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Black Holes and Spinning Test Bodies*

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The orbit equations of Papapetrou are the basis for a study of spinning test bodies in a Schwarzschild or Kerr metric field. The constant of the motion for a Killing vector of an arbitrary metric field is given. An analysis of test-body equatorial motion suggests that significant departures from the results of geodesic motion such as larger maximum binding and orbits completely stable against capture may be expected for objects with large intrinsic spin.

Currently much attention is focused on black holes as possible underlying sources for large energy production and/or gravitational wave emission.[1-6] In previous studies of the binding energy available in capture of a test body by a black hole, only geodesic motion has been considered. It is the purpose of this Letter to present results of a study of spinning test bodies which, as is well known, do not execute geodesic motion.

Since a significant fraction of the stars observed have intrinsic angular momentum, a study of a spinning test body in the field of a black hole may serve as a model for the possible star–black-hole interactions. The simple act of endowing a black hole with angular momentum has led to an unexpected richness of possible physical phenomena. It seems appropriate to ask whether endowing the test body with intrinsic spin might not also lead to surprises.[7] Indeed, since in a weak-field approximation the spin-spin interaction is $\propto 1/r^4$ in contrast to the Newtonian attraction which is $\propto 1/r^2$, the possibility exists for spin interactions to dominate completely for small separation.[8,9]

The equations of motion for a spinning test body most commonly used are those of Papapetrou.[10,11] For extended bodies, alternative equations have been suggested with differing supplementary conditions.[12-14] We will, however, confine our attention to a model of point test bodies and use the Papapetrou equations for a spinning point mass as developed by Taub.[15] Our primary interest is in those equations determining the orbit and not the spin equations of motion. There is general agreement, even for extended bodies, that the spin undergoes Fermi-Walker transport and it is just this spin behavior which has been subjected to the most detailed examination.[9,16] Study of the orbit equations has been limited to the Schwarzschild field.[17,18]

Ignoring radiation reaction, the equations describing the motion of a test body with mass m, four-velocity u, and spin four-vector S in a metric field $g_{\mu\nu}$ with curvature operator R are

$$(mu + u\wedge\dot{u}\wedge S)^{\cdot} + \tfrac{1}{2}g^{\rho\mu}R(u, e_\mu)(e_\rho\wedge S\wedge u) = 0, \quad (1a)$$

$$\dot{S}\wedge u = 0, \quad (1b)$$

$$S\cdot u = 0; \quad (1c)$$

e_μ ($\mu = 0, \ldots, 3$) denote the basis four-vectors.[19] m and $S\cdot S$ are known to be constants of the motion. It can also be shown that for an arbitrary Killing vector field ξ the scalar

$$(mu + S\wedge u\wedge\dot{u})\cdot\xi + \tfrac{1}{2}g^{\mu\rho}(e_\rho\wedge S\wedge u)\cdot\nabla_\mu\xi \quad (2)$$

is a constant of the motion.[20] The proof is straightforward and relies only on Eq. (1a) and well-known properties of Killing vector fields.

We choose that "specific internal energy" (ϵ of Taub's Eq. [1.5]) to be zero so that the test body is characterized completely by giving its mass and spin. This implies as a consequence that

$$\dot{u}\wedge S = 0. \quad (3)$$

Thus, the second term in Eq. (1a) vanishes and one is left with

$$mu\cdot\xi + \tfrac{1}{2}g^{\mu\rho}(e_\rho\wedge S\wedge u)\cdot\nabla_\mu\xi \quad (4)$$

as a constant of the motion.[21]

For an application with possible astrophysical significance, we examine in detail these results for the uncharged Kerr metric field written in Boyer-Lindquist coordinates[22]:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}[a\,dt - (r^2+a^2)\,d\varphi]^2, \quad (5)$$

where $\Delta = r^2 + a^2 - 2Mr$ and $\rho^2 = r^2 + a^2\cos^2\theta$. We will primarily be interested in the cases where the source of mass M has angular momentum per unit mass $a=0$ (Schwarzschild field) and $a=M$ (maximal Kerr field).

