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The physics of musical scales: Theory and experiment

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The theory of musical scales involves mathematical ratios, harmonic resonators, beats, and human perception and provides an interesting application of the physics of waves and sound. We first review the history and physics of musical scales, with an emphasis on four historically important scales: twelve-tone equal temperament, Pythagorean, quarter-comma meantone, and Ptolemaic just intonation. We then present an easy way for students and teachers to directly experience the qualities of different scales using MIDI synthesis. *© 2015 American Association of Physics Teachers.*
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I. INTRODUCTION

As a young musician my (DD) guitars never seemed to be completely in tune. Making one chord sound better made others sound worse. No amount of tuning, bridge and neck adjustments,[1] or the purchasing of more expensive guitars solved the problem. I later learned that what I was looking for is impossible.[2,3] This longstanding issue has been studied by famous physicists, mathematicians, astronomers, and musicians, including Pythagoras of Samos (570–495 BC), Ching Fang (78–37 BC), Claudius Ptolemy (90–168), Christiaan Huygens (1629–1695), Isaac Newton (1643–1727), Johann Sebastian Bach (1685–1750), Leonhard Euler (1707–1783), Jean le Rond d'Alembert (1717–1783), and Hermann von Helmholtz (1821–1894).[1,4–8] Much has been written on this topic.[2–7,9–17]

The history and development of musical scales are intimately connected with physics, and the physics of intonation has affected how music has been written throughout history.[6,9] Musical scales provide a fun and interesting application of wave physics, and this led the current authors to independently develop lectures on musical scales for their introductory physics classes. In addition, specialized courses in the physics of music are available at many institutions. Tools such as *Temperament Studio*,[18] the Java MIDI synthesizer we have developed (described below), can make complicated examples accessible to students.

The production of sound from musical instruments relates to the physics of standing waves and resonators. Introductory physics students are routinely taught that modes of an ideal vibrating string and a one-dimensional oscillating air column occur at integer multiples of a fundamental frequency. When a musical instrument is played, it oscillates in a superposition of many modes, with multiple frequency components. The note or "pitch" we hear is typically based on the lowest, or fundamental, frequency.[19] The higher harmonics affect the tone, giving instruments their distinctive sounds (their *timbres*).

Introductory physics students also learn about *beating*, a periodic change in amplitude that occurs when sinusoidal waves having similar frequencies, f_1 and f_2, are superimposed. The beat frequency f_{beat}, occurs at

$$f_{\text{beat}} = |f_1 - f_2|. \tag{1}$$

As different notes are played together, their harmonics can beat against each other. This typically sounds unpleasant in music and the desire to avoid beating is one of the main factors that led to the various musical scales discussed in this article.

II. SCALES AND INTERVALS

A musical scale is a set of pitches (notes) used to make music. The notes are often labeled using the letters A through G and the symbols ♯ (pronounced "sharp") and ♭ (pronounced "flat"). We hear pitch logarithmically, meaning that relationships between notes are defined by frequency *ratios* rather than frequency *differences*. For example, in most scales doubling the frequency increases the pitch by an *octave*, be it from C to a higher C or from D to a higher D. Musical scales can be defined in terms of the frequency ratio of each note to a reference pitch, called the *root* of the scale. The ratios used in several important tuning schemes are shown in Table I.

The scale used as an almost universal standard today is known as twelve-tone equal temperament (also called "12-TET" or simply "equal temperament"). This scale divides octaves into twelve equal *half steps* or *semitones*, each changing the frequency by a factor of $2^{1/12}$. A 12-note (*chromatic*) scale is reminiscent of modular arithmetic with modulus 12; its 12 notes can be represented on a circle, similar to a clock face, as illustrated in Fig. 1. The pitch increases as one moves around the circle in a clockwise manner; when the next higher octave is reached, the labeling restarts.[20]

Finer changes in pitch are often described in hundredths of a 12-TET half step, called *cents*. Because there are 1200 cents in an octave, a frequency change from f_1 to f_2 changes the pitch by

$$d_{\text{cents}} = 1200 \log_2(f_2/f_1) \text{ cents}. \tag{2}$$

An *interval* is the tonal distance spanned by two notes, while a *chord* is a group of notes played together. It is generally accepted that chords tend to sound more pleasant or *consonant* if the intervals in the chord have frequency relationships that can be written as ratios of small integers.[15,21] Such ratios are known as "just." Four particularly important intervals are the *octave* (2/1), a *just fifth* (3/2), a *just fourth* (4/3), and a *just major third* (5/4). To the extent that the harmonics are exact integer multiples of the fundamental, playing chords containing just intervals results in many of the higher harmonics of the different notes being at precisely the same frequencies. For example, in Fig. 2, which displays the harmonic series of a 100-Hz root note along with series for three other notes played at just intervals above

Table I. Frequency ratios used to create the notes of the scales used in various tuning schemes. This table lists the factor by which one must multiply the frequency of the root note to obtain each note in the scale. In the general meantone scheme, x represents the value chosen for the 5th ratio. The values in bold in the Ptolemaic column represent the Ptolemaic (just) values; the plain text values in that column have been filled in using the five-limit scale discussed in the text.

