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Citation: Physics of Fluids (1958-1988) **24**, 931 (1981); doi: 10.1063/1.863466
View online: http://dx.doi.org/10.1063/1.863466
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/24/5?ver=pdfcov
Published by the AIP Publishing

Solutions of the Helmholtz equation with boundary conditions for force-free magnetic fields

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(Received 31 August 1979; accepted 5 February 1981)

It is shown that the solution, with one ignorable coordinate, for the Taylor minimum energy state (resulting in a force-free magnetic field) in either a straight cylindrical or a toroidal geometry with arbitrary cross section can be reduced to the solution of either an inhomogeneous Helmholtz equation or a Grad–Shafranov equation with simple boundary conditions. Standard Green's function theory is, therefore, applicable. Detailed solutions are presented for the Taylor state in toroidal and cylindrical domains having a rectangular cross section. The focus is on solutions corresponding to the continuous eigenvalue spectra. Singular behavior at 90° corners is explored in detail.

I. INTRODUCTION

The toroidal axisymmetric reversed-field pinch is a concept that may lead to a fusion reactor with the potential advantages of high ohmic heating, beta values $\gtrsim 10\%$, low forces on field coils, and no physics restrictions on the choice of aspect ratio. The early observations on the British ZETA experiment[1] as well as on the more recent Italian ETA-BETA II experiment[2] show that the discharge has a natural tendency to form a reversed-field configuration with the low levels of current fluctuations. As a result, these configurations are of great interest.

In 1974, Taylor proposed a hypothesis[3] that appears to account for many of the gross features of the observed "quiescent" plasma behavior in ZETA. He suggested that small but finite resistivity in the presence of magnetohydrodynamic turbulence drives the plasma toward a state of minimum magnetic energy, $\int B^2 dv$, for a fixed magnetic helicity, $\int \mathbf{A}\cdot\mathbf{B}\,dv$, and a fixed magnetic flux. The Euler equation of this variational problem leading to the Taylor state is the equation describing a force-free magnetic field configuration

$$\nabla \times \mathbf{B}(\mathbf{r}) = \lambda \mathbf{B}(\mathbf{r}) . \tag{1}$$

The spatial constancy of λ is a direct consequence of Taylor's variational principle. Equation (1) currently provides the cornerstone of any description of the magnetic fields during the relaxation stage of a reversed-field pinch. Therefore, recognition of the existence of its solutions for a variety of geometries is crucial. After such recognition, one may consider introducing non-ideal effects. For example, one might expect the presence of a finite resistivity to reduce the tangential components of the current density in the neighborhood of a highly conducting wall. However, this reduction need have only a negligible effect on the magnetic field profiles within the plasma, the features of which remain well-described by Eq. (1).[4]

For a multiply-connected geometry such as that of a toroid or the plasma configuration considered by Taylor (a straight cylindrical geometry with a circular cross section bounded by perfectly conducting wall), the eigenvalue λ has a continuous range of values which specifies the ratio of net toroidal or longitudinal current to net toroidal or longitudinal magnetic flux. For a simply connected domain such as the spheromak configuration only a discrete geometrically determined spectrum for λ exists.

There has been recent concern regarding the possibility that the continuous nature of the spectrum of λ for the geometry considered by Taylor would not occur in the more general, multiply-connected geometries.[5] To allay these fears, we have extended the analysis by demonstrating that the existence of a continuous spectrum for λ is neither an artifact of the straight cylindrical geometry nor of the circular nature of the cross section.

We shall prove that for any value of λ, the solution of Eq. (1) with one ignorable coordinate in either a straight cylindrical or a toroidal geometry with arbitrary cross section can be reduced to the solution of either an inhomogeneous Helmholtz equation or a Grad–Shafranov equation with simple boundary conditions. Standard Green's function theory is therefore applicable.

We shall present analytical solutions for the case of rectangular cross sections, with relevance to force-free magnetic field regions within cusp geometries.[6] Figures depicting magnetic field lines for specific examples will also be represented.

In Sec. II, the basic equations are reviewed and the notion established. In Sec. III, we examine in detail the behavior of force-free fields in the neighborhood of corners. In Sec. IV, we consider force-free magnetic fields in cylinders and toroids.

