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Numerical Investigations of Solitons in a Long Nonneutral Plasma

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Abstract. For realistic density profiles we have obtained two-dimensional soliton solutions numerically for a cold-fluid (CF) model and as a BGK wave with finite temperature. The CF soliton profile agrees well with an earlier analytic approximation (K. C. Hansen, Master's Thesis, BYU, 1995), and for small temperatures($<$0.1eV) the profiles for the CF soliton and the BGK soliton agree as well. The effects of temperature are evident in the propagation velocities and differences in the models are also evident for large amplitude solitons.

INTRODUCTION

Solitons in nonneutral plasmas have been studied using simulations by Neu and Morales [1] in slab geometry and by Hansen [2] in cylindrical geometry. Solitons have also been observed experimentally by Moody and Driscoll [3] and by Hart [4]. Solitons in nonneutral plasmas offer the potential for careful study of nonlinear waves and two-dimensional soliton type structures in a system where they live and interact for a substantial duration of time.

SOLITONS IN THE COLD-FLUID MODEL

The familiar equations for the fluid density $n(\mathbf{x},t)$, velocity $\mathbf{v}(\mathbf{x},t)$, and electrostatic potential $\phi(\mathbf{x},t)$ are:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0,$$
$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E} + \frac{q}{mc}\mathbf{v} \times \mathbf{B},$$
$$\nabla^2 \phi = -4\pi q n.$$

We make the following assumptions:

CP498, *Non-Neutral Plasma Physics III*, edited by John J. Bollinger, et al.

$$v_r = 0, \qquad \frac{\partial}{\partial \phi} = 0, \quad \text{(no } \phi \text{ dependence)}$$

$$\mathbf{v} = r\omega_0(r)\hat{\boldsymbol{\phi}} + v_z\hat{\mathbf{z}}, \qquad \mathbf{B} = B_0\hat{\mathbf{z}},$$

where B_0 denotes a constant magnetic field. We then simplify to find

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv_z) = 0, \tag{1}$$

$$\frac{\partial v_z}{\partial t} + v_z\frac{\partial v_z}{\partial z} = -\frac{q}{m}\frac{\partial \phi}{\partial z}, \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) + \frac{\partial^2 \phi}{\partial z^2} = -4\pi q n. \tag{3}$$

We now transfer focus to the moving frame of the soliton. Assume the soliton is moving to the right with a velocity u and let ζ denote the coordinate in the moving frame along the direction of the magnetic field. Then

$$\zeta = z - ut, \quad v_z = v_\zeta + u, \quad n(r,z,t) = n(r,\zeta(z,t)), \quad \text{similarly for } v_\zeta \text{ and } \phi.$$

Equation (1) becomes

$$\frac{\partial}{\partial \zeta}(nv_\zeta) = 0 \Rightarrow nv_\zeta = \text{const (in } \zeta).$$

We assume the boundary conditions that $v_z = 0$ when z (or ζ) $\to \infty$ and that also that $n(r,\zeta) \to n_0(r)$ and $\phi(r,\zeta) \to \phi_0(r)$. Thus

$$n(r,\zeta)v_\zeta(r,\zeta) = -un_0(r). \tag{4}$$

Equation (2) becomes

$$\frac{\partial}{\partial \zeta}(\frac{1}{2}v_\zeta^2 + \frac{q}{m}\phi) = 0,$$

$$\Rightarrow \quad \frac{1}{2}v_\zeta^2(r,\zeta) + \frac{q}{m}\phi(r,\zeta) = \frac{1}{2}u^2 + \frac{q}{m}\phi_0(r). \tag{5}$$

We now solve for v_ζ and $n(r,\zeta)$ to find

$$v_\zeta(r,\zeta) = -u(1 - 2\bar{\psi}(r,\zeta))^{\frac{1}{2}} \quad \text{and} \tag{6}$$

$$n(r,\zeta) = n_0(r)/(1 - 2\bar{\psi}(r,\zeta))^{\frac{1}{2}}, \tag{7}$$

where $\bar{\psi}(r,\zeta) = q(\phi(r,\zeta) - \phi_0(r))/mu^2$. From Poisson's Equation (3) we then find

$$\nabla^2\bar{\psi}(r,\zeta) = \frac{\omega_{e0}^2(r)}{u^2}\left[1 - (1 - 2\bar{\psi}(r,\zeta))^{-\frac{1}{2}}\right] \tag{8}$$