Note	Interval from Root	Equal Temperament	Pythagorean	General meantone	QC meantone	Ptolemaic
0	Unison	1	1	1	1	**1**
1	Minor second	$2^{1/12}$	256/243	$2^3/x^5$	$8/5^{5/4}$	16/15
2	Major second	$2^{2/12}$	9/8	$x^2/2$	$5^{1/2}/2$	**9/8**
3	Minor third	$2^{3/12}$	32/27	$2^2/x^3$	$4/5^{3/4}$	6/5
4	Major third	$2^{4/12}$	81/64	$x^4/2^2$	5/4	**5/4**
5	Perfect fourth	$2^{5/12}$	4/3	$2/x$	$2/5^{1/4}$	**4/3**
6	Augmented fourth	$2^{6/12}$	729/512	$x^6/2^3$	$5^{3/2}/8$	45/32
…	Diminished fifth	…	1024/729	$2^4/x^6$	$16/5^{3/2}$	…
7	Perfect fifth	$2^{7/12}$	3/2	x	$5^{1/4}$	**3/2**
8	Minor sixth	$2^{8/12}$	128/81	$2^3/x^4$	8/5	8/5
9	Major sixth	$2^{9/12}$	27/16	$x^3/2$	$5^{3/4}/2$	**5/3**
10	Minor seventh	$2^{10/12}$	16/9	$2^2/x^2$	$4/5^{1/2}$	9/5
11	Major seventh	$2^{11/12}$	243/128	$x^5/2^2$	$5^{5/4}/4$	**15/8**
12	Octave	$2^{12/12}$	2	2	2	**2**

the root, many of the harmonics in the different notes overlap exactly. But if the frequency ratios of the intervals are slightly detuned from just intervals, the harmonics will then be at slightly different frequencies, resulting in beating.

One of the most important intervals is the fifth. It is the interval between the first and second "Twinkles" in the song "Twinkle, Twinkle, Little Star." As mentioned, a just fifth involves a frequency ratio of 3/2 (≈701.955 cents), causing the second harmonic of the upper note to have the same frequency as the third harmonic of the lower note. However, there are alternate, and conflicting, definitions of the fifth. For example, in the 12-TET scale, a fifth is the interval between two notes that are seven half steps apart (e.g., C and G), and therefore involves a frequency factor of $2^{7/12} = 700$ cents ≈1.4983. So, while the two notes in a *just* fifth will not produce noticeable beating, the notes in an *equal temperament* fifth will contain harmonics that beat.

As is discussed below, it is impossible to define a 12-note scale in which all intervals are just in all keys. Often musicians can adjust pitch dynamically to eliminate beating,[22] and some historic instruments were built with extra keys for some notes, giving different options to optimize consonance.[4,5,7,23–25] But instruments with fixed pitches, such as pianos and organs, are generally afflicted with dissonance due to the beating of overtones.

III. THE PYTHAGOREAN SCALE

One of the oldest known methods to generate a scale is Pythagorean tuning,[4] invented by the famous geometrist. The Pythagorean scale is built using only the ratios of the just fifth (3/2) and the octave (2/1). To see how this can be done consider Fig. 1 again. Starting at C and going up a fifth takes us to G on the chromatic circle. Going up another fifth from G takes us past a full circle to D. Going up additional fifths, we arrive at the notes A, E, B, and F♯. Similarly, starting with C and going down in fifths, one ends up at the notes F, B♭, E♭, A♭, D♭, and G♭. By going up and down in fifths in such a manner—each fifth representing a multiplication or division by 3/2—and multiplying or dividing by powers of 2 as needed to bring the note frequencies back into the original

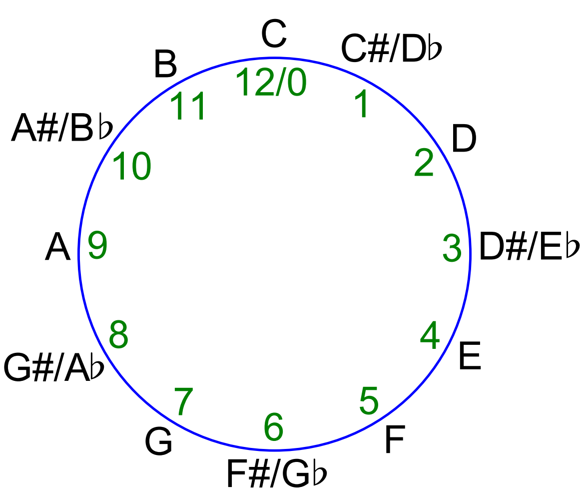

Fig. 1. The chromatic circle, showing the notes of the 12-tone scale.

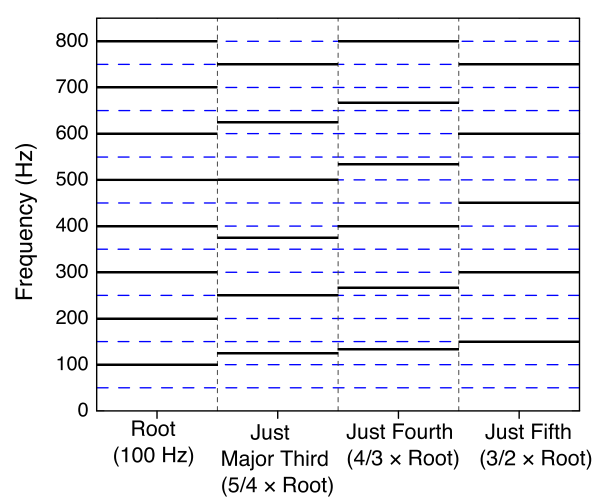

Fig. 2. Harmonic series (such as in an ideal string or one-dimensional organ pipe) of various notes, based on a 100-Hz root.

octave, we generate all of the ratios given in the Pythagorean column of Table I.

