applied optics

Light splitting with imperfect wave plates

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Received 25 October 2016; revised 2 January 2017; accepted 2 January 2017; posted 3 January 2017 (Doc. ID 279468); published 27 January 2017

We discuss the use of wave plates with arbitrary retardances, in conjunction with a linear polarizer, to split linearly polarized light into two linearly polarized beams with an arbitrary splitting fraction. We show that for non-ideal wave plates, a much broader range of splitting ratios is typically possible when a pair of wave plates, rather than a single wave plate, is used. We discuss the maximum range of splitting fractions possible with one or two wave plates as a function of the wave plate retardances, and how to align the wave plates to achieve the maximum splitting range possible when simply rotating one of the wave plates while keeping the other one fixed. We also briefly discuss an alignment-free polarization rotator constructed from a pair of half-wave plates.

OCIS codes: (230.1360) Beam splitters; (230.5440) Polarization-selective devices; (260.5430) Polarization; (260.1440) Birefringence.

https://doi.org/10.1364/AO.56.001062

1. INTRODUCTION

A common method for splitting a polarized laser beam into two parts is to use a half-wave plate to rotate the polarization to an arbitrary direction, and then pass the light through a polarizing beamsplitter, as shown in Fig. 1. If the half-wave plate generates a perfect 180 deg of retardance, the fraction of power exiting either port of the beamsplitter can be smoothly changed to any value from 0% to 100% of the light by simply rotating the wave plate.

The retardance of a wave plate is wavelength dependent. In a lab where lasers at multiple wavelengths are used, often one will not have a half-wave plate designed for a particular wavelength readily available, but may have access to half-wave plates designed for a different wavelength, or two quarter-wave plates made for that or another wavelength. In student labs and in optics demonstrations, often inexpensive polymer wave plates [1] with much looser tolerances are employed. Generally, unless a wave plate is manufactured to produce a precise half-wave shift at the specific wavelength to be used, an incorrect retardance will limit the range of splitting fractions (the fraction of light exiting a particular port of the beamsplitter) that can be achieved in this scheme.

Many methods have been devised to diminish the wavelength dependence of wave plates. Using zero-order wave plates [2] reduces, but does not eliminate, wavelength dependence. Similar to the way in which chromatic aberration is reduced in lenses, by combining different birefringent materials with different wavelength dependence, achromatic wave plates can be constructed [2–6]. Combinations of retarders made of the same material can also be used to reduce chromatic effects [7–10]. Other ways of realizing phase retarders with less chromatic dependence have also been developed [11,12]. However, achromatic phase retarders are typically more expensive, and sometimes introduce other side effects such as beam deflection.

In this paper we briefly discuss limitations caused by deviations from an ideal half-wave retardance in the single wave plate beamsplitting method described above. We then consider the advantages of using a *pair* of wave plates in place of a single half-wave plate, as shown in Fig. 2. We show that by using a pair of non-ideal wave plates together, a much greater range of splitting fractions can typically be achieved.

In addition to examining what is possible by rotating both wave plates arbitrarily, we also consider when it is possible to smoothly adjust the power in either beam from 0% to 100% by rotating only one of the two wave plates, keeping the other one fixed. When possible, this makes the beamsplitter easier to use. We determine what the angle of the fixed wave plate should be for a given pair of retardances, and how to practically align the fixed wave plate in such a configuration. We discuss the range of possible splitting fractions for an arbitrary pair of wave plates used in this manner, and we also consider what is possible when the two wave plates are rigidly fixed to each other and the pair is rotated together.

2. JONES ALGEBRA

To calculate the fraction of light in either beam exiting the polarizing beamsplitter, we will make use of Jones algebra [13].

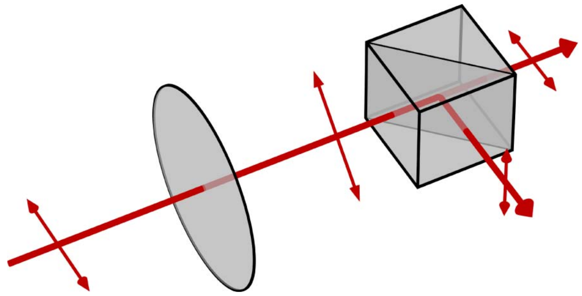

Fig. 1. Beamsplitting using an ideal half-wave plate and a polarizing beamsplitter. The beam entering on the left is horizontally polarized. After the polarization is rotated by the wave plate, a polarizing beamsplitter separates the horizontal and vertical components.

