Criticality and bifurcation in the gravitational collapse of a self-coupled scalar field

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We examine the gravitational collapse of a nonlinear σ model in spherical symmetry. There exists a family of continuously self-similar solutions parametrized by the coupling constant of the theory. These solutions are calculated together with the critical exponents for black hole formation of these collapse models. We also find that the sequence of solutions exhibits a Hopf-type bifurcation as the continuously self-similar solutions become unstable to perturbations away from self-similarity. [S0556-2821(97)00220-8]

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I. INTRODUCTION

The last few years have seen a renewed interest in gravitational collapse, particularly with regard to what numerical relativity is able to teach us about the general phenomenon. Choptuik's initial discovery of criticality and other behavior strikingly similar to that seen in statistical mechanical systems has suggested a deep property of the gravitational field equations.

A good deal of recent work has shown the existence of collapse solutions exactly at the threshold of the formation of a black hole for a variety of matter fields. These include both real and complex scalar fields [1,2], vacuum gravity [3], a perfect fluid [4], and an axion-dilaton model from low-energy string theory [5]. In each of these models, some common behavior emerges. For example, the growth of the black hole mass just off threshold is described by a power-law relation

$$M_{\rm BH}(p)\propto\begin{cases}0, & p\leqslant p^*,\\ (p-p^*)^\gamma, & p>p^*,\end{cases} \tag{1}$$

where p is any parameter which can be said to characterize the strength of the initial conditions and p^* is the threshold value, i.e., the value for the critical solution. The critical exponent γ is universal within a particular class of matter fields. For example, $\gamma\approx0.37$ for the real scalar field, $\gamma\approx0.36$ for perfect fluid collapse, and $\gamma\approx0.2641066$ for the axion-dilaton (axiodil) system [6]. The solutions may also exhibit an echoing behavior in that the features of the exactly critical solution are repeated on ever decreasing time and length scales. This self-similar behavior of the solutions has been found in both discrete and continuous versions. In particular, for vacuum gravity, discrete self-similarity, and echoing are observed, while in fluid collapse, continuous self-similarity with no echoing emerges. In scalar field collapse, both types have been shown to be present.

The main results of this paper unify the discrete vs continuous self-similarity known in the above models. Specifically, we examine a particular nonlinear σ model which smoothly interpolates between the complex scalar field model [2] and the axion-dilaton model [5] as the value of a certain dimensionless coupling constant κ varies. We find a family of continuously self-similar solutions parametrized by κ. Using linear perturbation theory, we study the stability of these solutions, and find that the sequence of solutions undergoes a bifurcation at a particular value, $\kappa\approx0.0754$, where the continuously self-similar solutions go from being stable to being unstable. The free complex scalar field ($\kappa=0$) is found to be on the unstable side of this bifurcation, while the axion-dilaton field ($\kappa=1$) is on the stable side. This is in agreement with previous results for both of these matter fields. Further, we find that for negative values $\kappa\lesssim-0.28$, the self-similar solutions become ever more unstable hinting at the possibility of further bifurcations and more complicated dynamics. Since we work only in perturbation theory, we cannot confirm these latter possibilities here, but our results are somewhat suggestive of the existence of more exotic behavior than may have previously been observed. For this reason, full scale numerical work on these models would undoubtedly be a very enlightening undertaking.

Prior to our work, Choptuik and Liebling [7,8] studied an apparently different model, namely, Brans-Dicke gravity coupled to a free real scalar field, for various values of the dimensionless Brans-Dicke coupling constant $-3/2<\omega_{\rm BD}<\infty$. They use a spherical collapse code, and their main result is a change of stability at $\omega_{\rm BD}\approx0$. After the continuously self-similar solution was found in the collapse of an axion-dilaton field [5], they realized that it was their more general Brans-Dicke model for a particular value of $\omega_{\rm BD}$. In fact, we find that their Brans-Dicke model is equivalent to some range of our nonlinear σ model ($\infty>\kappa\geqslant0$), with $\omega_{\rm BD}=\infty$ corresponding to the free complex scalar field and $\omega_{\rm BD}=-11/8$ corresponding to the axion-dilaton field. The bifurcation in stability we find here in linear perturbation theory then coincides with the change of stability previously found by Choptuik and Liebling; in particular, we agree with their result that, for axion-dilaton collapse, the continuously

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self-similar critical solution is stable and appears to be the attractor. The range $\kappa<0$ is not present in the Brans-Dicke model, however.

Hamade, Horne, and Stuart [6] gave both full numerical results and perturbation results on axion-dilaton collapse in spherical symmetry. Our results in linear perturbation theory for $\kappa=1$ agree with theirs with regard to real modes and critical exponents. They also find by their numerical collapse code that the continuously self-similar critical solution is stable and is the attractor, in agreement with the work of Choptuik and Liebling; this is also consistent with our results below on the complex modes for $\kappa=1$. This change in stability which occurs near $\omega_{\mathrm{BD}}\approx 0$ appears to be a ''Hopf bifurcation,'' as it is known in the dynamics literature [9].

The outline of this paper is as follows. In Sec. II, we give general arguments on what kinds of self-coupling of a scalar field may show new critical phenomena in gravitational collapse; likely candidates are the nonlinear σ models. In Sec. III, we introduce the particular nonlinear σ model to be studied in this paper, and discuss its relationship to matter fields which have been studied previously. Section IV introduces the equations of motion, derives their form in the presence of a continuous self-similarity, and sketches our numerical approach to solving them. Section V discusses the perturbation of the continuously self-similar solutions and the question of stability of these solutions. Section VI presents our results and conclusions. Appendix A summarizes our equations in detail. Finally, in Appendix B we provide a short bibliography on the related issue of nonlinear σ models in flat spacetime.