Going up an equal temperament fifth corresponds to a rotation of exactly 210° around the chromatic circle, whereas a just fifth corresponds to a rotation of 210.237°. One can say that the just fifth doesn't "close" the octave—powers of 3/2 will never exactly equal powers of 2. As a result, as one goes up by (just) fifths, the pitches start to deviate from the precise positions of the notes indicated in Fig. 1. And going up six just fifths (and then down three octaves) from C gives a different frequency for F♯ than for the G♭ obtained by going down six just fifths (then up four octaves). These two notes in the Pythagorean scale—which are the *same* note in an equal temperament scale—differ by 23.46 cents, or nearly a quarter of a half step [calculated via Eq. (2) with $f_1 = f_0(3/2)^6 \times 2^{-3}$ and $f_2 = f_0(3/2)^{-6} \times 2^4$, f_0 being the frequency of the root of the scale]. This difference is called the "Pythagorean comma."[4] (A comma is a small interval resulting from a note tuned two different ways.) The difference between these two notes necessitates an extra row for note 6 in Table I: the C–F♯ interval is called an *augmented fourth*, C–G♭ is a *diminished fifth*. (To simplify matters, all subsequent plots and calculations in this paper exclusively use the augmented fourth ratio.)

Another consequence of the Pythagorean comma is that one of the fifths in the Pythagorean scale is very different from the others, namely, the one whose lower note is F♯. This fifth involves a ratio of 262144/177147 instead of 3/2. If the lower note of this fifth has a fundamental frequency of 370 Hz, the third harmonic of the lower note and the second harmonic of the upper note will beat at 14.9 Hz. This is called a *wolf interval* or *wolf fifth*, because the horrible sounding beats are reminiscent of a howling wolf. The ×'s in Fig. 3 depict the differences from a 3/2 ratio for the twelve types of fifths in four scales; the Pythagorean wolf fifth is clearly visible as the lone nonzero fifth in Fig. 3(b).

A final problem with Pythagorean tuning is that, although most fifths involve just ratios, other intervals do not. One of the most used musical intervals is the major third,[13] the interval between two notes which in 12-TET are four half steps apart (e.g., between C and E). This is the interval between "Oh" and "when" at the start of the song "When The Saints Go Marching In." A just major third has a frequency ratio of

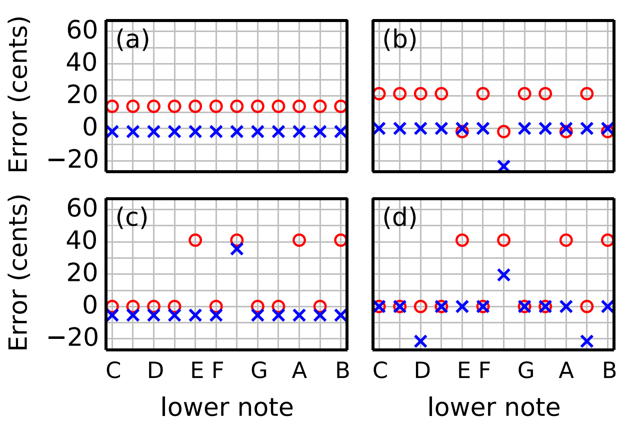

Fig. 3. Deviation of the twelve possible major thirds (○) and fifths (×) from just ratios of 5/4 and 3/2 for (a) equal temperament, (b) Pythagorean, (c) quarter-comma meantone, and (d) five-limit just intonation, all based on C as the root note for the tuning. The *x*-axis indicates the lower note of each interval. Wolf intervals are apparent as large deviations from 0.

5/4. As shown by the circles in Fig. 3(b), most major thirds in a Pythagorean scale differ from that by over 20 cents (more than one fifth of a half step). As with the wolf fifth, this causes substantial beating. The Pythagorean scale is therefore one in which some intervals are consonant, but others are very dissonant.

IV. MEANTONE SCALES

Major thirds became more important during the Renaissance when the use of chords rather than single-note melodies became more prominent.[4,13] "Meantone" temperaments were developed to address this issue. Meantone scales can be derived in the same manner as the Pythagorean scale—by going up and down in fifths to generate the notes of the scale—but using a fifth that is adjusted, or *tempered*, away from the just value of 3/2,[26] typically by reducing the fifth slightly.[27] In the "general meantone" column of Table I, the symbol x represents the specific frequency ratio of the tempered fifth used in tuning.[28]

The most common temperament used in organs built in the early Baroque era was quarter-comma (QC) meantone.[15] The comma referred to in this case is the difference between four just fifths and a just major third—the difference between going up in ratios of 3/2 from C-G-D-A-E (then dividing by 4 to return to the original octave), and going from C to E directly in a ratio of 5/4. This is called the "syntonic comma" and has a value of 21.506 cents. In QC meantone, each fifth is reduced from a just fifth by one quarter of that comma (i.e., tempered by 5.377 cents); this results in many of the thirds in the scale being perfectly just [as seen in Fig. 3(c)]. The tempered fifth ratio defined that way is equal to $5^{1/4} \approx 1.4953 = 696.578$ cents.