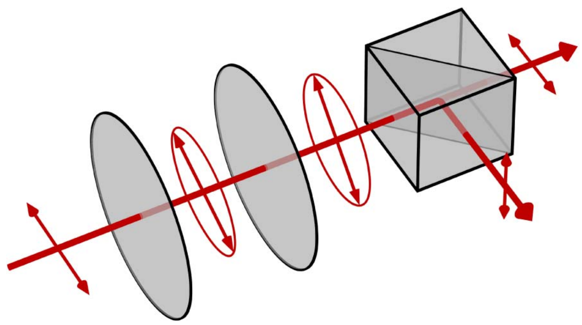

Fig. 2. Beamsplitting using two non-ideal wave plates. The beam entering on the left is horizontally polarized. After the polarization is modified by passing through a pair of wave plates, most generally resulting in an elliptical polarization state, a polarizing beamsplitter separates the horizontal and vertical components.

The polarization state of the light can be written as a Jones vector,

$$J = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{1}$$

where a and b are complex numbers that represent the magnitude and phase of the component of the light's oscillating electric field in the x and y directions, respectively. We will choose our coordinate system such that the light before passing through the first wave plate is represented by the Jones vector

$$J = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2}$$

The polarization of the light after passing through an optical element can be described by the multiplication of the Jones vector by a Jones matrix, $J' = MJ$. The Jones matrix that describes a wave plate with a retardance δ whose slow axis makes an angle of θ relative to the x axis is given (to within an arbitrary overall phase factor) by

$$M(\theta, \delta) = \begin{pmatrix} e^{i\delta} \cos^2 \theta + \sin^2 \theta & (e^{i\delta} - 1) \cos \theta \sin \theta \\ (e^{i\delta} - 1) \cos \theta \sin \theta & \cos^2 \theta + e^{i\delta} \sin^2 \theta \end{pmatrix}. \tag{3}$$

3. SPLITTING WITH A SINGLE WAVE PLATE

We will first consider the performance of a beamsplitter that utilizes a single wave plate, as shown in Fig. 1. To the extent that the beamsplitter approximates an ideal polarizer, light missing from the x polarization will exactly equal the power in the y polarization. As such, we only need to consider the splitting fraction for one of the polarizations. We have arbitrarily chosen to consider the fraction of light transferred to the y polarization.

After x-polarized light passes through a wave plate, the fraction of light power in the y polarization ranges from zero (when either the fast or slow axis is exactly aligned with the incoming polarization) to a maximum value that can be determined by considering the Jones vector of the light exiting the wave plate:

$$J' = M(\theta, \delta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4}$$

Utilizing the equations above, it can be shown that the fraction of power in the y direction is

$$P_y = \sin^2 \left(\frac{\delta}{2} \right) \sin^2(2\theta). \tag{5}$$

From this expression, it is clear that as the wave plate is rotated, the maximum fraction of light that can be moved to the y direction is $\sin^2(\delta/2)$. Only for the case of an ideal half-wave plate with a retardance of $\delta = \pi$ (or 3π, 5π, etc.) radians can the full range of splitting fractions be achieved.

As a side note, by measuring the maximum amount of light that can be transferred to the orthogonal polarization by rotating the wave plate in the single wave plate setup, the retardance of a particular wave plate can be determined. If we maximize Eq. (5) by setting θ to 45 deg and solve for δ, we get

$$\delta = 2 \arcsin\left(\sqrt{P_{y\ \max}} \right). \tag{6}$$

Of course, arcsin is a multi-valued function, such that for any solution δ, $\delta + 2n\pi$ or $2n\pi - \delta$, where n is an integer, are also solutions. The fact that we get new solutions by adding factors of 2π to δ simply illustrates that offsets by integer multiples of 2π do not affect the polarization of light passing through a wave plate. In fact, that is precisely the idea behind multiple-order wave plates.

The $2n\pi - \delta$ solutions can be explained by noting that $M(\theta + 90°, -\delta) = e^{-i\delta} M(\theta, \delta)$. As such, other than an overall phase factor, a wave plate with a retardance of $2n\pi - \delta$ behaves equivalently to a wave plate with a retardance of $2n\pi + \delta$, except that the angle at which it needs to be placed is offset by 90 deg.