II. CRITICAL BEHAVIOR AND SELF-INTERACTION

With the important exception of [3], all the work so far on critical phenomena in gravitational collapse has assumed spherical symmetry. In spherical symmetry, there is no gravitational collapse without matter, from Birkhoff's theorem. Therefore one might expect that critical behavior would depend importantly on the model of the matter. Indeed, the critical phenomenology and exponents differ among matter models such as real scalar field, ideal gas, complex scalar field, axiodil, ... However, studying a real scalar field ϕ, Choptuik found that inclusion of a nonlinear interaction term $V(\phi)$ in the action

$$L_{\mathrm{matter}}=\frac{1}{2}\nabla^{\alpha}\phi\nabla_{\alpha}\phi-V(\phi), \tag{2a}$$

$$V(\phi)\equiv\mu^2\phi^2/2+\lambda\phi^4/4 \tag{2b}$$

made no difference in the critical solution itself or in its phenomenology.

We can understand this result as follows. At least in all known cases, the critical solution is either ''echoing'' (discretely self-similar) or continuously self similar [(CSS) — admitting a homothetic Killing vector field]. In either case, by dimensional analysis, the solution cannot depend on any dimensionful parameters. Here we use dimensional analysis appropriate to classical general relativity, with a unit of length ℓ in some system of units where Newton's gravitational constant $G\equiv 1$. A scalar field ϕ (real or complex) then has dimensions ℓ^0, while a Lagrangian must have units ℓ^{-2}. It follows that the parameters μ and λ above have dimensions different from zero; in particular, μ is just the inverse compton wavelength of the particle. Since these parameters are dimensionful, the critical solution cannot depend on them, consistent with the numerical results.[1]

For this reason we turn attention to a different kind of self-coupling, one which multiplies the kinetic term instead of adding to it. The general form is

$$\frac{1}{2}G_{IJ}(\phi^K)\nabla^{\alpha}\phi^I\nabla_{\alpha}\phi^J \tag{3}$$

where there are now some number N of scalar fields ϕ^I ($I=1\ldots N$), and where G_{IJ} is some function of the fields, fixed once and for all to specify the model. The nonlinear functions G_{IJ} take the form of a Riemannian metric on the internal space of the ϕ^I, the *target space.* Such models are called nonlinear σ models (or ''harmonic map'' models, as discussed by Misner [10]), and much is known about them in high-energy physics, not least because they often appear in the low-energy limit of superstring theory. By dimensional analysis, the scalar fields ϕ^I are of dimension ℓ^0, as is the target space metric G_{IJ}. Therefore any parameters appearing in G_{IJ} may also be taken as dimensionless, and we can expect the critical solution to depend on them.

What is the simplest nonlinear σ model we can study? If $N=1$ then the matter action can be reduced to that of a free field by a field redefinition; a one-dimensional Riemannian space is always flat. So the simplest nontrivial value is $N=2$, wherein the two real scalar fields can be grouped into a single complex scalar field ϕ. For the target space metric, the simplest cases are the spaces of constant curvature, namely the two-sphere, flat two-space, or the two-hyperboloid, all with homogeneous metrics. This is the model we shall study.

III. THE MODEL

We work with a model defined by the action

$$S=\int d^4x\sqrt{-g}\left(R-\frac{2|\nabla F|^2}{(1-\kappa|F|^2)^2}\right). \tag{4}$$

The complex field $F(x^{\mu})$ is a scalar coupled to Einstein gravity with κ a real dimensionless coupling constant:

$$-\infty<\kappa<\infty. \tag{5}$$

The model given by Eq. (4) is a nonlinear σ model. As mentioned above, the target space of the model is a two-dimensional space of constant curvature. The curvature of this internal space is proportional to $-\kappa$ so that the space is hyperbolic for $\kappa>0$ and a two-sphere for $\kappa<0$. For the par-

[1]Choptuik has also tried adding a conformal coupling $\xi R\phi^2$ to the matter Lagrangian. In contrast, ξ is dimensionless, so that that critical solution should depend on it. This point deserves more investigation.

ticular case $\kappa=1$, our model becomes the axion-dilaton (axiodil) field $\tilde{\tau}$ coupled to gravity[2]:

$$F=\frac{1+i\tilde{\tau}}{1-i\tilde{\tau}}. \tag{6}$$

It turns out in quantum field theory that the value $\kappa=1$ is not affected by quantum corrections, as it is protected by extended supersymmetry. For $\kappa=0$ the model (4) reduces (after a further trivial rescaling of the field) to the free complex scalar field coupled to gravity. Thus this general model smoothly interpolates between the two particular matter models that we have already considered. In fact, for $0<\kappa<\infty$ we find that this nonlinear σ model is equivalent to the model of a massless real scalar field coupled to Brans-Dicke theory. Liebling has recently examined this theory using a version of Choptuik's adaptive mesh refinement algorithm. He finds behavior qualitatively similar to that found by Choptuik for the real scalar field [7]. The connection between the two theories can be seen in the relationship between the Brans-Dicke coupling constant [11] $\omega_{\rm BD}$ and our constant κ:

$$\omega_{\rm BD}=-\frac{3}{2}+\frac{1}{8\kappa}, \quad 0\leq\kappa<\infty. \tag{7}$$

This means that the axion-dilaton model ($\kappa=1$) corresponds to $\omega_{\rm BD}=-11/8$, while the free complex scalar field ($\kappa=0$) corresponds to $\omega_{\rm BD}=+\infty$. Also, as $\omega_{\rm BD}\to-3/2^{+}$, we have $\kappa\to+\infty$; however this may be a singular limit of the theory. For $-\infty<\kappa<0$ the model (4) appears not to be equivalent to any Brans-Dicke model; in particular Eq. (7) does not apply. The model behaves in a smooth way as κ passes through zero.

Returning to the model for general real κ, the field equations in covariant form as derived from the action in Eq. (4) are

$$R_{ab}=\frac{1}{(1-\kappa|F|^2)^2}(\nabla_aF\nabla_bF^*+\nabla_aF^*\nabla_bF), \tag{8a}$$

$$\nabla^a\nabla_aF=\frac{-2\kappa F^*}{1-\kappa|F|^2}\nabla_aF\nabla^aF. \tag{8b}$$

In this form, these equations are manifestly invariant under a global U(1) group of transformations, parametrized by a real constant Λ:

$$F'=e^{i\Lambda}F, \quad -\infty<\Lambda<\infty \tag{9}$$

and which leave the metric unchanged.