II. BASIC EQUATIONS

We confine our attention to the Taylor state where B satisfies Eq. (1) and λ is a nonvanishing constant. The usual method of finding force-free magnetic fields in a specific context is by application of the Chandrasekhar–Kendall[7] technique: If the potential ψ is a solution of the Helmholtz equation

$$\nabla^2\psi + \lambda^2\psi = 0 , \tag{2}$$

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then

$$\mathbf{B} = \nabla \times [a\psi + (1/\lambda)\nabla \times (a\psi)] \tag{3}$$

is a solution of Eq. (1). The vector a is either a constant unit vector or the position vector x.

The typical boundary condition is that $\mathbf{B} \cdot \hat{n} = 0$ at the surface of a conducting container with $\hat{n}$ being the usual outward normal. These boundary conditions, when implemented in terms of the potential ψ, invariably lead to inhomogeneous boundary conditions. Often these boundary conditions involve mixed partial derivatives of ψ, depending on the geometry involved and the symmetries assumed.

III. SOLUTION IN THE NEIGHBORHOOD OF CORNERS

In this section we consider a coordinate system (x^1, x^2, x^3) in which x^1 and x^2 are arbitrary curvilinear coordinates in a plane orthogonal to a fixed unit vector $\hat{x}^3$ corresponding to the direction of increase of x^3. We shall confine our attention to situations where the potential ψ and hence the vector field is independent of x^3. Taking **a** to be $\hat{x}^3$, we obtain, from Eq. (3), the (contravariant) components of the magnetic field

$$B^1 = g^{-1/2}\frac{\partial \psi}{\partial x^2}; \quad B^2 = g^{-1/2}\frac{\partial \psi}{\partial x^1}; \quad B_z \equiv B^3 = \lambda\psi \tag{4}$$

The quantity g is the determinant of the metric tensor g_{ij} and g^{ij} is the metric inverse.

Several authors have considered solutions of the Helmholtz equation in the neighborhood of corners.[8-15] For the most part, however, they have been concerned with standard homogeneous boundary conditions describing scattering of waves at the corner. Reference 8 is a notable exception and the results obtained in this section are consistent with the theorems of Wigley. We shall illustrate the behavior of force-free fields in the neighborhood of corners and shall show the singular nature of field derivatives for 90° corners.

Let two conducting surfaces which meet at a corner coincide with the coordinate surfaces $x^1 = \text{const}$ and $x^2 = \text{const}$. The boundary condition $\mathbf{B} \cdot \hat{n} = 0$ implies that first and second derivatives of ψ vanish on these surfaces, i.e., on $x^1 = \text{const}$,

$$\frac{\partial \psi}{\partial x^2} = 0, \tag{5}$$

for all x^2. Similarly on $x^2 = \text{const}$

$$\frac{\partial \psi}{\partial x^1} = 0, \tag{6}$$

for all x^1. At the corner both Eqs. (5) and (6) hold. Thus, from Eq. (2) we find

$$2g^{12}\frac{\partial^2 B_z}{\partial x^1 \partial x^2} + \lambda^2 B_z = 0. \tag{7}$$

For a 90° corner where $g^{12} = 0$, either $B_z = 0$ or $\partial^2 B_z/\partial x^1 \partial x^2 \to \infty$ at the corner. We illustrate this behavior explicitly by calculating $B_z = \lambda\psi$ in the neighborhood of 90° and 45° corners.

Using Green's functions,[16] we shall calculate the solution of Eq. (2), valid in the neighborhood of a corner. We require $\psi = \psi_0 = \text{const}$ on the boundary and then consider the function $\chi = \psi - \psi_0$. This function satisfies the inhomogeneous Helmholtz equation with the spatially constant source term, $-\lambda^2\psi_0$, but satisfies homogeneous boundary conditions. Then,

$$\psi(\mathbf{x}) = \psi_0 + (4\pi)^{-1}\lambda^2\psi_0 \int G(\mathbf{x}, \mathbf{x}')d^2x', \tag{8}$$

where $G(\mathbf{x}, \mathbf{x}')$ satisfies

$$\nabla^2 G + \lambda^2 G = -4\pi\delta^{(2)}(\mathbf{x} - \mathbf{x}'), \tag{9}$$

and the integral is over the region of interest with $G(\mathbf{x}, \mathbf{x}')$ vanishing if either x or x′ is on the boundary. In two dimensions the singularity in the solution for a point source is of the logarithmic type and thus the dominant contribution to the solution comes from $Y_0(\lambda r)$, the Bessel function of the Neumann type. To Y_0 one could add const $\times J_0$, but without closing off the region bounded by the corner, the constant remains undetermined. In the solutions to be given, such a constant would appear in the coefficient of the polynomial type terms and, for simplicity, we set the constant equal to zero,