Another thing to consider is what effect you get if the incoming light polarization is not aligned with one of the axes of the linear polarizer. If the light is polarized such that $J = (\cos(\phi), \sin(\phi))$, it turns out that the range of splittings is still just $\sin(\delta/2)^2$. However, rather than going from no power exiting the y-polarization port of the polarizer to a fraction of $\sin(\delta/2)^2$, the amount of light in the y-polarized beam goes as

$$f = f_0 - \frac{1}{2} \cos(2\phi - 4\theta) \sin \left(\frac{\delta}{2} \right)^2, \tag{7}$$

where

$$f_0 = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\delta}{2}\right)^2 \cos(2\phi). \tag{8}$$

4. MATCHED AND ANTI-MATCHED WAVE PLATES

We will now consider the use of two wave plates, as shown in Fig. 2. First we will discuss what happens when two "matched" wave plates with the same retardance δ (or, equivalently, two "anti-matched" wave plates with retardances of δ and $2\pi - \delta$) are used together to change the light polarization before the polarizing beamsplitter.

This case is relevant because often one is in possession of multiple wave plates with similar characteristics, perhaps purchased at the same time. We consider this case before the more general case because it is mathematically simpler and more intuitive than the treatment of two wave plates with arbitrary retardances.

We will denote the angle of the wave plate slow axis relative to the x axis as θ_1 and θ_2 for the first and second plates that the light goes through, respectively. We will assume that we start with light that is linearly polarized along the x axis before passing through the two wave plates.

We want to know under what conditions we will be able to smoothly shift all of the light from the x to the y polarization by rotating only *one* wave plate, keeping the other plate fixed. This makes for a much simpler arrangement than if both wave plates must be adjusted, and a two-dimensional parameter space must be explored to realize the desired splitting ratio. Also, when this is not possible, we want to determine what range of splitting fractions *is* possible with any arbitrary pair of matched wave plates when only one plate is allowed to rotate.

To determine what fraction of the light remains in the x polarization, or, conversely, what fraction is shifted to the vertical polarization, we can first find the Jones vector to describe the polarization of the light after passing through the two wave plates. This can be easily calculated as

$$J' = M(\theta_2, \delta)M(\theta_1, \delta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9}$$

When we do this we find that, except for certain cases, the phases of the two polarization components are not the same—the exiting light will not be linearly polarized. But for the application of splitting a beam, these phases are not important—only the magnitude of the two components matters.

To find the amount of light that will end up in either of the beams after the polarizing beamsplitter, we simply calculate the squared magnitude of one of the elements of the Jones vector. When we do this, we find that the fraction of power transfered to the y polarization is given by

$$\begin{aligned} P_y = {} & 2(1 - \cos(\delta))\cos^2(\theta_1 - \theta_2)[\cos^2(\theta_2)\sin^2(\theta_1) \\ & + 2\cos(\delta)\cos(\theta_1)\cos(\theta_2)\sin(\theta_1)\sin(\theta_2) \\ & + \cos^2(\theta_1)\sin^2(\theta_2)]. \end{aligned} \tag{10}$$

This is a fairly complicated expression, but it is clearly symmetric—it does not matter whether it is the first or second wave plate that is fixed while the other one rotates.

For any pair of matched plates, and with one wave plate fixed at any angle, it is possible to rotate the other wave plate such that the outgoing light is entirely polarized in the x direction. One simply has to place them with their slow axes at 90 deg to each other, such that they cancel each other out. This means that it is always possible with matched or anti-matched plates to make Eq. (10) zero, so that the splitting range is just determined by the largest value we can obtain from Eq. (10) for a given δ. This can be determined more easily by transforming Eq. (10) to the coordinates $\theta_{\mathrm{sum}} = \theta_1 + \theta_2$ and $\theta_{\mathrm{dif}} = \theta_1 - \theta_2$. When we do this, the above can be written as

$$\begin{aligned} P_y = {} & \sin^2(\delta/2)\cos^2(\theta_{\mathrm{dif}})[2 + (\cos(\delta) - 1)\cos(2\theta_{\mathrm{dif}}) \\ & - 2\cos^2(\delta/2)\cos(2\theta_{\mathrm{sum}})]. \end{aligned} \tag{11}$$

In this form, the parameter θ_{sum} only appears once. So it is clear that the maximum is obtained when $\cos(2\theta_{\mathrm{sum}}) = -1$. We can set $\theta_{\mathrm{sum}} = \pi/2$ without placing any limit on θ_{dif}, so we will do so. Then it is simply a matter of finding what value of θ_{dif} maximizes the expression

$$\cos^2(\theta_{\mathrm{dif}})[2 + (\cos(\delta) - 1)\cos(2\theta_{\mathrm{dif}}) + 2\cos^2(\delta/2)]. \tag{12}$$

This is maximized when $\theta_{\mathrm{dif}} = \arccos(1/\sqrt{1 - \cos(\delta)})$, if this value turns out to be real. This occurs as long as $\pi/2 \le \delta \le 3\pi/2$. In this case, the maximum value of P_y is 1, meaning that by rotating a single wave plate, the entire range of splitting fractions is possible.