To tune a QC meantone instrument, one typically starts with a C and an E tuned justly in a 5/4 ratio. The pitches for G, D, and A are obtained by going up in tempered fifths from C or (equivalently) down in tempered fifths from E.[29] The pitches for B, F♯, C♯, and G♯ can be obtained by going up in successive tempered fifths from the E, or by going up a just major third from G, D, A, and E. Similarly, the notes F, B♭, E♭, and A♭ can be obtained by going down in tempered fifths from the C or by going down a just major third from A, D, G, and C.[30] The notes of a QC meantone scale tuned this way are exactly the same as starting with a D and using the Pythagorean tuning method (going up and down by fifths, six times each) but with the tempered fifth instead of a just fifth. The ratios obtained this way are given in Table I.

As happens in the Pythagorean scale, the QC meantone scale ends up with a disagreement between two notes that are the same in equal temperament—in this case between G♯ and A♭. The QC meantone scale also results in a wolf fifth, in fact one that is even more egregious than the Pythagorean wolf fifth; this can be seen in Fig. 3(c).[31] Also, while eight of the twelve possible major thirds have been made just, the other four have become very bad wolf thirds.

As with the Pythagorean scale, music played in this temperament sounds different in different keys. This imposed limitations on Baroque composers,[32] but it also gave each key a unique "color," allowing them to use the dissonance in different chords to convey tension.[33] So, to hear a Baroque piece composed in QC meantone the way the composer intended it to sound, one should listen to it on an instrument that is tuned to QC meantone.

V. JUST INTONATION

Instead of using a fixed fifth for tuning, some scales explicitly define each note in terms of small integer ratios from the root. These scales, known as "just intonation" scales, minimize beats for intervals in which the lower note is the root of the scale. But intervals based on other notes can be very dissonant, as illustrated in Fig. 3(d).

The most well-known just intonation scale is "Ptolemy's intense diatonic scale,"[5] invented by and named for the famous astronomer. Because this scale only defines seven notes, the Ptolemaic column of Table I and Figs. 3(d) and 4(d) also include notes from a five-limit scale generated from intervals present in the Ptolemaic scale.[34]

VI. CIRCULATING SCALES

"Circulating" scales seek a compromise by giving up just ratios in certain intervals to reduce dissonance in others[13,15] so that the composer and performer may circulate freely from key to key without wolf intervals cropping up and without needing to retune the instrument. One such scale is 12-TET in which all keys are equivalent; it has no just intervals other than the octave—but neither are there any intervals that are as dissonant as the most dissonant intervals in other temperaments. Other circulating scales can still allow different colors for different keys; an example is the "well tempering" of Bach's famous composition "The Well-Tempered Clavier," that was likely built on a mixture of pure and tempered fifths.[35]

From Fig. 3(a), one can see that equal temperament has the advantage that all fifths are very close to just. The major thirds, however, are not particularly close to the just ratio of 5/4, differing by 13.7 cents. As the major third is considered by many to be the most important musical interval, this, among other things, has caused some to lament the widespread adoption of equal temperament.[36]

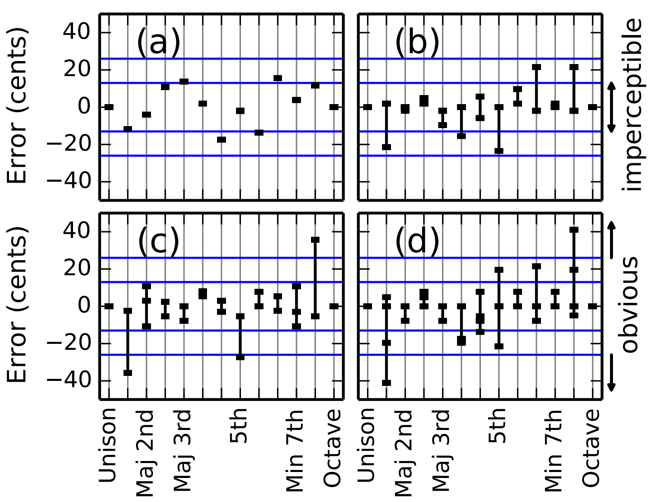

Fig. 4. Deviation of musical intervals from the "ideal" ratios (defined here as containing only integers $\leqslant 16$) for (a) equal temperament, (b) Pythagorean, (c) quarter-comma meantone, and (d) five-limit just intonation. The *x*-axis contains all intervals (major thirds, fifths, etc.), although due to space constraints only certain select ones have been labeled. The *y*-values plotted for a given interval include all twelve types of that interval (many overlapping). The horizontal lines at ± 13 cents indicate the value below which detuning is mostly imperceptible; at ± 26 cents, the value above which detuning in successive notes is obvious.

Figure 4 displays detunings of all intervals for the same four scales from the nearest "ideal" interval, which we have somewhat arbitrarily defined to be one involving a ratio of integers ≤ 16. Each vertical segment represents a musical interval (major thirds, fifths, etc.), and the points on each segment represent all 12 possible types of that interval. In other words, the "5th" column represents all fifths such as C-G, D♭–A♭, and D-A. There are fewer than 12 points on each segment because for a given scale and interval, intervals with different roots can have the same detuning from the nearest "ideal" ratio. By way of example, consider the QC meantone fifths plotted in Fig. 3(c). There are two type of fifths in that figure, one which differs from a just fifth by -5.377 cents (the tempered fifth, multiplicity eleven), and the other which differs from a just fifth by $+35.682$ cents (the wolf fifth, multiplicity one). Those two intervals are represented by the two dots in the "5th" column of Fig. 4(c), but are at different values than in Fig. 3(c) because the error is now calculated versus all possible "ideal" ratios as opposed to versus the specific just ratio of 3/2.