Approximate analytic solution to Eq. (8)

Following Hansen [2], assume $|\bar{\psi}| \ll 1$ so that $1/\sqrt{1-2\bar{\psi}} \simeq 1 + \bar{\psi} + \frac{3}{2}\bar{\psi}^2$. Substituting in Equation (8) find:

$$\nabla^2 \bar{\psi} = -\frac{\omega_{e0}^2(r)}{u^2}(\bar{\psi} + \frac{3}{2}\bar{\psi}^2). \tag{9}$$

Let $\bar{\psi}(r,\zeta) = R(r)f(\zeta)$, where we assume a knowledge of $R(r)$; with boundary conditions $R(0) = 1$ and $R(r_{\text{wall}}) = 0$. Substitute into Equation (9), multiply through by $rR(r)$ and then integrate from 0 to r_{wall}. We obtain the following equation:

$$\frac{d^2 f}{d\zeta^2} = (\beta^2 - \frac{\eta^2 \omega_e^2(0)}{u^2})f - \frac{3}{2}\frac{\alpha\omega_e^2(0)}{u^2} f^2, \tag{10}$$

where α, β^2, and η^2 are defined below. Let $\|g\|^2 \equiv \int_0^{r_{\text{wall}}} rg^2\, dr$. Then

$$\beta^2 = -\frac{1}{\|R\|^2}\int_0^{r_{\text{wall}}} \frac{d}{dr}(r\frac{dR}{dr})R\, dr = (\frac{\|\frac{dR}{dr}\|}{\|R\|})^2. \tag{11}$$

$$\eta^2 = \frac{1}{\omega_e^2(0)\|R\|^2}\int_0^{r_{\text{wall}}} \omega_{e0}^2(r) r R^2\, dr, \tag{12}$$

$$\alpha = \frac{1}{\omega_e^2(0)\|R\|^2}\int_0^{r_{\text{wall}}} \omega_{e0}^2(r) r R^3\, dr. \tag{13}$$

Equation (10) we recognize as the first integral of the KdV equation and is readily verified to have the soliton solution:

$$f(\zeta) = A\text{sech}^2(\frac{\zeta}{\Delta}), \quad \text{where} \tag{14}$$

$$A = \frac{u^2}{\alpha\omega_e^2(0)}(\beta^2 - \frac{\eta^2\omega_e^2(0)}{u^2}), \tag{15}$$

$$\frac{1}{\Delta} = \frac{1}{2}(\beta^2 - \frac{\eta^2\omega_e^2(0)}{u^2})^{-\frac{1}{2}}. \tag{16}$$

Numerical solution to Eq. (8)

We assume a tensor product spline approximation for $\bar{\psi}(r,\zeta) = \sum_{i,j}\psi_i(r)\psi_j(\zeta)$ and take a Galerkin approximation to Eq. (8). We assume symmetry about $\zeta = 0$ and $r = 0$ and thus require $\partial\bar{\psi}/\partial\zeta(r,0) = 0$ and $\partial\bar{\psi}/\partial r(0,\zeta) = 0$. Furthermore we take $\bar{\psi}(r_{\text{wall}},\zeta) = 0$ and $\bar{\psi}(r,\zeta_{\text{wall}}) = 0$, where ζ_{wall} is arbitrary but taken large enough to approximate ∞. The unperturbed radial density profile is taken to be of the form

$$n_0(r) = n_{00} \exp\left[-\left(\frac{r}{r_p}\right)^{\mu}\right].$$

Due to the nonlinear nature of Eq. (8), the numerical solution is obtained via Picard iteration. Let superscripts denote the iteration index, then symbolically,

$$\bar{\psi}^{(n+1)}(r,\zeta) = (\nabla^2)^{-1} f(\bar{\psi}^{(n)})$$

where $(\nabla^2)^{-1}$ represents the inverse of the matrix operator obtained in the Galerkin procedure to represent the Laplacian and $f(\bar{\psi}^{(n)})$ represents the right hand side of Eq. (8). An efficient algorithm is devised that converges rapidly without underrelaxation: compute $\bar{\psi}^{(n+1)}(r,\zeta)$, then adjust u according to

$$u^{(n+1)} = u^{(n)} \sqrt{\int \bar{\psi}^{(n)} \bar{\psi}^{(n+1)}\, da \Big/ \int (\bar{\psi}^{(n)})^2\, da},$$

The amplitude $\bar{\psi}(0,0)$ is fixed, $0 < \bar{\psi}(0,0) < 0.5$, and thus after finding $u^{(n+1)}$ the coefficients are adjusted to satisfy this constraint which then give us a new $\bar{\psi}^{(n)}$. Then cycle again until convergence is achieved.