For $\kappa>0$, this model also has a larger global symmetry not present in general relativity, namely, an SL(2,$\mathbb{R}$) symmetry that acts on F, but leaves the spacetime metric invariant; this is a classical version of the conjectured SL(2,**Z**) symmetry of heterotic string theory called S duality [12]. For the axiodil, $\kappa=1$, this symmetry acts on $\tilde{\tau}$ as

$$\tilde{\tau}\to\frac{a\tilde{\tau}+b}{c\tilde{\tau}+d}, \tag{10}$$

where $(a,b,c,d)\in\mathbb{R}$ with $ad-bc=1$, while leaving $g_{\mu\nu}$ invariant. The corresponding transformation of F for general $\kappa>0$ is

$$F\to\frac{1}{\sqrt{\kappa}}\frac{\alpha\sqrt{\kappa}F+\beta}{\beta^*\sqrt{\kappa}F+\alpha^*}, \tag{11}$$

where $(\alpha,\beta)\in\mathcal{C}$ with $|\alpha|^2-|\beta|^2=1$. The transformations of Eq. (9) form a special case.

In the case where $\kappa=0$, the larger global symmetry consists of rigid translations and rotations in the two flat directions of the target space, the group E(2). Finally, for $\kappa<0$, the group of motions on the two-sphere, SO(3), constitutes the larger global symmetry.

IV. THE CONTINUOUSLY SELF-SIMILAR SOLUTIONS

We briefly review the process of setting up the equations such that they are compatible with a continuous self-similarity. To begin, we work in spherical symmetry so the metric can be taken as

$$ds^2=(1+u)[-b^2dt^2+dr^2]+r^2d\Omega^2, \tag{12}$$

where $b(t,r)$ and $u(t,r)$ are the metric functions. This is essentially Choptuik's metric in radial gauge with some minor redefinitions. The timelike coordinate t is chosen so that the collapse on the axis of spherical symmetry happens at $t=0$ and the metric is regular for $t<0$.

We are interested in finding collapsing solutions of our model. In particular we ask whether, as in the complex scalar, axiodil, and fluid collapse cases, there exist continuously self-similar (CSS) solutions to these equations for arbitrary κ. That a spacetime admits a continuous self-similarity is described covariantly by the existence of a homothetic Killing vector field ξ satisfying

$$\mathcal{L}_\xi g_{ab}=\nabla_a\xi_b+\nabla_b\xi_a=2g_{ab}, \tag{13}$$

where $\mathcal{L}$ denotes the Lie derivative. A coordinate system better adapted to our assumption of self-similarity involves the coordinates $z=-r/t$ and $\tau=\ln|-t|$. In these coordinates, the metric takes the form

$$ds^2=e^{2\tau}\{(1+u)[-(b^2-z^2)d\tau^2+2zd\tau dz+dz^2]+z^2d\Omega^2\}, \tag{14}$$

and the homothetic Killing vector is then expressed simply in these coordinates as

$$\xi^a\partial_a=\partial_\tau. \tag{15}$$

In these coordinates, Eqs. (8) can be written as

$$zu'-\dot{u}=\frac{z(u+1)}{\rho^2}[F'(zF'-\dot{F})^*+F'^*(zF'-\dot{F})], \tag{16a}$$

[2]Notation: We use $\tilde{\tau}$ here for the axiodil field, instead of τ as in [5], to avoid confusion with logarithmic time coordinate τ below.

$$u'=\frac{z(u+1)}{\rho^2}\left[|F'|^2+\frac{1}{b^2}|zF'-\dot{F}|^2\right]-\frac{u(u+1)}{z}, \tag{16b}$$

$$b'=\frac{ub}{z}, \tag{16c}$$

$$0=F''\Delta-\ddot{F}+2z\dot{F}'+F'\left[z(u-2)+\frac{b^2}{z}(u+2)-z\frac{\dot{b}}{b}\right] \tag{16d}$$

$$+\dot{F}\left(\frac{\dot{b}}{b}+1-u\right)+\frac{2\kappa}{\rho}F^*(\Delta F'^2+2zF'\dot{F}-\dot{F}^2), \tag{16e}$$

where the overdot here means $\partial/\partial\tau$ and the prime denotes $\partial/\partial z$ and we define the functions

$$\Delta=b^2-z^2, \quad \rho=1-\kappa|F|^2. \tag{17}$$

For completeness, we include the field equations in (t,r) coordinates in Appendix A. However, they are not crucial to our current discussion.

The boundary conditions we use are that the solution is regular on the time axis $z=0$ and on the so called similarity horizon $\Delta=b^2-z^2=0$. Regularity on the time axis $z=0$ at the center of spherical symmetry allows us to write the boundary conditions for the metric functions $b(\tau,z)$ and $u(\tau,z)$ as

$$b(\tau,0)=1, \quad u(\tau,0)=0. \tag{18}$$

The hypersurface defined by $\Delta=0$ is where the homothetic Killing vector becomes null. As this hypersurface is in the Cauchy development of the initial data, we expect everything to be perfectly regular there even though this is a singular point of Eqs. (16).

The existence of the homothetic Killing vector simplifies these equations somewhat. For the general collapse problem without self-similarity, the metric coefficients u and b will be functions of z and τ, but our assumed symmetry restricts these coefficients to be functions of z alone. We could also let the dimensionless field F be invariant under the action of the vector field ξ, but that would then fail to incorporate the SL(2,$\mathbb{R}$) symmetry which the field equations also possess. We therefore assume that ξ act on F with an arbitrary infinitesimal SL(2,$\mathbb{R}$) transformation, which generates some U(1) subgroup of SL(2,$\mathbb{R}$). Without loss of generality, we can assume that this U(1) transformation acts by a pure phase rotation of F, so that

$$\mathcal{L}_\xi F=\xi^a\partial_a F=i\omega F, \tag{19}$$

where ω is a real constant. This allows us to give the form of F under our assumption of self-similarity as

$$F(\tau,z)=e^{i\omega\tau}f(z). \tag{20}$$

The continuously self-similar (CSS) fields are now

$$F(\tau,z)=e^{i\omega\tau}f_0(z), \tag{21a}$$

$$b(\tau,z)=b_0(z), \tag{21b}$$

$$u(\tau,z)=u_0(z), \tag{21c}$$

where ω is a real eigenvalue, determined by solving the field equations. The subscript zero that we have appended denotes unperturbed values in anticipation of the perturbation calculation below in Sec. V.