The horizontal lines in Fig. 4 are at detunings of ± 13 cents, below which most people would not notice detuning, and ± 26 cents, above which detuning is obvious to most people when the two notes in an interval are played successively.[6] (When the notes are played together, however, beating makes detunings much more apparent.) Equal temperament has become dominant because none of the intervals sound *too* horrible—they all come close to falling inside the "imperceptible" region—whereas in other temperaments there are at least some intervals that are quite far from ideal.

VII. TUNING MIDI SYNTHESIZERS WITH PITCH BENDS

Knowing the physical basis and the mathematics behind different scales is not sufficient for one to internalize how physics affects the way music is written, performed, and enjoyed. Only by *listening* to the musical intervals produced by different tuning schemes and by (for example) hearing how baroque and modern compositions sound on organs from different eras can one really understand tuning and temperament. However, most physics and music classes do not have access to instruments tuned using historical temperaments.

Because of the difficulty in obtaining such instruments, we developed *Temperament Studio*,[18] a piece of free, open-source software that uses the MIDI (Musical Instrument Digital Interface) audio standard to make it easy for teachers to demonstrate, and for students to explore the physical characteristics of different scales and to test the various mathematical predictions. In this section, we explain how we implemented alternate tuning schemes using MIDI. This information may be useful to others who wish to implement this technique; however, understanding the details of the MIDI synthesis technique is not required to utilize the *Temperament Studio* software (see Sec. VIII) or to work the exercises available as supplementary material.[37]

MIDI is a standard developed in 1983 to allow electronic musical instruments to communicate with each other.[38] Figure 5 illustrates the fundamental difference between transmitting sound information in a waveform stream (a) versus a MIDI stream (b). A waveform stream, such as the data contained on an audio CD, gives a list of numbers that indicate the displacement of the sound wave to be produced at sequential times. A MIDI stream, on the other hand, contains digital commands

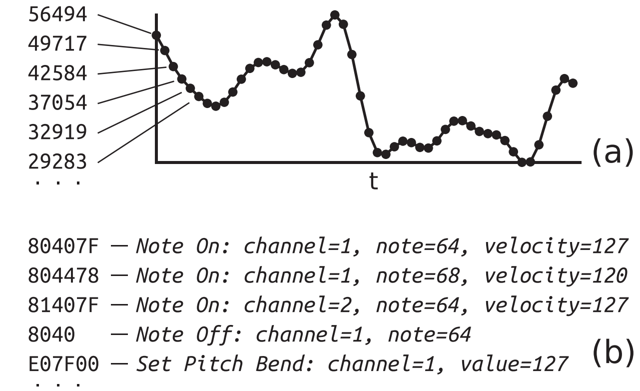

Fig. 5. Comparing a MIDI stream to a waveform stream. (a) A waveform stream can be represented by a sequence of numbers indicating the displacement of the wave at successive moments in time. (b) In a MIDI stream, a sequence of numbers (shown above in hexadecimal format) gives information not about the precise waveform to be produced, but about musical events, such as playing or silencing a note, changing which instrument sound is to be used on a channel, setting the pitch bend for the channel, etc.

describing notes being turned on and off, volume changes, etc. Each event is assigned to a channel, and each channel is assigned to sound like a particular instrument. When MIDI data are sent to a synthesizer or played on a personal computer, the synthesizer or computer must interpret these commands. For example, if the computer reads a "note on" command for a given note on a given channel, it first selects an appropriate waveform based on data from instruments stored in its memory, and then it adjusts the volume and pitch and starts the waveform playing. When a MIDI stream is being generated by a source such as a musical keyboard, the commands can be sent directly to a synthesizer (perhaps part of the keyboard or perhaps an external component) that generates the appropriate sounds in the same manner as the personal computer. A MIDI stream can also be saved to a file, which stores MIDI events and the times that they occur for future editing and/or playback.

Nearly all personal computers sold today come preconfigured to do MIDI musical synthesis and to play MIDI files. MIDI synthesis on personal computers has been used to do things such as generate background music and sound effects in games, to compose music digitally, or to teach piano performance.[39] Advantages of MIDI synthesis over waveform formats such as MP3 or WAV files include much smaller files—a MIDI file only contains information about what notes are on at what time, etc., not the entire waveform to be played—and the ability to edit individual notes, change instruments, etc.

