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Bias error analysis for phase and amplitude gradient estimation of acoustic intensity and specific acoustic impedance

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Sound intensity measurements using two microphones have traditionally been processed using a cross-spectral method with inherent error in the finite-sum and finite-difference formulas. The phase and amplitude gradient estimator method (PAGE) has been seen experimentally to extend the bandwidth of broadband active intensity estimates by an order of magnitude. To provide an analytical foundation for the method, bias errors in active intensity and specific acoustic impedance are presented and compared to those of the traditional method. Bias errors are reported for a plane-wave field and sound radiated from a monopole and a dipole. Additionally, bias errors are reported for reactive intensity, the estimation of which is unchanged by the PAGE method for the two-microphone case. © *2017 Acoustical Society of America*. https://doi.org/10.1121/1.5007834

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I. INTRODUCTION

Sound intensity is a vital measurement in energy-based acoustics, as can be seen from its use in sound power and source characterization and localization. Acoustic intensity cannot be measured directly, but is rather estimated using pressure and particle velocity. One method measures particle velocity directly using heated wires.[1] However, this sensor is sensitive to mean flow effects, which may make it less robust for outdoor measurements involving wind or temperature fluctuations (e.g., jet noise measurements).[2,3] Another method for estimating acoustic intensity uses microphone pairs and their cross spectra. In this formulation, pressure is estimated as the average measured pressure, and particle velocity is estimated using the pressure gradient across the microphones. This method, referred to in this article as the traditional method, was developed in the 1970s and is still in use today.[4–7] In an attempt to extend the frequency bandwidth of intensity calculations, the phase and amplitude gradient estimator method (PAGE) has been developed.[8] To provide an analytical foundation for the PAGE method, bias errors in calculations of active and reactive intensity and specific acoustic impedance using the PAGE method are compared to those of the traditional method.

Several sources of error limit the bandwidth of intensity estimates using the traditional method. Low-frequency errors arise from phase and amplitude mismatch present in non-ideal microphones,[6,9] whereas high-frequency errors arise from calculation bias errors inherent to the method caused by limitations in the finite-difference and finite-sum formulas,[7,10,11] as well as scattering from the microphones.[12–14] Of particular note to this paper is the work done by Fahy[7] and Thompson and Tree,[10] who report bias errors from the traditional method of calculating active acoustic intensity for the fields created by several analytical sources.

A related quantity, specific acoustic impedance, can be used to determine the absorption of materials. As with intensity, the measurement depends on pressure and particle velocity, and can be estimated using two microphones in a similar manner. This method was developed first for use in tubes,[15,16] and later for free-field measurements.[17,18] An error analysis similar to that of Thompson and Tree has been carried out by Champoux and L'espérance[19] for free-field measurements of specific acoustic impedance.

Recently, the PAGE processing method has been shown experimentally to reduce, and in some cases completely remove, high-frequency calculation bias error from energy-based acoustic quantities.[20] Initial laboratory experimental validation of the method has been done,[21,22] and the method has been applied to jet and rocket noise measurements.[22–24] With the use of phase unwrapping, the PAGE method has extended the bandwidth of intensity measurements for broadband sources to be an order of magnitude greater than that of the traditional method.[22] In this work, we seek to validate the PAGE method analytically by examining its improvement of calculation bias errors over the traditional method.

Just as in the traditional method, the PAGE method estimates particle velocity by estimating the pressure gradient across multiple microphones. However, the amplitude and the phase of the pressure are treated separately, resulting in a more robust method. In this paper, the two-microphone bias errors for the PAGE method are reported for three ideal fields: a plane wave, and fields from a monopole source and a dipole source. This investigation complements work of Fahy, Thompson and Tree, and Champoux and L'espérance by doing a similar analysis for the PAGE method, and extends their work by including reactive intensity. The PAGE method is shown to be more accurate at higher frequencies than the traditional method for active intensity and specific acoustic impedance, both with and without phase unwrapping.