The lower limit intuitively makes sense. A retardance of less than 1/4 wave is insufficient to make an equal superposition of both linear polarizations. If the fixed wave plate cannot make an equal superposition, the other wave plate will not be able to complete the process of moving the light entirely to the y polarization. The upper limit makes sense when we consider again that a wave plate with a retardance of $3\pi/2$, or 3/4 waves, behaves just like a quarter-wave plate that is rotated an additional 90 deg.

Furthermore, following the reasoning above, if the retardance of the plates is between $\pi/2$ and $3\pi/2$ radians, and if the first plate is placed at an angle such that light initially polarized in the x direction ends up in an equal superposition of x and y, it makes sense that by rotating the second wave plate it is possible to smoothly change the light from being entirely in the x direction to being entirely in the y direction.

The appropriate angle for the fixed wave plate is the one that yields $\theta_{\mathrm{sum}} = \pi/2$ when θ_{dif} is equal to the value given above. This means that the fixed wave plate should be set to the angle

$$\theta_{\mathrm{EQ}} = \frac{1}{4}\left[\pi \pm 2\arccos\left(\sqrt{\frac{1}{1 - \cos(\delta)}}\right)\right]. \tag{13}$$

For a wave plate with a retardance in the range of $\pi/2 \le \delta \le 3\pi/2$, this angle is, coincidentally, the angle that will change x-polarized light into an equal superposition of x and y. This makes sense in light of the arguments given above; if the first wave plate creates an equal superposition, the second wave plate can undo the effects of the first plate, or double them, bringing all of the light power into the y polarization component.

For retardances outside of the range $\pi/2 \le \delta \le 3\pi/2$, the optimum occurs when $\theta_{\mathrm{dif}} = 0$, such that the angle of the fixed wave plate is $\pi/4$ radians. This results in a maximum splitting fraction range of $\sin^2(\delta)$. For a wave plate with a retardance less than $\pi/2$ or greater than $3\pi/2$, it is impossible to move half of the light power into the y polarization direction. An angle of $\pi/4$ radians, or 45 deg, is the angle that will move the maximum possible amount of light power into the y polarization direction.

In general, aligning the angle of the first wave plate is simple in practice. To set up a system with matched wave plates, one first places one wave plate into the system, and adjusts it such that as close as possible to half of the light exits the polarizing beamsplitter with the opposite polarization as the incoming light. The second wave plate is then placed either before or after the first wave plate. This wave plate can then be rotated to produce the desired splitting fraction. If you can get exactly half of the light into the other polarization using just the first wave plate, you will be able to explore the full range of splitting fractions smoothly by rotating the second plate. If not, the maximum fraction of light you will be able to transfer to the orthogonal polarization will be $\sin^2(\delta)$.

As discussed above, a wave plate with a retardance of $2\pi - \delta$ should perform equivalently to a wave plate with a retardance of δ for this application, provided that the angle of the plate is rotated by an additional 90 deg. As such, the results and procedures for a pair of anti-matched plates (one having a retardance of δ, and the other having a retardance of $2\pi - \delta$) are identical to those of a pair of wave plates with matched retardance.

Experimentally measured splitting fractions for several matched wave plates, along with theoretical curves, are shown in Fig. 3.

5. ARBITRARY RETARDANCES

In the more general case, where the retardances of the two plates could be different, an analytical solution is more difficult. In addition to involving an additional parameter, it is no longer enough to consider just the maximum amount of light transferred to the y polarization, because you can no longer guarantee that, for any given angle for the fixed wave plate, it is possible to return all of the light to the x polarization simply by rotating the other wave plate. Instead, to determine the maximum range of splitting fractions possible by rotating a single wave plate, one must consider what both the maximum and minimum of the y component of polarization will be as the moving plate is rotated, as a function of the angle of the fixed plate, and then find the angle for the fixed plate that produces the largest difference between maximum and minimum.