Our equations are now just Eqs. (16) with the τ derivatives of $u(z)$ and $b(z)$ vanishing, F and F' being replaced by f_0 and f_0', and $\dot{F}$ and $\ddot{F}$ being replaced by $i\omega f_0$ and $-\omega^2 f_0$, respectively. Note that with $\dot{u}_0=0$, we can eliminate u_0' and we are left with an algebraic relation for $u_0(z)$. The equations of motion now reduce to[3]

$$b_0'=\frac{b_0u_0}{z} \tag{22a}$$

$$\Delta_0 f_0''=f_0'\left(-2i\omega z-z(u_0-2)-\frac{b_0^2}{z}(u_0+2)-\frac{4i\kappa\omega z}{\rho_0}|f_0|^2\right) \tag{22b}$$

$$-f_0[\omega^2+i\omega(1-u_0)]-\frac{2\kappa}{\rho_0}f_0^*(\Delta_0 f_0'^2+\omega^2 f_0^2), \tag{22c}$$

where we have defined

$$\Delta_0=b_0^2-z^2, \tag{23a}$$

$$\rho_0=1-\kappa|f_0|^2, \tag{23b}$$

$$u_0=\frac{z^2}{\rho_0^2}\left(\frac{1}{b_0^2}|i\omega f_0-zf_0'|^2+|f_0'|^2\right) \tag{23c}$$

$$+\frac{z}{\rho_0^2}[f_0'(i\omega f_0-zf_0')^*+f_0'^*(i\omega f_0-zf_0')] \tag{23d}$$

and where the prime now denotes d/dz.

The boundary conditions at $z=0$ for the CSS problem now reduce to

$$b_0(0)=1, \quad f_0=\text{ free real constant}, \quad f_0'(0)=0, \tag{24}$$

where we have used our U(1) phase symmetry to fix f_0 as real. We define the value of z where Δ_0 vanishes as z_2. As mentioned earlier, we demand regularity at $\Delta_0(z_2)=0$ and this leads to the additional boundary conditions

$$b_0(z_2)=z_2=\text{ free real const}, \quad f_0(z_2)=\text{ free complex const}, \tag{25}$$

with the constant $f_0'(z_2)$ being determined by Eq. (22) at the similarity horizon.

Now with the equations and boundary conditions, we can numerically integrate these equations. Once we reduce our

[3]Our notation here more closely follows the paper [5] on the axiodil, $\kappa=1$, and not the earlier papers [2,13] on the complex scalar field, $\kappa=0$.

second order ordinary differential equation (ODE) to two first order ODE's and include the real eigenvalue ω we have five real equations and five real unknowns. We use our standard technique of solving this two-point boundary value problem by shooting with an adaptive ODE solver from both boundary points to a point z_1 in the middle. The free boundary values are then found using a Newton's solver for the nonlinear matching conditions [14].

We then follow the CSS solution as κ varies, and we find that a CSS solution exists for

$$-0.60 \lesssim \kappa < +\infty; \tag{26}$$

for $\kappa = 0,1$ the CSS solution is the same one found in previous work. Our computations only extend to $\kappa \leqslant 15$, but the behavior is smooth and the CSS solutions seem likely to extend all the way to $\kappa = \infty$. On the other hand, our calculations of CSS solutions appear to terminate somehow at $\kappa \approx -0.60$. We are unsure what exactly goes wrong there, but we tend to believe that our numerical routine fails and it is not the case that the CSS solutions cease to exist for smaller κ. It is, however, worth recalling that Maison [15] found that his sequence of CSS gas collapses terminated at a maximal value $k_{\max} \approx 0.88$, where k appears in the equation of state for an Eulerian fluid $p = k\rho$. The reason in his case was a change in the nature of the eigenvalues associated with the singular sonic point. At $k_{\max}$, two of the eigenvalues degerate. But we have no evidence that a similar thing occurs here.

As far as we know, there is only one eigenvalue ω possible for the CSS solution for a given κ; however, we have not looked very carefully for others. We also mention that although we describe the spacetime only up to the similarity horizon, the spacetime can be continued in these coordinates to $z = +\infty$. This corresponds to the spacelike hypersurface $t = 0$. We expect everything to be regular on this hypersurface except at the axis of spherical symmetry since it too is in the Cauchy development of the initial data. Thus the apparent singularity in our equations at $z = +\infty$ is merely a coordinate singularity. By changing coordinates, we can continue the spacetime through $t = 0$. We will not detail the explicit construction of this extension here. It is similar to that found in [2,5]. Suffice it so say that we have made this construction and the spacetime is indeed extendible for all values of κ for which we find a solution. Hence the spacetime can be continued to and beyond the future similarity horizon.

V. PERTURBATIONS AND STABILITY

As interesting as the CSS solutions are, they do not tell us everything we would like to know about the gravitational collapse. After all, these are the exactly critical solutions $p = p^*$ and comprise a set of measure zero in the space of initial conditions of the collapse. To reach them, the initial conditions must be tuned with exquisite care. In addition, such things as the critical exponents of the black hole scaling relation are found only with information gained by collapse slightly away from the critical solution.

For these reasons, we look to perturbation theory for additional understanding of the CSS solutions. It too is not the last word, but it can shed some light on questions of stability and in particular allow us to calculate the critical exponents of the black hole growth.

As described in [13], the very construction of a Choptuon involves stabilization — a balancing between subcritical dissipation and supercritical black hole formation with the critical exponent γ measuring the strength of this black hole or dissipation instability. More specifically, for initial data close to, but not exactly on the critical solution, the critical solution serves as an intermediate attractor with near-critical solutions approaching it but eventually running away from it to form a black hole or dissipate the field to infinity. However, in addition to this particular instability, we would like to know if there are *additional* instabilities which would rather drive the near-critical solutions completely away from the Choptuon to another, perhaps very different, attractor. Thus by appealing to perturbation theory, we are looking for both the black hole instability (i.e., the critical exponent) and possibly other instabilities indicating the existence of other, stronger attractors.