The metric field (5) has Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$. Restricting the analysis to equatorial orbits and substituting into Eq. (4) gives the constants of the motion

$$\gamma = \frac{E_\infty}{m} = \dot t\left(1-\frac{2}{\mathfrak r}\right) + \frac{M\dot\varphi}{\mathfrak r}\left(1+\frac{\mathfrak a^2}{\mathfrak r^2}\right)s + \frac{2M\mathfrak a\dot\varphi}{\mathfrak r} - \frac{\dot t\mathfrak a s}{\mathfrak r^2} \quad (6a)$$

and

$$\mathfrak j \equiv \frac{J}{mM} = \mathfrak a\left[\frac{M\dot\varphi s}{\mathfrak r}\left(3+\frac{\mathfrak a^2}{\mathfrak r^2}\right) - \frac{2\dot t}{\mathfrak r}\right] + \dot t s\left(1-\frac{2}{\mathfrak r}-\frac{\mathfrak a^2}{\mathfrak r^3}\right) + \mathfrak r^2 M\dot\varphi\left(1+\frac{\mathfrak a^2}{\mathfrak r^2}+\frac{2\mathfrak a^2}{\mathfrak r^3}\right) \quad (6b)$$

We have defined the dimensionless quantities $\mathfrak r = r/M$, $\mathfrak a = a/M$, and $s = S^z/mM$. S^z is the spin of the test body in the z direction and from Eq. (1b) is readily shown to be constant for equatorial orbits. Equation (1a), taking into account Eq. (3), shows that equatorial orbits are stable, i.e., equatorial orbits remain equatorial when the spin of the test body is entirely along the z direction. We readily recognize the nongeodesic terms appearing in (6a) as spin-orbit and spin-spin contributions to the specific energy constant. Similarly the angular momentum constant has contributions from the spin of the source and the spin of the test body as well as from the orbital motion.

Equations (6) can be inverted to obtain $\dot\varphi$ and $\dot t$ in terms of γ and $\mathfrak j$ provided the quantity $D \equiv (\mathfrak r - s^2/\mathfrak r^2)(1 - 2/\mathfrak r + a^2/\mathfrak r^2)$ does not vanish. In order that the results of a test-body analysis not become meaningless, we require that $(m/M) < 1$ and $s < 1$ which implies that $D \neq 0$ outside the event horizon.[23] The expressions for $\dot\varphi$ and $\dot t$ when substituted into the relation $u\cdot u = 1$ lead to an effective radial potential which is just the value of γ when $\dot{\mathfrak r} = 0$. Here γ satisfies $\alpha\gamma^2 - 2\beta\gamma + \delta = 0$ where

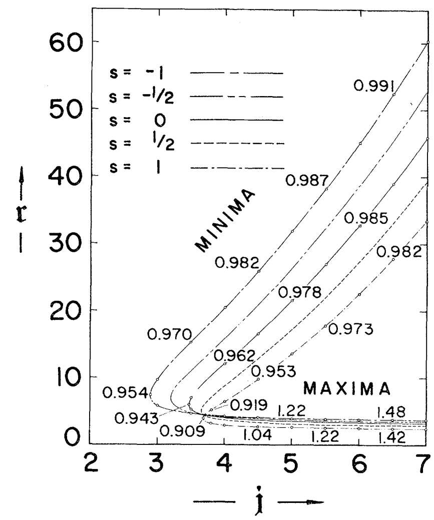

FIG. 1. As a function of the angular momentum constant $\mathfrak j$ [cf. Eq. (6b)], the radial position coordinate, $\mathfrak r = r/M$, of the effective potential maxima and minima for a spinning test particle executing equatorial motion in a Schwarzschild field. The curves for negative values of $\mathfrak j$ are obtained by letting $\mathfrak j \to -\mathfrak j$ and $s \to -s$.