Using the method of images we find for a 90° corner

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') = -\pi(&Y_0\{\lambda[(x-x')^2 + (y-y')^2]^{1/2}\} \\ &- Y_0\{\lambda[(x-x')^2 + (y+y')^2]^{1/2}\} - Y_0\{\lambda[(x+x')^2 \\ &+ (y-y')^2]^{1/2}\} + Y_0\{\lambda[(x+x')^2 + (y+y')^2]^{1/2}\}). \end{aligned} \tag{10}$$

Substitution of result (10) into Eq. (8) and a tedious but straightforward calculation gives, for $\psi = \psi_0 + \chi$, where

$$\chi = \chi^{(1)} + \chi^{(2)},$$

$$\begin{aligned} \chi^{(1)} = (\lambda^2\psi_0/\pi)\{&[-2(\ln(\lambda/2) + \gamma) + 3]xy - xy\ln(x^2 + x^2) \\ &- x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)\}, \end{aligned} \tag{11}$$

$$\begin{aligned} \chi^{(2)} = (\lambda^4\psi_0/12\pi)\{&-[-2(\ln(\lambda/2) + \gamma) + 11/3]xy(x^2 + y^2) \\ &+ xy(x^2 + y^2)\ln(x^2 + y^2) \\ &+ x^4\tan^{-1}(y/x) + y^4\tan^{-1}(x/y)\}. \end{aligned} \tag{12}$$

The Bessel functions of Eq. (10) have been expanded to quadratic order in x and y. γ is Euler's constant $= 0.5772$. It is straightforward, using Eqs. (11) and (12), to show that the Helmholtz equation is indeed satisfied and that for $\psi_0 \neq 0$, $\partial^2\psi/\partial x\,\partial y$, diverges logarithmically at the corner, i.e., as x and $y \to 0$.

In analogous fashion using a Green's function obtained by the image technique we find a solution for a 45° corner. With $\psi = \psi_0 + \chi$ and χ only calculated to lowest order,

$$\begin{aligned} \psi = \psi_0 + \frac{\lambda^2\psi_0}{\pi}\Bigg[&x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y} \\ &- \frac{1}{4}\int_{-x+y}^{x-y}\zeta\left(\tan^{-1}\frac{2y+\zeta}{\zeta} + \tan^{-1}\frac{2y-\zeta}{\zeta}\right)d\zeta \\ &+ \frac{1}{2}\int_{-x-y}^{-x+y}\zeta\left(\tan^{-1}\frac{2x+\zeta}{\zeta} - \tan^{-1}\frac{2y+\zeta}{\zeta}\right)d\zeta\Bigg]. \end{aligned} \tag{13}$$

In this case, we use as coordinates, $(x,\eta)=(x^1,x^2)$, where $\eta=(x+y)/\sqrt{2}$.

It follows easily that $g^{12}=1/\sqrt{2}$ and

$$\frac{\partial^2\psi}{\partial x^1\,\partial x^1}=\frac{\partial^2\psi}{\partial x\,\partial\eta}=\sqrt{2}\left(\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial^2\psi}{\partial y^2}\right). \tag{14}$$

Using the results (13) in Eq. (14), we find at the corner that

$$\frac{\partial^2 B_z}{\partial x^1\,\partial x^2}=\frac{-\sqrt{2}\lambda^2 B_z}{2},$$

which explicitly verifies Eq. (7), and which yields a well-behaved $\partial^2 B_z/\partial x^1\,\partial x^2$ for this case.

IV. CYLINDERS AND TOROIDS WITH RECTANGULAR CROSS SECTION

For a region completely enclosed by a conducting shell, we obtain the Green's function in terms of a sum of normalized eigensolutions of the homogeneous problem, rather than by using the method of images as in Sec. III.

The analysis of this section was also carried out for a cylinder of circular cross section and we obtained results in exact agreement with the solutions for arbitrary λ given in Ref. 3.