The MIDI standard gives guidelines related to tuning changes, but specifics are left to synthesizer manufacturers and are frequently not implemented at all. Therefore, to realize the historical scales described above on MIDI synthesizers, we used a circuitous method. The MIDI standard provides the ability to *bend* pitches. We can use these pitch bends to adjust the tuning of different notes. A difficulty in this approach is that pitch bends cannot be applied to individual notes or groups of notes, but only to an entire channel. This is an issue, since the different notes in different scales must be "bent" away from equal temperament by differing amounts. Our solution was to move all of the C notes in all octaves from all channels to channel one, all of the C♯ notes to channel two, etc., as illustrated in Fig. 6. Then we apply the appropriate pitch bend to each channel in order to make all of the C's, C♯'s, etc., have the appropriate pitches for the desired tuning scheme.[40]

To calculate the appropriate pitch bend for a non-equal temperament tuning scheme, let us consider the n^{th} note of the scale (i.e., n half steps above the root), which has a frequency of $2^{n/12}f_0$ in equal temperament but some other frequency f_n in a different tuning scheme. Here, f_0 is the frequency of the root note of the scale. The difference in cents between these two frequencies is found via Eq. (2)

$$\begin{aligned} d_{\text{cents}} &= 1200\log_2\left(\frac{f_2}{f_1}\right)\text{ cents}, \\ &= 1200\log_2\left(\frac{f_n}{2^{n/12}f_0}\right)\text{ cents}, \\ &= 100\left[12\log_2(r_n) - n\right]\text{ cents}, \end{aligned} \tag{3}$$

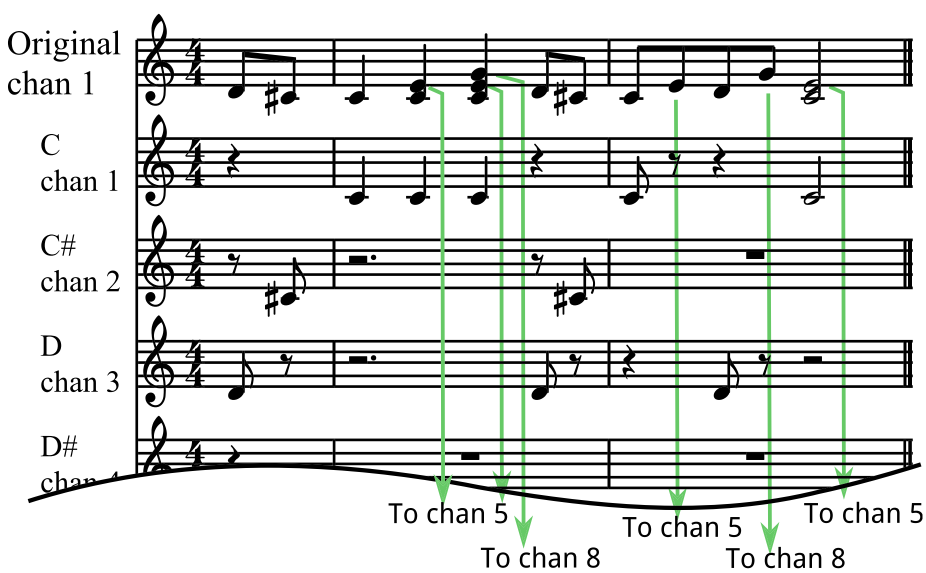

Fig. 6. How notes are moved to individual channels in *Temperament Studio*. To implement a tuning scheme, different notes must be moved to different channels, and then appropriate pitch bends applied to each channel. In this example, all of the notes are originally on MIDI channel 1, but are moved to other channels during playback. (Channel 10 is skipped because in the General MIDI standard it is reserved for percussion.)

where $r_n = f_n/f_0$ is the ratio of the desired frequency of the note (in the new scale) to the root note of the scale (listed for several scales in Table I).

The MIDI standard uses a 14-bit number to describe pitch bend, with a mid-range value of 8192 signifying no pitch bend. Most MIDI players conform to the General MIDI standard,[41] which specifies a pitch bend range of ± 2 half steps (± 200 cents). Assuming this range, the MIDI pitch bend one must apply to a channel is then just Eq. (3) scaled by a factor of 8192/200 and offset by 8192

$$d_{\mathrm{MIDI}} = \{4096[12\log_2(r_n) - n]\} + 8192, \quad (4)$$

where the braces represent rounding to an integer. Rounding gives a resolution of 0.024 cents, well below what humans can perceive, and for typical situations affects beat rates only by millihertz or fractions of a millihertz.

VIII. TEMPERAMENT STUDIO

One of the authors (DD) applied the method described above to demonstrate intonation in physics classes about 7 years ago. The early efforts involved writing scripts in a commercial music editing application. The scripts could not be adjusted interactively in class and were not available to students unless they had access to the same commercial editing application used to create the scripts. To solve these problems, *Temperament Studio* was created as a stand-alone package with a more user-friendly and flexible interface.

The *Temperament Studio* program, packaged with its source and several demo MIDI files, can be downloaded from the internet.[18] It is freely available, open source, and may be distributed and modified as desired. The program is written in Java and should run on nearly all personal computers. The software has a virtual musical keyboard that allows users to play and transpose intervals and chords in different tuning schemes. Users can also load MIDI files of songs and play them using different tuning schemes, even changing tuning schemes on the fly. MIDI files can be resaved in any tuning scheme so that they can be played back with any MIDI player, i.e., without needing this software. (A command line mode is also available, which allows scripted tuning conversions of MIDI files.) Because the software is free, easy to use, and runs on every major platform, it is suitable both for classroom presentations and for student use in homework, lab work, or independent exploration.

Temperament Studio comes with several built-in tuning schemes: equal temperament, Pythagorean, QC meantone, five-limit just intonation, and a well-temperament scheme known as Werckmeister III.[42] Additional arbitrary tuning schemes can be added by the user, either by specifying the notes of the scale (via frequency ratios or detunings from equal temperament) or (for meantone scales) by providing the desired frequency ratio of the fifth interval. Since most world scales consist of five to seven notes per octave,[6] non-Western scales can also be easily emulated using a subset of the twelve tone scale (though none are included). Also, while not implemented in *Temperament Studio*, this pitch-bending technique could also be used to realize scales with more than 12 notes per octave; the MIDI standard allows for 16 different channels, but one is typically reserved for rhythm instruments so scales with up to 15 notes could be synthesized.