Because the analytical treatment is complicated and nonintuitive, we instead performed a numerical study. The results of the study are shown in Figs. 4 and 5. For matched or

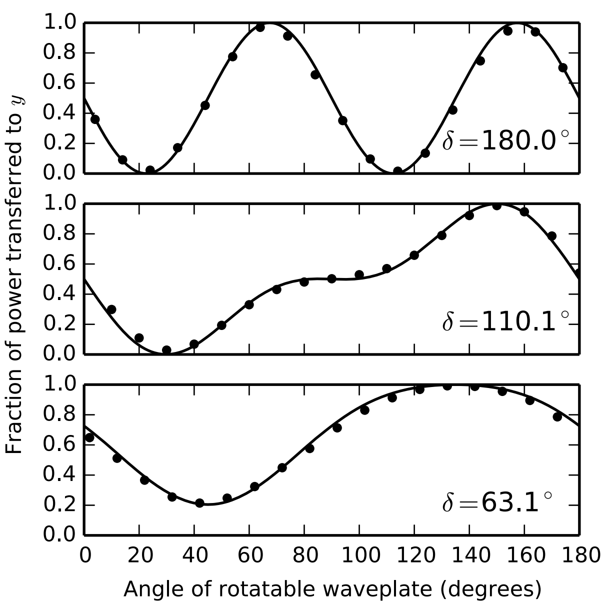

Fig. 3. Theory and experimental data for matched wave plates. The dots are experimentally measured data. The lines represent the theoretical splitting fraction as a function of the second wave plate's rotation angle given in Eq. (10), assuming that the first wave plate is fixed at the ideal angle for matched wave plates discussed in the text. The measured retardance for each plate is indicated on each plot. In the topmost plot, two half-wave plates designed to be used at 408 nm were used at 408 nm. In the middle plot, the same two wave plates were used, but the laser used was a 633 nm HeNe laser. The lower plot was taken using two quarter-wave plates designed for 408 nm and a HeNe laser at 633 nm.

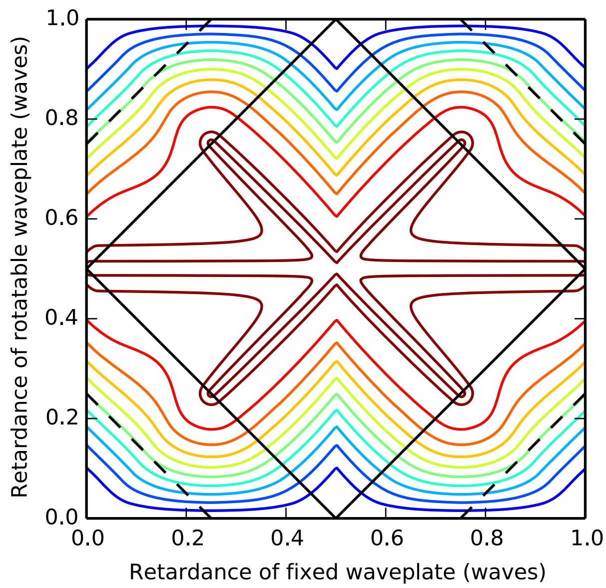

Fig. 4. Range of splitting possible using an arbitrary pair of wave plates. The plot shows the maximum range of splittings (the highest fraction transferred to the y polarization minus the minimum fraction) possible by holding one wave plate fixed and rotating the other, as a function of the retardances of the two wave plates. The contour lines are drawn at 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99, and 0.999, with the center of the plot having a value of 1, and the corners having a value of zero. Inside the diamond outlined by a solid line is the region of parameter space in which it is possible to achieve the entire range of splitting by *rotating both* wave plates. The dotted line indicates the region where a splitting range greater than 50% can be obtained by rotating both plates.

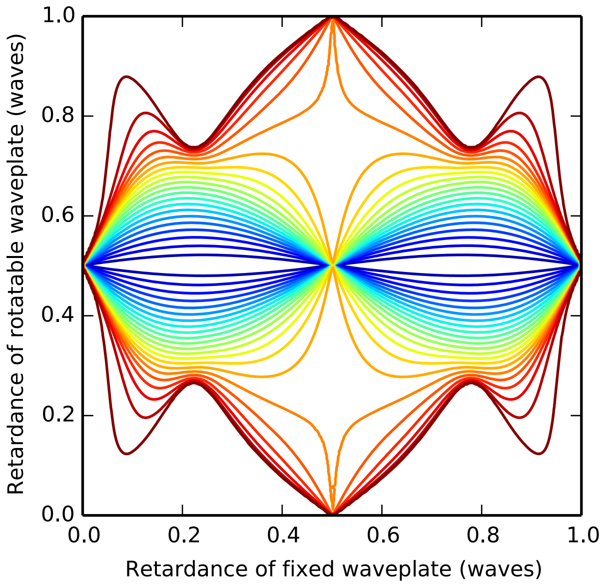

Fig. 5. Optimum angle for the fixed wave plate. This shows the optimum angle for the fixed wave plate in order to achieve the largest range of splitting fractions by rotating the rotatable wave plate. The contour lines are drawn every 2 deg from 2 to 44 deg. The value at the center of the plot is 0 deg, and the value at the corners is 45 deg. These values were found numerically, and used to calculate the maximum range shown in Fig. 4.

anti-matched wave plates, the results of the numerical study are in agreement with the equations in the previous section. The results displayed in Fig. 4 show that for a fairly wide range of retardances, the range of splitting fractions possible while holding one plate fixed is quite large.