As an example we choose $r_{\text{wall}} = 4.0$cm and $\zeta_{\text{wall}} = 30.0$cm. For the density profile we choose $r_p = 2.0$cm, $\mu = 4.5$, and $n_{00} = 4.0 \times 10^6 \text{cm}^{-3}$. We choose $\bar{\psi}(0,0) = 0.4$ and then find the numerical solution to Eq. (8). Figure 1 shows the two-dimensional soliton function $\psi(r,\zeta)$ for $\zeta > 0$.

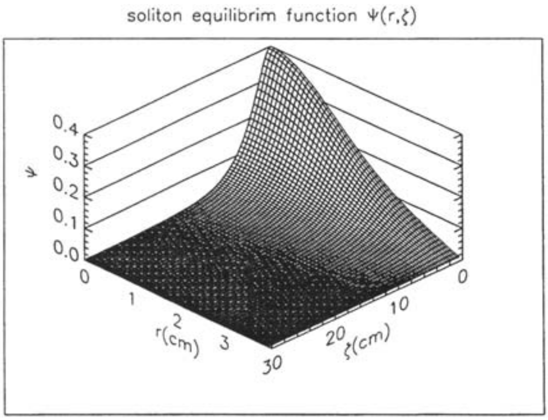

FIGURE 1. Potential soliton for $\zeta > 0$

Figure 2 compares the numerical solution to Eq. (8) to the approximate analytic solution as given in Eqs. (14)-(16) for $r = 0$.

Using Eqs. (11)-(13) with the function $R(r)$ replaced by $\bar{\psi}(r,\zeta)$ and then choosing an average over ζ, we estimate $\alpha = 0.52$, $\beta^2 = 0.41$, and $\eta^2 = 0.68$ for the soliton computed above. With these values Eq. (15) gives $u/(r_p\omega_e(0)) = 0.74$ whereas the numerical solution has $u/(r_p\omega_e(0)) = 0.80$.

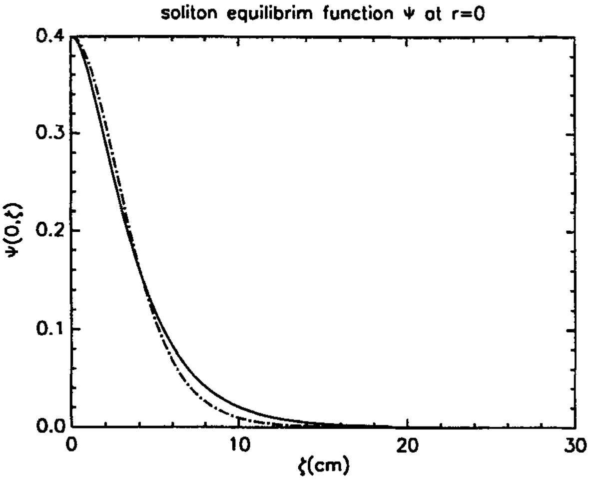

FIGURE 2. Comparison of numerical(solid), Eq. (8), and analytic(dashed), Eq. (14), solitons for $\zeta > 0$