A. Cylinders

To obtain the boundary condition on ψ at the conducting wall bounding a straight cylinder of arbitrary cross section oriented along the z axis, we write out Eq. (3) explicitly where $\mathbf{a}=\hat{\mathbf{z}}$ and restrict our considerations to z-independent solutions. We find $\mathbf{B}=\nabla\psi\times\hat{\mathbf{z}}+\hat{\mathbf{z}}\lambda\psi$ and that $\mathbf{B}\cdot\hat{\mathbf{n}}=\nabla\psi\cdot(\hat{\mathbf{z}}\times\hat{\mathbf{n}})$. Thus, the requirement $\mathbf{B}\cdot\hat{\mathbf{n}}=0$ implies the constancy of ψ on the boundary.

Confining our attention to a rectangular cross section, we choose the z axis to coincide with one corner of the cylinder and take $\psi=\psi_0=\text{const}$ at the conducting boundary.

For $\psi_0=0$, we obtain the eigenvalues and normalized eigensolutions of Eq. (2),

$$\lambda_{mn}^2=\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2};\quad m,n=1,2,\ldots, \tag{15}$$

$$U_{mn}=\frac{2}{\sqrt{ab}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{16}$$

where the sides of the rectangular cross section in the x and y directions are a and b, respectively. Using Eq. (8) where

$$G(\mathbf{x},\mathbf{x}')=-4\pi\sum_{m,n}\frac{U_{mn}(\mathbf{x})U_{mn}(\mathbf{x}')}{\lambda^2-\lambda_{mn}^2}, \tag{17}$$

and with $\lambda\neq\lambda_{mn}$ for any m and n, we obtain, as a solution to Eq. (2),

$$\begin{aligned}\psi=\psi_0+\frac{4\lambda^2\psi_0 a^2}{\pi}\sum_{n=0}^{\infty}(2n+1)^{-1}\sin\frac{(2n+1)\pi y}{b}\\ \times[\lambda^2a^2-(2n+1)^2\pi^2a^2/b^2]^{-1}[\cos\{(\tfrac{1}{2}-x/a)\\ \times[\lambda^2a^2-(2n+1)^2\pi^2a^2/b^2]^{1/2}\}\\ \times(\cos\{\tfrac{1}{2}[\lambda^2a^2-(2n+1)^2\pi^2a^2/b^2]^{1/2}\})^{-1}-1]\end{aligned} \tag{18}$$

In obtaining the final form of Eq. (18), we have used the result[17]

$$\begin{aligned}\sum_{n=0}^{\infty}[(2n+1)^2a^2-b^2]^{-1}\cos(2n+1)x\\ =\frac{\pi}{4ab}\sin\left[\left(\frac{\pi}{2}-x\right)\frac{b}{a}\right]\sec\frac{\pi b}{2a},\end{aligned} \tag{19}$$

which holds for $0\leq x\leq\pi$.

Equation (18) is the solution of Eq. (2) for λ not equal to one of the eigenvalues of Eq. (15). If $\psi_0=0$, then $\lambda=\lambda_{mn}$ for some m and n, and the solution of Eq. (2) is, of course, just $\lambda=\text{const}\times U_{mn}$.

Figure 1 gives sample field lines for a solution given by Eq. (18).

B. Toroids

The rectangular toroid is the region defined in cylindrical coordinates by $R_0\leq r\leq R_b$, $0\leq\theta\leq 2\pi$, and $0\leq z\leq a$. Choosing $\mathbf{a}=\hat{\mathbf{z}}$ and restricting our attention to azimuthally independent ψ, we obtain, using Eq. (3),

$$\mathbf{B}=\nabla\psi\times\hat{\mathbf{z}}+\hat{\mathbf{z}}\lambda\psi+\frac{1}{\lambda}\nabla\left(\frac{\partial\psi}{\partial z}\right),$$

or in component form

$$B_r=\frac{1}{\lambda}\frac{\partial^2\psi}{\partial r\,\partial z};\quad B_\theta=-\frac{\partial\psi}{\partial r};\quad B_z=-\frac{1}{\lambda r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right). \tag{20}$$

To obtain the appropriate boundary condition on ψ we may proceed as follows. We let $r_0(z)=\text{const}$ denote the curve of intersection for a poloidal plane with a toroid of *arbitrary* cross section. Then, we note that the normal to the conducting surface is proportional to $\hat{\mathbf{r}}-\partial r_0/\partial z\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ is the unit radial vector. Expressing $\mathbf{B}\cdot\hat{\mathbf{n}}=0$ at the boundary in terms of ψ, using the com-