So, with the continuously self-similar solutions in hand, we now carry out a linear perturbation analysis of the CSS solutions, still in spherical symmetry. We define the perturbed fields as

$$b(\tau,z) \approx b_0(z) + \epsilon b_1(\tau,z), \tag{27a}$$

$$u(\tau,z) \approx u_0(z) + \epsilon u_1(\tau,z), \tag{27b}$$

$$F(\tau,z) \approx e^{i\omega\tau}[f_0(z) + \epsilon f_1(\tau,z)], \tag{27c}$$

where again, the subscript zero denotes the zeroth order critical solution, the subscript one denotes the first order perturbation, ω is the (unique) eigenvalue of the unperturbed equations (which depends on the coupling constant κ), and where $\epsilon > 0$ is an infinitesimal constant, a measure of how far away the solution is from the critical solution in the space of initial conditions. Using Choptuik's terminology, we consider the supercritical regime for infinitesimal

$$\epsilon \propto p - p^*. \tag{28}$$

We now perturb the Einstein equations through first order in ϵ, to obtain a set of linear partial differential equations for the perturbed fields b_1, u_1, $\hat{f}_1$, in the independent variables τ, z. Following the standard approach, we Fourier transform the 1st order fields with respect to the ignorable coordinate $\tau = \ln(-t)$:

$$\hat{u}_1(\sigma,z) = \int e^{i\sigma\tau} u_1(\tau,z)\, d\tau, \tag{29a}$$

$$\hat{b}_1(\sigma,z) = \int e^{i\sigma\tau} b_1(\tau,z)\, d\tau, \tag{29b}$$

$$\hat{f}_1(\sigma,z) = \int e^{i\sigma\tau} f_1(\tau,z)\, d\tau; \tag{29c}$$

throughout, a caret will denote such a Fourier transform. The transform coordinate σ is in general complex. The first order field equations now become ordinary differential equations (ODE's) in z, and under appropriate boundary conditions, become an eigenvalue problem for σ. Solutions of the eigenvalue problem are then normal modes of the critical solution. Generally speaking, there will be many different normal modes $\hat{f}_1$, each belonging to a different eigenvalue σ. Eigenvalues in the lower half plane $\mathrm{Im}\sigma<0$ belong to unstable (growing) normal modes. Eigenvalues in the upper half σ plane correspond to quasinormal (dying) modes of the critical solution. The eigenvalue σ is related to the critical exponent by $\gamma=-1/\mathrm{Im}\sigma$. [15,16,13]

We now want to integrate our equations numerically so we need to determine the boundary conditions. It is important to bear in mind that in addition to solving the equation for $\hat{f}_1(\sigma,z)$, we must also solve the analogous equation for $\hat{f}_1(-\sigma^*,z)^*$. Thus, we will have two second order ODE's which must be reduced to four first order ODE's, we will have a total of six complex equations to integrate. For the perturbation problem, the boundary conditions at $z=0$ are found to be

$$\hat{b}_1(0)=0, \quad \hat{u}_1(0)=0, \quad \hat{f}_1'(\sigma,0)=0, \quad \hat{f}_1'^*(-\sigma^*,0)=0, \tag{30}$$

$$\hat{f}_1(\sigma,0)= \text{ free complex const,}$$

$$\hat{f}_1^*(\sigma^*,0)= \text{ free complex const.} \tag{31}$$

At the similarity horizon, $z=z_2$, the boundary conditions are as follows. Both $\hat{b}_1(z_2)$ and $\hat{u}_1(z_2)$ are free complex constants. Either $\hat{f}_1(\sigma,z_2)$ or $\hat{f}_1'(\sigma,z_2)$ is a free complex constant with the other describable in terms of the other boundary conditions at z_2. We chose to let $\hat{f}_1'(\sigma,z_2)$ to be free and $\hat{f}_1(\sigma,z_2)$ fixed as this facilitated examining the lower half complex σ plane. The same is true for the values $\hat{f}_1'(-\sigma^*,z_2)^*$ and $\hat{f}_1(-\sigma^*,z_2)^*$. Counting the eigenvalue σ, we now have seven pieces of complex boundary data to go with the six complex equations we need to integrate. Since the perturbation equations are linear, we expect the solutions to scale, so the extra piece of data is merely a reflection of the linearity of the equations. Solutions will come in families which will be parametrized by a single complex parameter. Thus we have an eigenvalue problem which is well posed and which should yield a discrete spectrum of eigenvalues σ.

To solve the first order problem we used a Runge-Kutta integrator with adaptive step size as part of a standard two point shooting method [14], shooting from $z=0$ and from both boundaries and matching in the middle $z=z_1$. For convenience we solved the zeroth order system, Eqs. (22), and the first order system, Eqs. (A2), simultaneously with the same steps in z. As discussed elsewhere, the similarity horizon z_2 is a demanding place to enforce a boundary condition, and a second order Taylor expansion of the regular solution was used for this purpose.