It can be shown analytically that, as with the matched case, the order of the wave plates (whether the light passes through the fixed or rotatable plate first) does not matter. However, from Fig. 4 it is clear that, when the plates are not matched or anti-matched, the largest range of splitting fractions occurs when the fixed plate is the one with a retardance that most differs from a half-wave, and the rotating plate is the one that has a retardance closest to a half-wave.

We also numerically tested whether the range of splitting fractions could be increased if the final polarizing beamsplitter is not aligned to the polarization direction of the incoming light before passing through the wave plates. We found that some improvement was possible for most wave plate retardances, but the improvement was small—a fraction of a percent for most wave plate combinations, and never larger than a few percent.

6. ROTATING BOTH WAVE PLATES

If, rather than fixing one wave plate, we allow both wave plates to be moved, we gain greater flexibility at the expense of increased experimental complexity. As we will see, we can get large ranges of splitting fractions with a wider range of retardances with this method. But instead of a smoothly changing splitting fraction as a function of one parameter, with this method two plates will have to be adjusted such that realizing a particular splitting fraction involves searching a two-dimensional parameter space.

Allowing both plates to move affords a simplification in our analysis. No matter what the retardances of the two plates may be, we can always align them such that either the fast or the slow axis of each plate is in the x direction, such that no power is transferred out of the x polarization. Thus the range of splitting fractions is simply the maximum fraction that can be moved to the y polarization by a combined rotation of the two wave plates.

Performing another numerical calculation, we found that a full range of splitting fractions is possible as long as $\pi \leq \delta_1 + \delta_2 \leq 3\pi$ and $|\delta_1 - \delta_2| < \pi$, where δ_1 and δ_2 are the retardances of the two wave plates. This is indicated by the region enclosed by the black diamond in Fig. 4. Otherwise, the maximum range of splitting fractions is just $\sin^2((\delta_1 \pm \delta_2)/2)$, where the upper sign is used if both retardances are less than or greater than π, the lower being used when one is greater and one less than π.

7. CO-ROTATING WAVE PLATES

Another approach would be to mount two wave plates rigidly together, their slow axes fixed at a relative angle ϕ such that the angles of the two wave plates track each other as the pair is rotated: $\theta_2 = \theta_1 + \phi$. An obvious example of this would be mounting two ideal quarter-wave plates ($\delta = \pi/2$) with their slow axes in the same direction ($\phi = 0$) to effectively make one ideal half-wave plate.

Again, for simplicity, we will first consider the case of two matched or anti-matched plates. The amount of light in the y polarization after passing through a pair of matched wave plates is given again by Eq. (11) if we let $\theta_{\mathrm{sum}} = \theta_1 + \theta_2 = 2\theta_1 + \phi$ and let $\theta_{\mathrm{dif}} = -\phi$. For a given retardance δ and relative angle ϕ, the range of possible splitting fractions (the maximum P_y minus the minimum P_y) as theta is rotated is just $\cos^2(\phi)\sin^2(\delta)$. For anti-matched wave plates, the range of splitting fractions is just $\sin^2(\phi)\sin^2(\delta)$.

From this, we see that the best relative angle for this beamsplitting application, for any retardance, is $\phi = 0$ (or $\phi = 90$ deg for anti-matched wave plates). The best you can do in this configuration is add the retardance of the wave plates, and it is clear that the example of using two ideal quarter-wave plates (or, equivalently, two 3/4-wave plates or one 1/4- and one 3/4-wave plate) is the only case where two matched wave plates can be rotated together to get the maximum range of splitting fractions.

The more general case of two co-rotating wave plates with arbitrary retardances is more complicated. We numerically verified that the best that can be done is to combine the wave plates, with the slow axes either aligned or at 90 deg to each other, to add or subtract retardance to get as close as possible to a perfect half-wave plate. As such, we can treat the combination like a single wave plate, as discussed in Section 3, and it is clear that the full range of splitting values is only possible when the two retardances of the plates add up to or differ by a half-wave.