For MIDI sound creation from the virtual keyboard or from playback of a file, the user can select any instrument tone that is installed on the computer. For most computers, this would include the 128 standard instruments in the General MIDI instrument set. The default instrument of *Temperament Studio* is an organ tone; organ waveforms tend to produce clear demonstrations because of their extremely harmonic overtone series.[43] Furthermore, the precisely harmonic overtone series makes it relatively easy to synthesize organ sounds, such that they are generally some of the more realistic instruments on most standard computer synthesizers.

Advanced features of the software include enabling the display of different information on the virtual keyboard (such as the frequency of each note in the selected tuning scheme or the ratio of each note's frequency to the root note), transposing MIDI files into different keys, and allowing the user to set and jump to cue points in the playback of a MIDI file to quickly compare the same passage of music with different tuning methods.

Teachers and students can use this software to hear the physics of intonation. For example, an equal temperament C-major chord produces some beating, while the same chord, when the instrument is tuned to the five-limit just intonation scale with a C root, does not have any beating and therefore sounds subtly more pure and powerful to most listeners.[44] But while equal temperament chords have the same character in any key, when we use the other scales we find that moving the chord up by half steps results in some chords that are very pleasant, some that are less consonant, and some that are extremely dissonant.

As another example, students can explore the fifths in the Pythagorean scale. Due to the just ratio of 3/2 for most of the fifths, one will hear no beating as fifths are played moving up the scale, until the wolf fifth is reached (which sounds very strange). The information given above for the various scales allows one to calculate wolf intervals and beat frequencies, and these predictions can be tested experimentally (the supplementary material to this article[37] contains many examples). Beating can be explored with MIDI instruments, such as the default organ setting, or notes can be built from a harmonic series of sine waves, allowing the user to add or remove harmonics to verify which ones are the dominant source of beating for the various intervals.

One can also use *Temperament Studio* to play modern and ancient music in different intonations. For example, by loading a MIDI file and setting the tuning scheme to QC meantone with a root of D, it is possible to simulate what an organ work by Bach or Handel would have sounded like on the organs that Bach and Handel actually played. (If the user additionally wants to explore historical pitch standards, which varied wildly across Europe in the Baroque era,[45] there is an Advanced setting that allows users to give an overall shift up or down in pitch from today's A440 standard.) Several example MIDI songs are included in the package. Moreover, numerous MIDI files can be found online,[46] and there are many programs to create or modify MIDI files including several free, open-source applications.[47]

This method of implementing alternate tunings via pitch bends in MIDI channels does have some limitations. Since the software has to move different notes to specific channels, the channels cannot be used to realize different instrument sounds. As such, *Temperament Studio* can only play MIDI files with one instrument sound.[48] Results using MIDI instruments that employ "stretch tuning" rather than equal temperament may also be problematic. (In stretch tuning, sometimes used for stringed instruments such as pianos or harps, octaves are tuned to be slightly wider than a pure factor of two in order to reduce beating caused by dispersion in the strings.) Also, if a synthesized MIDI instrument employs vibrato, the vibrato will tend to mask or imitate beating, making differences between scales less apparent. Finally, a limitation of MIDI synthesis in general is that many physical instruments, including most string and wind instruments, can have their pitch dynamically adjusted by the musician during a performance. Therefore, no fixed tuning scheme will accurately simulate what one would hear in a live performance.

IX. CONCLUSION

Musical scales involve the physics of resonators, harmonics, and beats, and the mathematics of irrational numbers, integer ratios, and logarithmic intervals. These issues have affected the definitions of notes in scales for centuries. Understanding and experimenting with music can increase student engagement when studying waves and sound. With MIDI synthesis via *Temperament Studio*, students can easily hear the mathematically predicted differences between scales and gain intuition and appreciation for the mathematical and physical nuances of intonation.[37]