To solve the first order system, we collected all the boundary values but σ into a complex six-vector

$$X \equiv [\hat{f}_1(\sigma,0),\hat{f}_1(-\sigma^*,0)^*,\hat{b}_1(z_2),\hat{u}_1(z_2),$$
$$\hat{f}_1'(\sigma,z_2),\hat{f}_1'(-\sigma^*,z_2)^*].$$

Because the equations are linear, the matching conditions at $z=z_1$ are likewise linear in the boundary values. A solution is found when the values at z_1 of $[\hat{b}_1,\hat{u}_1,\hat{f}_1(\sigma),\hat{f}_1(-\sigma^*)^*,\hat{f}_1'(\sigma),\hat{f}_1'(-\sigma^*)^*]$ upon integrating from $z=0$ match with those found by integrating from $z=z_2$, for some boundary values X. We can express this matching condition

$$A(\sigma)X=0, \tag{32}$$

where $A(\sigma)$ is a 6×6 complex matrix which is a nonlinear function of σ, constructed numerically by integrations of the first order equations, Eqs. (A2), for six linearly independent choices of boundary values X. The condition on σ for a solution is then

$$\det A(\sigma)=0. \tag{33}$$

Once a value for σ was found that satisfies this condition, the corresponding boundary values X were found as a zero eigenvector of the matrix A; these come in one (complex) parameter families, as observed above. Solution of Eqs. (A2) with boundary values X yields the normal mode itself. Now, $A(\sigma)$ has been carefully constructed so that it is a complex *analytic* solution of σ. This follows from the fact that all equations leading to A contain σ but not σ^*, together with some standard theorems about ODE's. Moreover, $A(\sigma)$ has no singularities in the closed lower half σ plane. These properties allow us to use a number of ideas from scattering theory to study $\det A(\sigma)$. In particular, there is a theorem for counting the number N_C of zeros of $\det A$ within any closed contour $\mathcal{C}$ in the closed lower half σ plane:

$$\Delta_{\mathcal{C}} \arg\det A = 2\pi N_{\mathcal{C}} \tag{34}$$

where $\arg\det A$ is the phase of $\det A$, and $\Delta_{\mathcal{C}}\arg\det A$ is the total phase wrap (in radians) around the closed contour $\mathcal{C}$, a result similar to Levinson's theorem for counting resonances in quantum scattering theory.

Furthermore, a conjugacy relation holds,

$$A^*(-\sigma^*)=A(\sigma), \tag{35}$$

which means that A need only be evaluated for $\mathrm{Re}\sigma\geqslant 0$ in the lower half plane.

The nonlinear equation $\det A(\sigma)=0$ was solved by the secant variant of Newton's method [14]. The equation being complex-analytic, the one-complex-dimensional realization of the method was used, and it performed well.

Since our field equations possess gauge invariance due to general coordinate invariance, and also possess a three-dimensional group of global invariances acting on F, some unphysical pure gauge modes will appear at first order, to the extent that the gauge conditions implicit in our boundary conditions Eqs. (30),(31) fail to be unique.

A pure gauge mode arises from an infinitesimal phase rotation $\phi \to e^{i\epsilon}\phi$ in the zeroth order critical solution

$$\hat{b}_1(z)=0, \tag{36a}$$

$$\hat{u}_1(z)=0, \tag{36b}$$

$$\hat{f}_1(z)=i f_0(z). \tag{36c}$$

This gives a time independent solution of Eqs. (A2) that satisfies the boundary conditions; hence it corresponds to an unphysical mode at $\sigma=0$.

Another pure gauge mode results by adding an infinitesimal constant to time $t\to t+\epsilon$ at constant r in the zeroth order critical solution. This is possible because our coordinate conditions, Eqs. (18) normalize t to proper time along the negative time axis $(t<0,z=0)$, but the zero of time is not specified. Then the solution is perturbed by

$$b_1(\tau,z)=\partial b_0/\partial t|_r=-(z/t)b'(z)=e^{-\tau}zb'(z), \quad (37a)$$

$$u_1(\tau,z)=\partial u_0/\partial t|_r=-(z/t)u'(z)=e^{-\tau}zu'(z), \quad (37b)$$

$$f_1(\tau,z)=e^{-i\omega\tau}\partial(e^{i\omega\tau}f_0)/\partial t|_r=e^{-\tau}[-i\omega f_0(z)+zf_0'(z)]. \quad (37c)$$

This pure gauge mode has time dependence $e^{-i\sigma\tau}=e^{-\tau}$ and so has negative imaginary $\sigma=-i$.

There are also two more gauge modes which appear as a pair on the real axis. In the case $\kappa=0$, these come from the addition of an infinitesimal complex constant c to our zeroth order solution: $F\to F+\epsilon c$. The perturbed fields are then

$$b_1(\tau,z)=0, \quad (38a)$$

$$u_1(\tau,z)=0, \quad (38b)$$

$$f_1(\tau,z)=ce^{-i\omega\tau}. \quad (38c)$$

This mode has a time dependence of $e^{-i\sigma\tau}=e^{-i\omega\tau}$ and so has $\sigma=\omega$. Of course, since we have $A^*(-\sigma^*)=A(\sigma)$, the value $\sigma=-\omega$ will also solve the equation $\det A=0$ and be the fourth gauge mode.[4] A similar but more complicated argument shows that two gauge modes persist at the same frequency even for $\kappa\neq 0$.

Thus, for all values of κ, there exist four gauge modes in the σ plane, and it can be shown that there are no others. These modes should appear as numerical solutions—therefore serving as calibrations—but are unphysical.

VI. RESULTS AND CONCLUSIONS

On integrating and solving for the eigenvalues $\sigma(\kappa)$, we found some novel behavior. We confirmed the existence of the gauge modes thereby checking the consistency of our method. We also found the critical exponent $\gamma(\kappa)$ over the range of κ values for which we found a solution. Figure 1 is a graph of this exponent as a function of the coupling constant. As can be seen, the critical exponent for the CSS solution depends strongly on the value of the coupling constant.

In addition, we evaluated $\det A(\sigma)$ around a large rectangular contour in the lower half plane and used Eq. (34) to count the zeros lying within. This allowed us to determine if there were additional modes in the lower half σ plane. Our results were as follows. We did find many more modes in the complex σ plane. These additional modes are initially in the upper half plane for large positive κ and approach the real axis as κ decreases. Once one of these modes crosses the axis into the lower half plane we infer that the leading normal mode of the CSS solution has a change of stability. This first occurs at $\kappa\approx 0.0754$. We thus have