BGK WAVE SOLITON

To find the appropriate nonlinear BGK wave we need the distribution function $f(r, \zeta, v)$ where

$$n(r, \zeta) = \int_{-\infty}^{\infty} f(r, \zeta, v)\, dv.$$

We obtain this distribution function by assuming that far away from the soliton, $\zeta \to \infty$, the distribution function should be a Boltzmann distribution centered about $-u$, $f \sim \exp[-\frac{(u+v_\infty)^2}{2v_T^2}]$, where v_T is the thermal velocity given by $\sqrt{kT/m}$ and $v_\infty(v, \zeta)$ is defined below. In other words, we inject a Boltzmann distribution toward the soliton from the right. This distribution function we get everywhere by noting that the distribution function is preserved along particle orbits and using conservation of energy, $\frac{1}{2}mv^2 + q\phi(r, \zeta) = \frac{1}{2}mv_\infty^2 + q\phi_0(r)$. Thus we find

$$f(r, \zeta, v) \propto \exp\left[-\frac{1}{2v_T^2}(u \pm [v^2 + 2u^2\bar{\psi}(r, \zeta)]^{\frac{1}{2}})^2\right],$$

where the $\pm$ must be decided according to whether the particle at ∞ has positive or negative velocity. This distribution function is normalized by demanding the $n(r, \zeta) \to n_0(r)$ as $\zeta \to \infty$. The net result of this procedure is the following density distribution which then goes into Poisson's equation. The overbars on the velocities denotes that they have been scaled by u.

$$n(r, \zeta) = \frac{n_0(r)}{\bar{v}_T}\sqrt{\frac{2}{\pi}}\left[2(1 + \text{erf}(\frac{1}{\bar{v}_T\sqrt{2}})) - \text{erf}\left(\frac{1 - \bar{v}_0(r)}{\bar{v}_T\sqrt{2}}\right) - \text{erf}\left(\frac{1 + \bar{v}_0(r)}{\bar{v}_T\sqrt{2}}\right)\right]^{-1}$$

$$\times \left\{ \int_0^{\infty} \exp \left[\frac{-1}{2} \left(1 - [\bar{v}^2 + 2\bar{\psi}(r,\zeta)]^{\frac{1}{2}} \right)^2 \right] d\bar{v} \right.$$
$$+ \int_0^{\bar{v}_0(r,\zeta)} \exp \left[\frac{-1}{2} \left(1 - [\bar{v}^2 + 2\bar{\psi}(r,\zeta)]^{\frac{1}{2}} \right)^2 \right] d\bar{v}$$
$$\left. + \int_{\bar{v}_0(r,\zeta)}^{\infty} \exp \left[\frac{-1}{2} \left(1 + [\bar{v}^2 + 2\bar{\psi}(r,\zeta)]^{\frac{1}{2}} \right)^2 \right] d\bar{v} \right\} \tag{17}$$

In this expression we use $\bar{v}_0(r) = \sqrt{2\bar{\psi}(r,0)}$ and $\bar{v}_0(r,\zeta) = \sqrt{2|\bar{\psi}(r,0) - \bar{\psi}(r,\zeta)|}$. The right-hand side of Eq. (8) becomes then

$$\frac{\omega_{e0}^2(r)}{u^2}(1 - n(r,\zeta)/n_0(r))$$

With the above right-hand side we solve Eq. (8) for the BGK solution. Under-relaxation is now required for convergence. As an example we choose $\bar{\psi}(0,0) = 0.1$ and $T = 1.0$eV. Figure 3 compares this soliton with the analytic approximation. The corresponding soliton velocities are $u/(r_p\omega_e(0)) = 0.67$ for the analytic approximation, 0.68 for the CF numerical solution, and 0.80 for the BGK soliton.

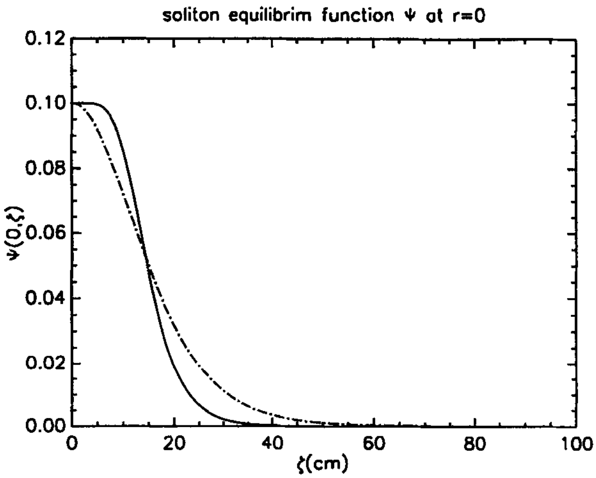

FIGURE 3. Comparison of a BGK numerical soliton(solid) and the analytic approximation (dashed) for $\zeta > 0$

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