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[27]Defining the scale in terms of the just third directly is problematic because going up in thirds will not reach every note.
[28]Equal temperament can be thought of as a meantone scale where the fifth is chosen to be $2^{7/12}$, just the right value to close the octave and make the augmented fourth and diminished fifth equal in frequency.
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[30]Tempered intervals are tuned until beating occurs at a specific rate (which depends on the note being tuned), while just intervals can be tuned by simply eliminating beats. This favors using just major thirds over tempered fifths.
[31]The wolf fifth in Fig. 3(c), F$\sharp$ – D$\flat$, differs from the way QC temperament was actually tuned, which gives a wolf fifth of G$\sharp$ – E$\flat$, because the figure uses C as the root instead of D (for consistency with the other scales).
[32]In this temperament, B, F$\sharp$, C$\sharp$, and A$\flat$ chords have dissonant major thirds. We looked at 302 organ pieces by Bach and Pachelbel and found that all major keys *except these four* were represented in both composers' works. Also, the keys of E and E$\flat$ major were very rarely used in these compositions, perhaps because the B and A$\flat$ chords are important chords in those keys. Patterns were less obvious for minor keys.
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[34]This scale is generated by considering all the different intervals present in the Ptolemaic scale and choosing the most just interval to define the missing notes relative to the root, subject to the limitation that the integers used in the ratios of notes can only be numbers whose prime factorization involves primes ≤ 5 (hence the term "five-limit"). For example, there are two different minor third intervals in the Ptolemaic scale: from note 2 to 5 in Table I is a ratio of 32/27, but from note 4 to 7 is a ratio of 6/5. The most just of the two, 6/5, is used to define the minor third above the root. This scale is referred to as "the duodene of C" in D. B. Doty, "The ladder of primes, part one: Two, three, and five," in *The Just Intonation Primer* (Other Music, Inc., San Francisco, 2002), Chap. 4, pp. 35–50.
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[37]Several student exercises are available as supplementary material to this article at http://dx.doi.org/10.1119/1.4926956.
[38]MIDI Manufacturers Association, "MIDI 1.0 Detailed Specification," Technical Report (MIDI Manufacturers Association, 1995) http://www.midi.org/techspecs/midispec.php (last accessed July 29, 2014).
[39]See http://www.midi.org/aboutmidi/funwithmidi.php for several examples of MIDI use (last accessed July 29, 2014).
[40]Since authoring the *Temperament Studio* software, we have discovered that at least four other programs have used this same pitch bend method, namely an unnamed program by Jules Siegel (Refs. 49 and 50), Helm (Ref. 51), Miditune (Ref. 52), and Scala (Ref. 53). (Their features, however, differ from *Temperament Studio* with regards to real-time playback and/or ease-of-use for students).

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[44]An exception to this was when we demonstrated the software to a student with perfect pitch. She perceived any non-equal-temperament chords to be slightly out of tune and disliked them.
[45]C. Wolff and M. Zepf, *The Organs of J.S. Bach: A Handbook* (University of Illinois Press, Urbana, 2012), pp. 1–2.
[46]One large archive of free MIDI files is http://www.mutopiaproject.org (last accessed July 29, 2014).
[47]"MuseScore" is a popular free, open-source program available on all major platforms. It can be downloaded from: http://musescore.org (last accessed January 26, 2015).
[48]If multiple instrument sounds are specified in a MIDI file, the user can select to have all notes played using the first instrument set up in the file, to apply all instrument changes to all channels, or to ignore instrument changes and use a specific MIDI instrument selected by the user.
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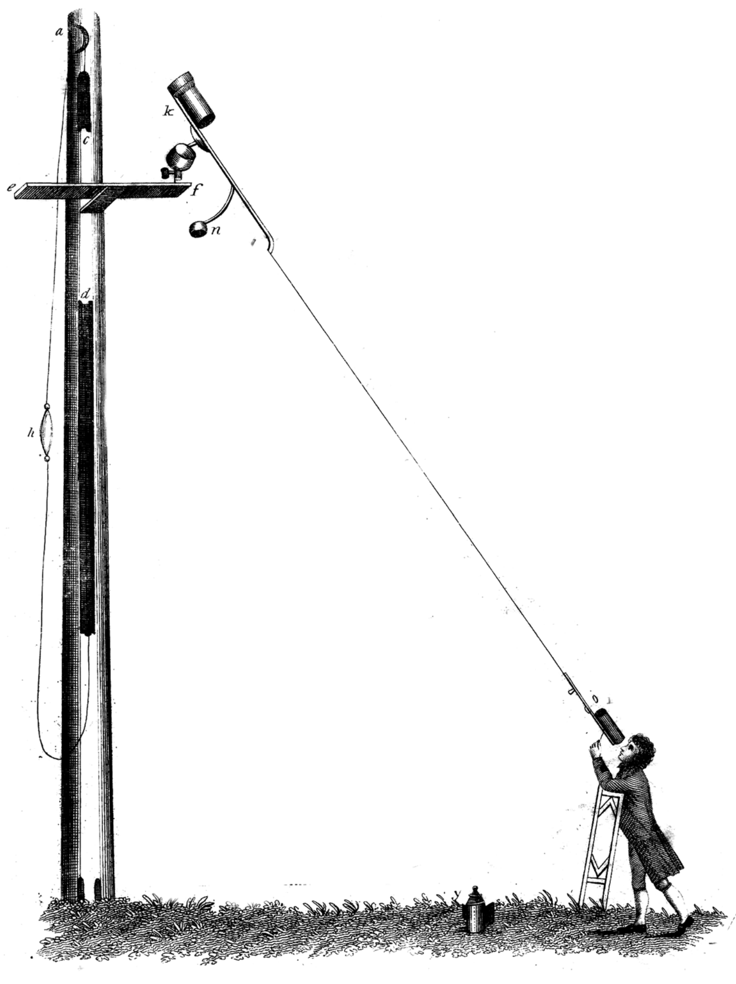

Long Focal Length Telescope

Single lenses suffer from chromatic aberration, which, for an astronomical telescope, results in colored halos around the images. This problem was alleviated by the invention of the achromatic lens in the later years of the 18th century. However, if a single lens has a long focal length for a given lens diameter (i.e., has a large enough f/ratio), these chromatic effects are less prominent. The design in the picture, with widely separated objective and eyepiece lenses, is due to Christian Huygens who, with his brother Constantine, built aerial telescopes with lengths up to 210 ft. This image is from The *New Encyclopedia* ca. 1800. (Notes by Thomas B. Greenslade, Jr., Kenyon College)