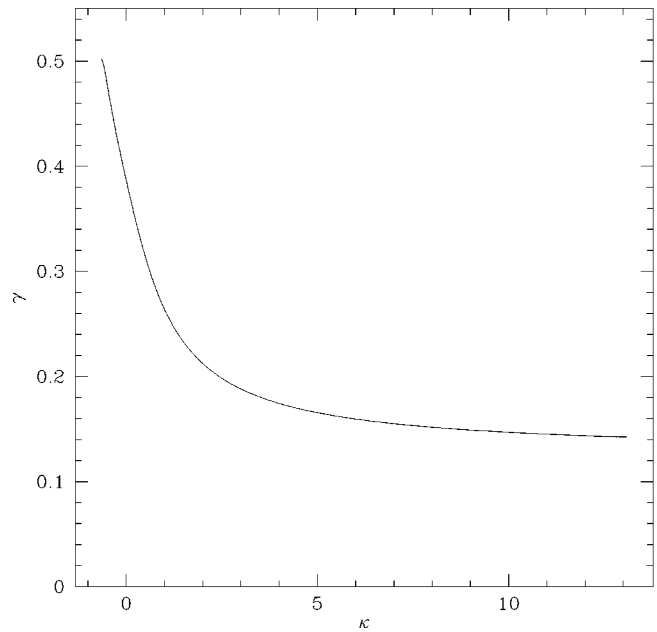

FIG. 1. The critical exponent γ, as in Eq. (1), of the continuously self-similar solution as a function of κ. See text.

$$0.0754\lesssim\kappa<+\infty, \qquad \text{CSS stable}, \quad (39a)$$

$$-0.60\lesssim\kappa\lesssim 0.0754, \qquad \text{CSS unstable}. \quad (39b)$$

This confirms the discovery by Choptuik and Liebling of a change of stability at $\omega_{BD}\approx 0$; from Eq. (7) the value would be $\omega_{BD}\approx 0.158$ [7,8]. Note that these results are in good agreement with earlier work. The CSS solution for the complex scalar field $(\kappa=0)$ was shown to be unstable by a similar analysis [13] while the CSS solution for the axion-dilaton field $(\kappa=1)$ was shown in [6] to be the attractor in gravitational collapse and hence agrees with what we have found here, namely that the solution found in [5] is stable. An important question is if the CSS solution becomes unstable at $\kappa\approx 0.0754$, what is the attractor for the collapse. Our conjecture, borne out by collapse calculations of Choptuik and Liebling, is that the attractor between $0<\kappa\lesssim 0.0754$ (i.e., $\omega_{BD}\gtrsim 0$) is the more dynamically interesting discretely self-similar (DSS) or echoing solution analogous to the echoing solution originally seen by Choptuik in the collapse of a real scalar field.

Since everything in our model is smooth at $\kappa=0$, as we decrease κ below zero, we expect the relevant attractor for the collapse to continue to be the echoing solution. However, the above mentioned unstable mode turns out not to be the only mode to move into the lower half plane. We have some evidence for more eigenvalues going unstable by $\kappa\approx -0.28$. We have constructed the corresponding perturbation modes and they appear to be legitimate solutions of the perturbation equations and not numerical artifacts. These additional

[4]When we did a similar analysis [2,13] for the complex scalar field, we were insensitive to these modes, since we worked with the derivatives of $\phi(\tau,z)$, and these modes vanished identically.

TABLE I. Range of the model (parametrized by κ), its relation to the Brans-Dicke/scalar model (parametrized by ω_{BD}), critical exponent γ, and stability.

Nonlinear σ,κ	BD/scalar ω_{BD}	γ	Stability of CSS
$+\infty$	$-3/2$	$\lesssim 0.14(?)$	stable?
10.0	-1.4875	0.1469	stable
1	$-11/8$	0.2641	Stable
$\lesssim 0.0754$	$\gtrsim 0.158$	$\gtrsim 0.373$	becomes unstable
0	$+\infty$	0.3871	unstable
$\lesssim -0.28$	n/a	$\gtrsim 0.435$	more modes become unstable?
$\lesssim -0.60$	n/a		(not known if CSS exists)

modes suggest that the model becomes ever more unstable, that is more nonlinear, as κ decreases. So what happens in gravitational collapse as κ decreases below ≈ -0.28? The CSS solution will certainly not be the attractor but, we speculate, the existence of similar unstable modes may trigger further bifurcations in the echoing solution. Since our calculations in this paper are limited to perturbation theory, we cannot pursue this question further here, but a possibility is that the echoing solution becomes unstable and bifurcates into an even more dynamically complicated solution. One way to determine what happens here with greater assurance would be to take a numerical solution for the DSS solution and perform a perturbation analysis. That this would be feasible is suggested by Gundlach's results in which he calculates the echoing solution as an eigenvalue problem resulting from the assumption of discrete self-similarity in the collapse of a real scalar field [17]. However, another, more direct approach would be to perform a full scale numerical collapse calculation in order to understand what is going on in this regime.

In this paper, we have combined a few of the previously studied models of gravitational collapse into a single model of a self-coupled complex scalar field. The model is parameterized by a single coupling constant κ. In Table I, we give a summary of some of the key values of κ. As the value of the coupling constant decreases, the continuously self-similar solution which we find undergoes a change in stability. For the regime where the CSS solution is unstable, we believe that the attractor for gravitational collapse is an echoing and discretely self-similar solution. This change in stability which occurs near $\kappa=0.0754$ appears to be a ''Hopf bifurcation,'' as it is known in the dynamics literature [9]. As κ continues to decrease, we find evidence for additional instabilities in the model, suggesting that another bifurcation of the collapsing solution may exist. From the lore on other dynamical systems, this further conjectural bifurcation might lead to a doubly periodic attractor, or might lead to full blown dynamical chaos in gravitational collapse. Additional work will be able to determine whether this is indeed the case.

As this paper was being readied for publication, there were indications in Liebling's work that the attractor in this region of parameter space, though DSS, is different from the family of DSS solutions originally discovered in the Brans-Dicke model [18].