8. ALIGNMENT-FREE POLARIZATION ROTATOR

A pair of perfect half-wave plates co-rotating with a fixed relative angle ϕ produces a range of possible splitting fractions of zero; when $\delta = \pi$, the θ_{sum} term drops out of the equation, and the amount of power in the y polarization does not change as the pair of wave plates rotates. It turns out that the phases of the two polarization components are unchanged as well, with the full Jones vector simplifying down to

$$M(\theta_1 + \phi, \pi)M(\theta_1, \pi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(2\phi) \\ \sin(2\phi) \end{pmatrix}. \quad \textbf{(14)}$$

In this special case, linearly polarized light will be rotated by an angle of 2ϕ, regardless of how the pair of wave plates is rotated relative to the direction of the polarization of the incident light. As such, this combination could be useful to create a device that could be dropped into place to rotate light polarization by a fixed angle without the need for careful alignment (other than the alignment done to lock the wave plates together to create the device), similar to an optically active polarization rotator.

9. COMPARING TECHNIQUES

We can use Fig. 4 to compare the use of two wave plates to the single wave plate scheme by noting that the single wave plate setup is equivalent to a two wave plate setup for which the fixed wave plate has zero retardance. Looking at the figure, we see that for a given rotating wave plate retardance, increasing the retardance of the fixed plate from zero can drastically increase the range of splitting fractions. The improvement is greatest when the retardance of the rotating wave plate is not too close to an ideal half-wave (which makes sense, because with an ideal half-wave plate the full range of splitting is possible with a single wave plate)—and when the fixed plate has a retardance close to a quarter-wave.

By integrating the area inside a given contour in Fig. 4 and comparing that to the total area of the figure, we can quantitatively determine what the odds would be to achieve at least a given range of splitting fractions if two wave plates with arbitrary retardances were chosen randomly with an even distribution from zero to a full wave of retardance. This gives us another way to compare the different beamsplitting methods discussed in this paper. The results are shown in Fig. 6.

From Fig. 6(a) we see that, for a random selection of wave plates, using a two wave plate scheme typically increases the range of possible splitting fractions significantly. For example, when a single wave plate with a random retardance is used, there is only a 50% chance that a range of splitting ratios as large as 0.5 or larger will be achieved. For a pair of random wave plates, there is an 88% chance of a ratio as large as or larger than 0.5. Furthermore, except for the very highest ranges of splitting ratios, a setup with one fixed and one rotatable plate has probabilities comparable to a setup in which both plates are moved to change the splitting ratio.

The probability of randomly selecting plates that will allow a full range of splitting ratios from 0 to 1 goes to zero for the single plate scheme, as it requires the selection of an ideal half-wave plate from an infinite range of possibilities. The probability similarly goes to zero for the two-plate scheme in which only one plate is rotated, as a full range of splitting ratios requires the random selection of two plates with precisely matched or anti-matched retardance. The probability for a full range of splitting ratios goes to 0.5 for a pair of randomly selected wave plates if both plates are rotated.

Because one often purchases multiple wave plates at the same time, it is not unlikely to have a pair of matched wave plates available to use in this scheme. In Fig. 6(b) we see the same curves plotted in Fig. 6(a) except with the assumption that both wave plates are matched. For much of the curve, the scheme in which one wave plate is fixed is lower than in Fig. 6(a). This is because there is only one chance of selecting a plate near to an ideal half-wave plate. But it is higher at high splitting ratio ranges, with a 50% chance of a full range of splitting ratios for both two-plate schemes.