$$\alpha = A\left[\mathfrak r^2 + \mathfrak a^2\left(1+\frac{2}{\mathfrak r}\right) + \frac{2s\mathfrak a}{\mathfrak r}\left(3+\frac{\mathfrak a^2}{\mathfrak r}\right) + \frac{s^2}{\mathfrak r^2}\left\{\mathfrak a^2\left(\frac{1}{\mathfrak r^2}+\frac{2}{\mathfrak r}\right) + \mathfrak r(2-\mathfrak r)\right\}\right], \quad (7a)$$

$$\beta = \mathfrak j A\left[\frac{2\mathfrak a}{\mathfrak r} - s\left(1-\frac{3}{\mathfrak r}-\frac{2\mathfrak a^2}{\mathfrak r^3}\right) + \frac{\mathfrak a s^2}{\mathfrak r^3}\left(1+\frac{1}{\mathfrak r}\right)\right], \quad (7b)$$

$$\delta = A\left[\frac{\mathfrak j^2}{\mathfrak r^2}\left(a+\frac{s}{\mathfrak r}\right)^2 - A\left\{\mathfrak j^2 + \left(\mathfrak r - \frac{s^2}{\mathfrak r^2}\right)^2\right\}\right], \quad (7c)$$

and $A = 1 - 2/\mathfrak r + \mathfrak a^2/\mathfrak r^2$.

Modeled after Fig. 19 of Ref. 6, Figs. 1 and 2 present the radial position coordinate for potential maxima and minima as a function of the angular momentum constant $\mathfrak j$ for selected values of s. Figure 3 plots the minimum value of the specific energy for a stable circular orbit versus test-body spin. All plots were obtained numerically.

We point out some of the interesting features: (i) For test-body spin aligned opposite to the spin of the Kerr source and having an optimum value of $s \equiv S^z/mM \simeq 0.92$, approximately 78% of the rest mass-energy may be radiated away before reaching the stable circular orbit of minimum energy. (ii) Furthermore, such an orbit is superstable in the sense that a test body in such an orbit will never undergo capture because of

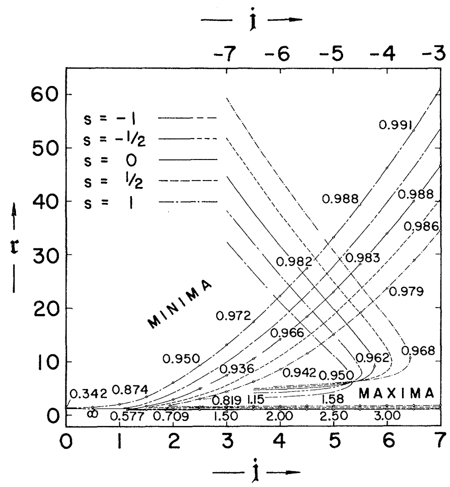

FIG. 2. As a function of the angular momentum constant ȷ [cf. Eq. (6b)], the radial position coordinate, $\mathfrak{r}=r/M$, of the effective potential maxima and minima for a spinning test particle executing equatorial motion in a maximal Kerr field. The curves which are open to the left are for negative ȷ values.

the existence of an infinite potential barrier outside the horizon. The existence of these superstable orbits may be qualitatively understood as the culminating effect of the spin-spin interaction force which is repulsive for oppositely aligned spins and equatorial motion.[8,9] (iii) For the Schwarzschild source, not only is the amount of rest mass-energy radiated altered but the coordinate position of closest approach before capture takes place is also significantly changed. (iv) For the Kerr case, inclusion of test-body spin moves the maximum in the potential off the horizon much like the Schwarzschild case.

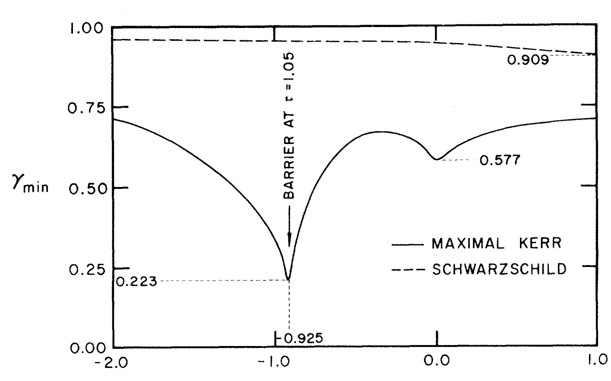

FIG. 3. Value of the effective potential minimum for the last stable circular orbit as a function of test body spin, $s \equiv S^z/mM$. Values of $s < -1$ are only included for display purposes.