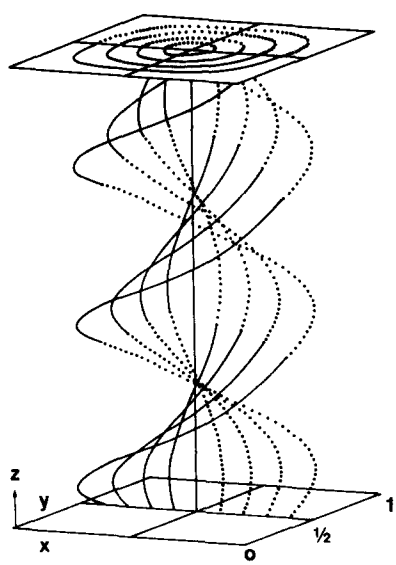

FIG. 1. Sample B-field lines satisfying $\nabla\times\mathbf{B}=3\mathbf{B}$ in a cylinder of square cross section having unit area. Field lines are dotted when $y>0.5$. The intersection curves of a $z=\text{const}$ plane and the magnetic surfaces for the chosen field lines are shown on the upper plane.

ponents of **B** given by Eq. (20), leads to $r\partial\psi/\partial r = \text{const}$ on the boundary. For the rectangular cross section this implies that if we set $\psi = \chi + \psi_0 \ln(r/r_0)$, where ψ_0 and r_0 are constants, then χ will satisfy the inhomogeneous Helmholtz equation

$$(\nabla^2 + \lambda^2)\chi = -\lambda^2 \psi_0 \ln(r/r_0), \tag{21}$$

with mixed homogeneous boundary conditions: $\chi = 0$ at $z = 0$ and $z = a$, and also $\partial\chi/\partial r = 0$ at $r = R_0$ and $r = R_b$.

We construct a Green's function from the eigensolutions to Eq. (21) with $\psi_0 = 0$.

Let

$$f_0(\rho r) = Y_1(\rho R_0) J_0(\rho r) - J_1(\rho R_0) Y_0(\rho r). \tag{22}$$

The roots ρ_m are determined from the equation

$$\frac{1}{\rho}\frac{df_0}{dr}\bigg|_{r=R_b} = 0, \tag{23}$$

with the eigenvalues λ_{mn} given by

$$\lambda_{mn}^2 = \rho_m^2 + n^2\pi^2/a^2 \tag{24}$$

The normalized eigensolutions are

$$U_{mn}(r,z) = A_m f_0(\rho_m r)\sin(n\pi z/a), \tag{25}$$

with

$$A_m = [(\pi a/2)(R_b^2 f_0^2(\rho_m R_b) - 4/\pi^2\rho_m^2)]^{-1/2}. \tag{26}$$

Using these solutions as an orthonormal basis for the Green's function of the form specified by Eq. (17), we then obtain, with $\lambda \neq \lambda_{mn}$ for any m and n,

$$\psi(r,z) = \psi_0 \ln\frac{r}{r_0} + \pi a\lambda^2 \sum_{m=1}^{\infty} \frac{A_m^2 f_0(\rho_m r)}{\rho_m^2(\lambda^2 - \rho_m^2)} \times (f_0(\rho_m R_b) + 2/\pi\rho_m R_0)(\cos[(a/2 - z)(\lambda^2 - \rho_m^2)^{1/2}] \times \{\cos[(a/2)(\lambda^2 - \rho_m^2)^{1/2}]\}^{-1} - 1) \tag{27}$$

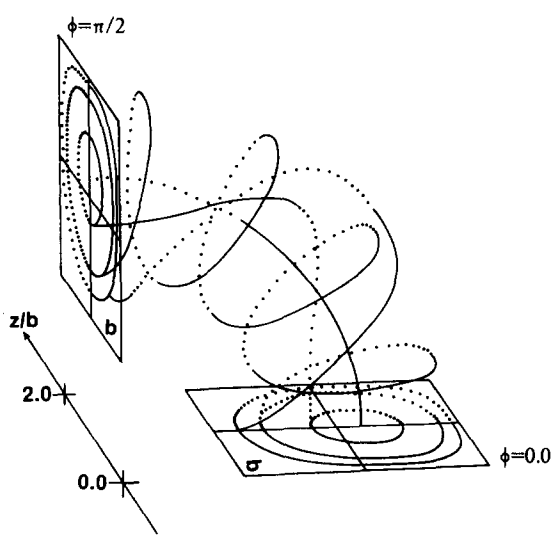

FIG. 2. Sample B-field lines satisfying $\nabla \times \mathbf{B} = 3\mathbf{B}$ in a toroid of square cross section having area $4b^2$. The aspect ratio is 2 and only a 90° section of the toroid is shown. The field lines are dotted when $(z/b) > 1.0$. The curves of intersection of the $\psi = 0$ and $\psi = \pi/2$ planes with the magnetic surfaces for the chosen field lines are shown. The toroidal component of the B field reverses at $(r/b) = 1.3$ and $(r/b) = 2.9$ for $(z/b) = 1.0$.