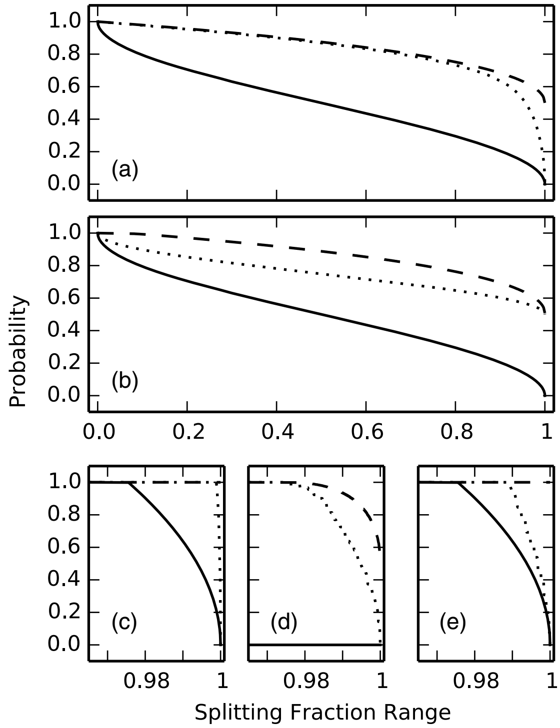

Fig. 6. Probability of having at least a given range of splitting fractions. In each plot the horizontal axis represents the range of possible splitting fractions, and the vertical axis is the probability that, when selecting wave plates with a random retardance, a given setup will be able to be adjusted over a range of splitting fractions that is as large as or larger than the value on the horizontal axis. The solid line represents a single wave plate used to change the light polarization. The dotted line represents two randomly selected wave plates such that the one with a retardance closest to a half-wave is rotated while the other is fixed at the optimum angle discussed in Section 5. The dashed line is for two randomly selected wave plates that are both allowed to rotate. The data plotted in (a) were calculated assuming that any retardance was possible for the randomly selected wave plates. Plot (b) assumed that the wave plates in the two wave plate schemes are matched. Plots (c) and (d) do not assume matched wave plates, but assume the wave plates have retardances that are within 10% of an ideal (c) half-wave or (d) quarter-wave. Plot (e) does not assume matched wave plates, but assumes that the fixed wave plate has a retardance within 10% of an ideal quarter-wave, and the rotatable plate has a retardance within 10% of an ideal half-wave shift.

In Figs. 6(c)–6(e) we consider the case in which wave plates are being used at a wavelength that does not significantly differ from the design wavelength. In these plots we show the probability of getting at least a given range of splitting ratios for wave plates with retardances randomly selected to be within 10% of an ideal half- or quarter-wave. Note that the horizontal axis on these plots is zoomed in to show the range of 0.965 to the full range of splitting fractions.

In Fig. 6(c), since the plates are assumed to have a retardance near to an ideal half-wave, even with a single wave plate a splitting range of at least 0.976 is guaranteed. Still, by using two plates, a nearly perfect range of splitting ratios is guaranteed. For a retardance near to a quarter of a wave, as is shown in Fig. 6(d), a single plate cannot generate a range of splitting ratios much higher than 0.5. For a pair of plates, however, very high ranges are possible. Figure 6(e) shows improvements when a wave plate with a retardance near to a quarter-wave is added to a setup using a single plate with a retardance near to a half-wave.

10. CONCLUSION

We considered the performance of an adjustable beamsplitter in which a linearly polarized beam of light passes through one or two wave plates followed by a polarizing beamsplitter. We calculated the range over which we could change the amount of light exiting each port of the beamsplitter by rotating the wave plates, as a function of the wave plate retardances, subject to different limitations on how the wave plates are rotated.

For a single wave plate, we found that the range of possible splitting fractions was $\sin(\delta/2)^2$, such that the fraction of light exiting one port of the beamsplitter could go from 0 up to $\sin(\delta/2)^2$ while the fraction in the other port went from 1 to $1 - \sin(\delta/2)^2$.

For a pair of identical wave plates (or, equivalently, a pair of anti-matched wave plates), we found that the entire range of splitting could be achieved by simply rotating one wave plate while keeping the other fixed as long as the retardance of the plates was between $\pi/2$ and $3\pi/2$. Otherwise, the range of splitting fractions was limited to $\sin(\delta)^2$. For any pair of matched or anti-matched wave plates, the largest range of splitting fractions is obtained when the fixed wave plate is rotated to produce as close as possible to an equal amount of light in both polarization directions. If the two wave plates do not have matched or anti-matched retardances, it is impossible to get a full splitting range unless one of the wave plates is an ideal half-wave plate. However, it is typically possible to get much larger splitting ranges than one would get with a single, non-ideal wave plate.

If both wave plates are rotated, a full splitting range can be achieved as long as the two retardances have a sum between π and 3π radians and differ by less than π radians. Otherwise, the range of splitting fractions is $\sin((\delta_1 \pm \delta_2)/2)^2$.

If two wave plates are rigidly fixed to one another and rotated together, we found that the best that can be done is to align the plates such that their slow axes are either parallel or perpendicular, such that together they act as a single wave plate whose retardance is the sum or difference of the individual retardances, in the manner that produces a net retardance as close as possible to a half-wave. As such, unless $\delta_1 + \delta_2 = \pi$ or $|\delta_1 - \delta_2| = \pi$, the full range of splitting fractions cannot be achieved with this method.

Finally, we showed that two ideal half-wave plates, with their slow axes fixed at a relative angle of ϕ, will rotate an incoming linear polarization by 2ϕ, regardless of how the pair of wave plates is rotated, creating an alignment-free polarization rotator.

Funding. National Science Foundation (NSF) (PHY1205736).

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