=\zeta_{,2}$, where ζ is an arbitrary function ζ may be absorbed into x^0, so that $f_1=f_2=0$.

Finally, we evaluate $P_{\alpha\beta}$ by formulas suitable for diagonal metrics[7] and write out and simplify Eq. (17). The P_{33} equation becomes

$$\nabla^2C=4Pe^{-2U}. \tag{20}$$

The combination

$$\begin{aligned}\mathbf{u}^1e^{-A}[&-e^{-2B}C_{,2}P_{12}+\tfrac{1}{2}C_{,1}(-e^{-2A}P_{11}+e^{-2B}P_{22}\\ &-e^{2C}P_{33})]+\mathbf{u}^2e^{-B}[-e^{-2A}C_{,1}P_{12}\\ &+\tfrac{1}{2}C_{,2}(e^{-2A}P_{11}-e^{-2B}P_{22}-e^{-2C}P_{33})]\end{aligned}$$

yields

$$\begin{aligned}\tfrac{1}{2}\nabla[(\nabla C)^2]+\nabla C(\nabla C)^2=2Pe^{-2U}\nabla C+\tfrac{1}{4}e^{-4U}[\nabla C(\nabla\xi\cdot\nabla\xi^*)\\ -\nabla\xi(\nabla C\cdot\nabla\xi^*)-\nabla\xi^*(\nabla C\cdot\nabla\xi)].\end{aligned} \tag{21}$$

The curl of Eq. (21) reduces to an identity by virtue of Eqs. (14)–(16) and (20). The divergence of Eq. (21) reduces to an equation obtained from the combination $e^{-2A}P_{11}+e^{-2B}P_{22}-e^{-2C}P_{33}$:

$$\begin{aligned}e^{-2B}[A_{,22}+A_{,2}(A_{,2}-B_{,2})]+e^{-2A}[B_{,11}+B_{,1}(B_{,1}-A_{,1})]\\ =2Pe^{-2U}-\tfrac{1}{4}e^{-4U}\nabla\xi\cdot\nabla\xi^*.\end{aligned} \tag{22}$$

Equations (8), (14)–(16), and (20)–(22) are now the complete set of equations [with $f_1=f_2=0$ and Eqs. (4)–(7) understood].

SIMPLIFYING CHOICES OF COORDINATES

(1) We may choose $x^1=f(U)$. U then becomes an independent variable, labelling the isobaric surfaces. The equation of surface of the fluid object, $P=0$, is now simply $U=$const. Thus this coordinate choice simplifies the description of the surface. Since P decreases monotonically from center to surface of fluid, U increases monotonically. This behavior of U continues outside the fluid. The value of U can be taken as zero at infinity, so that $U<0$ in finite regions. U is similar to a radial distance coordinate; x^2 now becomes similar to a polar angle. Thus this choice of coordinates is very suitable for treatment of spherically symmetric and slow rotation problems (problems previously treated[8]).

(2) The angular velocity $\mathbf{\Omega}$ ($\sim\nabla\psi$) points along the axis of rotation, so that the variable ψ is similar to a cylindrical z coordinate. We can see easily from Eq. (8) that $\nabla Q\cdot\nabla\psi=0$ ($Q=f_3$), so that Q is similar to cylindrical r. If we take $\psi=-2\lambda x^2$ (λ determines the magnitude of $\mathbf{\Omega}$), then Eq. (8) gives $Q=Q(x^1)$. Choose x^1 so that $Q=2\lambda x^1$; then Eq. (8) yields $A-B+C-4U=0$. We have in this way reduced the number of dependent variables to three. This provides a simplification over previous published results (p. 328 in Hartle and Sharp[5]).

(3) One may set $A=B$, in analogy to Weyl canonical coordinates. One may *not*, however, put $e^C=x^1$, as is usually done in vacuum,[9] because of the inhomogeneous term in Eq. (20). If we now write $DF=F_{,1}\mathbf{u}^1+F_{,2}\mathbf{u}^2$, $D^2F=F_{,11}+F_{,22}$, then we see that $\nabla F=e^{-A}DF$, $\nabla^2F=e^{-2A}\times(D^2F+DF\cdot DC)$, for any F. A simplification of appearance in the gradient and Laplacian operators is thus achieved; the corresponding physical significance is, however, obscure.