where we have again used Eq. (19) and a Wronskian relation for the Bessel functions. [Note that $\psi(r,z) = \psi_0 \ln(r/r_0) + (\lambda^2/4\pi)\psi_0 \int G(\mathbf{x},\mathbf{x}')\ln(r'/r_0)d^2x'$, where $d^2x' = dr'dz'$ and the domain of integration is the rectangular cross section.] Again if $\psi_0 = 0$, then $\lambda = \lambda_{mn}$ for some m and n, and the solution becomes $\psi = \text{const} \times U_{mn}$. Figure 2 shows sample field lines for a solution given by Eq. (27).

This force-free problem may also be formulated in terms of the familiar Grad–Shafranov equation by defining the flux function $\Phi = r\partial\psi/\partial r$. The Helmholtz equation for ψ then becomes the Grad–Shafranov equation for Φ. The boundary conditions correspond to $\Phi = \text{const}$ on the boundary. Similar comments can be made for the straight cylinder.

V. SUMMARY

In the neighborhood of 45° and 90° corners, and within cylinders and tori of rectangular cross sections, we have obtained the scalar Helmholtz potential satisfying boundary conditions relevant to force-free magnetic fields. Differentiation of the potential then gave the fields explicitly. Both continuous and discrete eigenvalues were considered.

ACKNOWLEDGMENTS

The authors express their gratitude to Don Baker, Ronald Moses, and Harvey Rose for many productive discussions.

Certain results obtained herein were checked with MACSYMA which is supported, in part, by the U.S. Department of Energy under Contract E(11-1)-3070 and by the National Aeronautics and Space Administration under Grant NSG 1323.

This work was supported by the U.S. Department of Energy. One of us (S.N.R.) expresses gratitude to Brigham Young University for leave support.

[1] E. P. Butt and A. A. Newton, in *Pulsed High Beta Plasmas*, edited by D. E. Evans (Pergamon, Oxford, 1976), p. 419.
[2] A. Buffa, S. Costa, R. De Angelis, L. Giudicotti, C. W. Gowers, G. F. Nalesso, S. Ortolani, M. Puiatti, P. Scarin, and M. R. C. Watts, in *Ninth European Conference on Controlled Fusion and Plasma Physics* (Culham Laboratory, Oxford, United Kingdom, 1979), p. 544.
[3] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).
[4] L. Turner and J. P. Christiansen, Phys. Fluids (to be published).
[5] G. Vahala, presented at the Sherwood Meeting on Theoretical Aspects of Controlled Thermonuclear Research, Mount Pocono, Pennsylvania (1979) paper 2B44; C. Chu and M. Chu, Bull. Am. Phys. Soc. **24**, 1036 (1978).
[6] R. N. Sudan, Phys. Rev. Lett. **42**, 1277 (1979).
[7] S. Chandrasekhar and P. C. Kendall, Astrophys. J. **126**, 457 (1957).
[8] N. M. Wigley, J. Math. Mech. **13**, 549 (1964).
[9] R. J. Riddell, J. Comput. Phys. **31**, 21 (1979).
[10] R. J. Riddell, J. Comput. Phys. **31**, 42 (1979).
[11] R. E. Kleinman, Z. Angew. Math. Mech. **58**, T379 (1978).
[12] H. M. Nussenzveig, Philos. Trans. R. Soc. London Ser. A **252**, 1 (1959).
[13] F. Oberhettinger, Commun. Proc. Appl. Math. **7**, 551 (1954).
[14] A. W. Maue, Z. Phys. **126**, 601 (1949).

[15]J. Meixner, Ann. Phys. **6**, 1 (1949).
[16]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New York, 1953), Vol. 1, Chap. 7.
[17]E. R. Hansen, *A Table of Series and Products* (Prentice–Hall, Englewood Cliffs, N. J., 1975), p. 243, No. 17.3.13.