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APPENDIX A: EQUATIONS OF MOTION AND THEIR PERTURBATION

In this appendix, we list some equations that were too cumbersome for the main portion of the paper. We have the general collapse equations for our model (4) in the usual (t,r) coordinates:

$$\dot{u}=\frac{r(u+1)}{\rho^2}[\dot{F}^*F'+\dot{F}F'^*], \tag{A1a}$$

$$u'=\frac{r(u+1)}{\rho^2}\left[|F'|^2+\frac{1}{b^2}|\dot{F}|^2\right]-\frac{u(u+1)}{r}, \tag{A1b}$$

$$b'=\frac{ub}{r}, \tag{A1c}$$

$$0=r^2\left(\frac{1}{b}\ddot{F}-\frac{\dot{b}}{b^2}\dot{F}\right)-(r^2bF''+2rbF'+r^2b'F') \tag{A1d}$$

$$-\frac{2\kappa r^2}{\rho}F^*\left(bF'^2-\frac{1}{b}\dot{F}^2\right), \tag{A1e}$$

where an overdot means $\partial/\partial t$ and a prime means $\partial/\partial r$, and where

$$\rho=1-\kappa|F|^2.$$

The Eqs. (16) when perturbed as given in Eqs. (27) become our original set as well as the following Fourier-transformed first order equations:

$$z\hat{u}_1'+i\sigma\hat{u}_1 \tag{A2a}$$

$$=\frac{z(u_0+1)}{\rho_0^2}[f_0'(z\hat{f}_1'-i(\omega+\sigma^*)\hat{f}_1)^*$$

$$+f_0'^*(z\hat{f}_1'-i(\omega-\sigma)\hat{f}_1) \tag{A2b}$$

$$+\hat{f}_1'(zf_0'-i\omega f_0)^*+\hat{f}_1'^*(zf_0'-i\omega f_0)] \tag{A2c}$$

$$+\frac{z(u_0+1)}{\rho_0^2}\left(\frac{\hat{u}_1}{u_0+1}-\frac{2\hat{\rho}_1}{\rho_0}\right)[f_0'(zf_0'-i\omega f_0)^*$$

$$+f_0'^*(zf_0'-i\omega f_0)], \tag{A2d}$$

$$\hat{u}_1' = \frac{z(u_0+1)}{\rho_0^2}\Bigg[f_0'\hat{f}_1'^* + f_0'^*\hat{f}_1' - \frac{2\hat{b}_1}{b_0^3}|zf_0' - i\omega f_0|^2$$

$$+\frac{1}{b_0^2}[(zf_0' - i\omega f_0)(z\hat{f}_1' - i(\omega+\sigma^*)\hat{f}_1)^* \tag{A2e}$$

$$+[(zf_0' - i\omega f_0)^*(z\hat{f}_1' - i(\omega-\sigma)\hat{f}_1)]\Bigg] \tag{A2f}$$

$$+\frac{z(u_0+1)}{\rho_0^2}\left(\frac{\hat{u}_1}{u_0+1} - \frac{2\hat{\rho}_1}{\rho_0}\right)\left(|f_0'|^2 + \frac{1}{b_0^2}|zf_0' - i\omega f_0|^2\right), \tag{A2g}$$

$$\hat{b}_1' = \frac{1}{z}(u_0\hat{b}_1 + u_1 b_0) \tag{A2h}$$

$$0 = \hat{f}_1''\Delta_0 + \hat{f}_1'\Bigg\{2iz\left[(\omega-\sigma) + \frac{2\kappa\omega}{\rho_0}|f_0|^2\right] + \frac{1}{z}[z^2(u_0-2)$$

$$+ b_0^2(u_0+2)] + \frac{4\kappa\Delta_0}{\rho_0}f_0^* f_0'\Bigg\} \tag{A2i}$$

$$+\hat{f}_1\Bigg\{(\omega-\sigma)\left[(\omega-\sigma+i(1-u_0)) + \frac{4\kappa}{\rho_0}f_0^*(izf_0' + \omega f_0)\right] \tag{A2j}$$

$$+\frac{2\kappa^2}{\rho_0^2}f_0^{*2}[b_0^2 f_0'^2 + (izf_0' + \omega f_0)^2]\Bigg\} \tag{A2k}$$

$$+\hat{f}_1^*\frac{2\kappa}{\rho_0^2}\{b_0^2 f_0'^2 + (izf_0' + \omega f_0)^2\} \tag{A2l}$$

$$+\hat{b}_1\Bigg\{2b_0 f_0'' + \frac{4\kappa b_0}{\rho_0}f_0^* f_0'^2 + \frac{\sigma}{b_0}(\omega f_0 + izf_0')$$

$$+\frac{2b_0}{z}(u_0+2)f_0'\Bigg\} \tag{A2m}$$

$$+\hat{u}_1\left\{-i\omega f_0 + zf_0' + \frac{b_0^2}{z}f_0'\right\}, \tag{A2n}$$

APPENDIX B: GLOBAL EXISTENCE THEORY OF NONLINEAR σ MODELS IN FLAT SPACETIME

The nonlinear σ models can of course be studied in flat spacetime, as a nonlinear wave equations in their own right. There is a considerable literature on them, both in physics and mathematics. For present purposes the important issue is the possible evolution if singularities in the field from regular initial conditions. After all, a curved spacetime containing a matter field should not be considered a counterexample to cosmic censorship of that same matter field can evolve a singularity in flat spacetime. A related question of considerable interest is the possible existence of critical solutions and similarity solutions, which typically are singular.

In mathematics, SO$(k+1)$ nonlinear σ models $(k=1,2,3,\ldots)$ are studied in general $(n+1)$-dimensional spacetimes, with much attention on the case $k=n$, where there exist nontrivial topological configurations, such as the ''equivariant wave maps'' — these are approximately the same as what the physicists call ''texture.'' Similarity solutions are known to exist in the case $n=3=k$ and these solutions are singular. A recent paper that contains both extensive results and a thorough discussion of previous literature is [19].

In physics, SO(3) nonlinear σ models have been studied as models of ''texture'' in (3+1)-dimensional cosmology. Texture can collapse, and the analytic form of the similarity solution is known [20].

In our paper we have used the SO(3) nonlinear σ model in (3+1) dimensions. One naturally wonders whether or not the flat-spacetime restriction of this particular model admits similarity solutions, or admits singularites at finite time that evolve from regular initial conditions. However, these questions appear to remain open, except of course in the case $\kappa=0$, where the model becomes the free wave equation.

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