It is hoped that the above approaches will suggest useful numerical computational schemes for solving the rapidly rotating dense star problem. An iterative technique, in which one first guesses values for the

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[6] F. J. Ernst, Phys. Rev. **167**, 1175 (1968).

[7] B. K. Harrison, Phys. Rev. **116**, 1285 (1959).

[8] For the spherically symmetric case, see B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago, Chicago, Ill., 1965). For the slowly rotating case, see J. B. Hartle, Astrophys. J. **150**, 1005 (1967); J. B. Hartle and K. S. Thorne, *ibid.* **153**, 807 (1968); J. B. Hartle and D. H. Sharp, *ibid.* **147**, 317 (1967).

[9] See, for example, H. Levy, Nuovo Cimento **56**, 253 (1968), and references to Lewis, etc., cited therein.

metric coefficients and evaluates the nonlinear terms and the operators, then solves the Poisson equations (16) and (20), may possibly be fruitful.[10,11]

[10] For an application of this approach to Newtonian rapidly rotating stars, see J. P. Ostriker and J. W. K. Mark, Astrophys. J. **151**, 1075 (1968).

[11] It has come to the author's attention that similar equations have been discussed by G. E. Tauber and J. W. Weinberg, Phys. Rev. **122**, 1342 (1961). However, their paper does not consider the reduction of the number of dependent variables to three; nor does it present the equations in this particular form.

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Satellite Terms in Veneziano Models*

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It is shown that many of the satellite terms possible in Veneziano models are linearly dependent, and that, in fact, specification of the residues of the poles is arbitrary, but completely determines all the coefficients of the satellite terms. Implications of this for ghosts, parity doubling, daughters, asymptotic behavior, Adler zero, and finite-energy sum rules are discussed. A specific numerical example for π-π scattering is given.

I. INTRODUCTION

A CLOSED-FORM expression to describe the scattering of pions has been proposed by Veneziano[1] and modified by others.[2,3] In all suggested forms, the usual desired criteria[1] save unitarity were satisfied, but a number of less vital characteristics needed consideration. Some of the more popular points of investigation have been studies of j-plane analyticity and daughter resonances,[4] positivity of the residues at a pole,[5] the constraints imposed by invoking the nonexistence of parity doublets, for example, in π-ρ scattering,[6] and the difficulties inherent in phenomenologically fitting experiment to the one-term form.[7,8]

The results indicate that higher-order terms, named "satellites" by some authors, must be included in the amplitude in order to provide consistency within the model and with experiment. For example, in one investigation[9] it was shown that in order to eliminate wrong-parity daughters at a pole, higher-order terms in the Veneziano amplitude must be employed. This was to be done by assigning the "arbitrary" coefficients of each term a value according to a definite prescription. It has also been remarked that various other constraints, such as the finite-energy sum rule (FESR) or the Adler condition in π-π scattering, can be trivially satisfied by adding suitable satellites, without significantly affecting properties of the leading trajectories.

In this paper we wish to examine these questions, and in particular to show the following: that the most general (four-point) Veneziano amplitude can be written in "diagonal" form, and, consequently, that specification of the residues at the poles completely determines the amplitude everywhere. Since each pole residue is a completely arbitrary polynomial, the assignments of widths (zero or otherwise) to parents and daughters in a particular model is subject only to questions of convergence. Thus, no parity doubling and/or no ghosts and/or no daughters at all can be built in; but all other constraints (such as the Adler condition) now become sum rules on the widths, with the trajectory function playing a parametric role.

We first give some reduction formulas,[10] and then the solution for the most general coefficients in terms of arbitrary residues. Next we touch on questions of asymptotics and convergences; and then we give, as is customary, an explicit example for π-π scattering, discussing its properties in terms of daughter structure. In our conclusion we summarize our main results and point out some implications for other models.

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[7] K. Vasavada, Phys. Rev. D **1**, 88 (1970).

[8] N. Antoniou *et al.*, Universität Tübingen Report (unpublished).

[9] S. Mandelstam, Phys. Rev. Letters **21**, 1724 (1968).

[10] The existence of a reduction formula has been invoked by some authors (cf. Ref. 7) and ignored by many others. To the best of our knowledge, no explicit proof of this fact exists in the literature; since this result yields vast simplifications in calculations with satellite terms, we should like to set this matter to rest once and for all.