The effects of intrinsic spin become dramatic just when the test-body analysis begins to become suspect.[23] A convincing demonstration of the existence or nonexistence of superstable orbits awaits an analysis not based on a perturbative approach. Nevertheless these results suggest that, in considerations of energy release by gravitational capture of orbiting bodies, significant departures from the results of geodesic motion are possible if S/mM differs significantly from zero.

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[21]The restriction of Eq. (3) reduces Eqs. (1) to those of W. G. Dixon, Nuovo Cimento 34, 317 (1964), established in the pole-dipole approximation for an extended body where terms higher than first order in the spin have been neglected. The neglect of terms of order s^2 and smaller is consistent with our test body approach.

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ERRATA

LOCALIZED STATES IN AMORPHOUS TELLURIUM. L. D. Laude, R. F. Willis, and B. Fitton [Phys. Rev. Lett. 29, 472 (1972)].

The vertical scale on Fig. 1 is too low by a factor of 10 as a result of a scale error. We are indebted to Professor W. E. Spicer for bringing this to our attention and providing comparison data. This correction does not affect the point of this Letter, which is to show the behavior of amorphous-Te yields compared to trigonal-Te yields, i.e., the nonmonotonicity of the amorphous-Te curves at about $h\nu = 5.0$ eV.

EVIDENCE FOR THE MOTT MODEL OF HOPPING CONDUCTION IN THE ANNEAL STABLE STATE OF AMORPHOUS SILICON. Adam Lewis [Phys. Rev. Lett. 29, 1555 (1972)].

In the discussion on p. 1557, the sentence "Brodsky and Gambino[1] ⋯ criticized Mott's model because they obtained reasonable numbers for γ and N" should read "Brodsky and Gambino[1] ⋯ criticized Mott's model because they obtained unreasonable numbers for γ and N."

O(4) TREATMENT OF THE ELECTROMAGNETIC-WEAK SYNTHESIS. A. Pais [Phys. Rev. Lett. 29, 1712 (1972)].

The following misprints occur: On the second line of page 1712, read $\bar{\nu}_\mu \mu$ for $\bar{\nu}_\mu \nu$. In Eq. (2), $\{W_\mu^1(t_+ - \rho_+) + \text{H.c.}\}$ should read $\{W_\mu^1(t_+ - \rho_+) + W_\mu^2(t_+ + \rho_+) + \text{H.c.}\}$. On the second line of Eq. (5), read $-\frac{1}{2}[\mathfrak{N} - \lambda + r^0\sqrt{2}]$ for $\frac{1}{2}[\mathfrak{N} - \lambda + r^0\sqrt{2}]$; and Q_L^2 for Q_L^1 on the second half of that line. In Eq. (7), $f^+f^+ - f^-f^-$ should read $\bar{f}^+f^+ - \bar{f}^-f^-$. In Eq. (11), $\bar{\mathfrak{N}} + A_1^0$ should read $\bar{\mathfrak{N}} + \bar{A}_1^0$. Seven lines below Eq. (11), $(M_{0,1,2})$ should read $(M_{0,1,2})^2$.

Further, nine lines from the end of page 1714, read "options[12] $(\vec{u}, \vec{v}) = (\vec{t}, \vec{t})$, $(\vec{p}, \vec{p})$" for "options[12] $(\vec{u}, \vec{v}) = (\vec{t}, \vec{t})$, $(\vec{p}, \vec{p})$, $(\vec{t}, \vec{p})$, $(\vec{p}, \vec{t})$." The lines between Eqs. (12) and (13) should read as follows: "$(\vec{p}, \vec{p})$ interchanges $x^+ \leftrightarrow e$ in Eq. (12). If $H(a, a')$ is a self-adjoint scalar quartet, then $a = a'$, so that then $(\vec{t}, \vec{t})$ yields $a = a' = -b$, hence[3] ⋯." On page 1715, line 11, read $(\vec{p}, \vec{p})$ for $(\vec{t}, \vec{p})$; on line 12 read iy^+, ϵy^0 for $-iy^+$, $-\epsilon y^0$.