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II. METHODOLOGY

In order to show the advantages of using the PAGE processing method, the bias errors of the PAGE method for estimating intensity and specific acoustic impedance are reported and compared to those of the traditional method. An overview of the two methods is provided in this section.

Although more than two microphones can be used as shown by Cazzalato and Hansen[25] and Pascal and Li[26] for the traditional method, this paper considers only a two-microphone probe for one-dimensional quantity estimation. The frequency-dependent complex pressures at the locations of the two microphones are

$$p_1 = P_1 e^{j\phi_1}$$
$$p_2 = P_2 e^{j\phi_2}, \tag{1}$$

with P denoting the magnitude and ϕ the phase at the location of the microphone. The microphones are in line with the source (as illustrated in Fig. 1), with microphone 1 closer to the source, and a distance d between the microphones. This paper considers only ideal point microphones, whereas in practice, high-frequency error would be introduced by scattering, which may include scattering from a solid spacer placed intentionally between the microphones.[12]

Complex pressures obtained from microphone measurements are used to estimate acoustic quantities at the center of the probe, including pressure, acoustic particle velocity, active and reactive vector intensity and specific acoustic impedance. The center pressure is estimated as the average of the measured pressures. The acoustic particle velocity is found using Euler's equation,

$$\boldsymbol{u} = \frac{j}{\rho_0 \omega} \nabla p, \tag{2}$$

where ρ_0 is the density of air and ω the angular frequency. The bold font indicates a vector. The complex acoustic vector intensity is

$$\boldsymbol{I}_c = \frac{1}{2} p \boldsymbol{u}^*, \tag{3}$$

with * indicating complex conjugate. $\boldsymbol{I}_c$ can be separated into the active (real) and reactive (imaginary) parts,

$$\boldsymbol{I} = \frac{1}{2} \mathrm{Re}\{p \boldsymbol{u}^*\}, \tag{4}$$

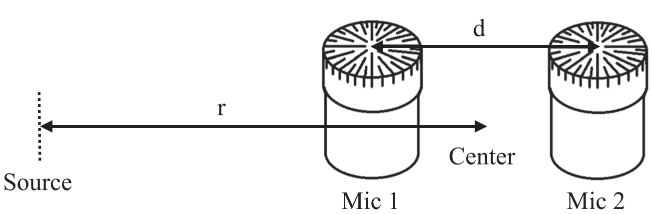

FIG. 1. Schematic of a one-dimensional intensity probe consisting of two microphones. The probe axis points towards the source, such that the sound first passes microphone 1. The distance between the microphones is d.

$$\boldsymbol{J} = \frac{1}{2} \mathrm{Im}\{p \boldsymbol{u}^*\}. \tag{5}$$

The factor of $\frac{1}{2}$ is due to the time or ensemble averaging of complex peak amplitudes [see Fahy[7] Eqs. (4.34c) and (4.34d)]. If using root-mean-square values instead of amplitudes, the factor of $\frac{1}{2}$ disappears. Finally, the complex specific acoustic impedance is

$$z = \frac{p}{u_e}, \tag{6}$$

where u_e is the particle velocity in the direction the specific acoustic impedance is to be measured.

Once estimates of pressure and particle velocity are obtained, the intensity and impedance quantities in Eqs. (3)–(6) can be estimated. The following two subsections explain how these quantities are estimated for both the traditional and the PAGE processing methods.

A. Traditional method

The traditional method of measuring intensity and specific acoustic impedance has been in use for decades and is used in many measurement standards.[27,28] In this method, the complex pressure at the center of the probe, p, is estimated by averaging the real and imaginary parts of the complex pressures p_1 and p_2,

$$p^{\mathrm{TRAD}} = \frac{1}{2}(p_1 + p_2). \tag{7}$$

The traditional estimate of particle velocity is found from finite-differencing both the real and imaginary parts of the complex pressure,

$$\boldsymbol{u}^{\mathrm{TRAD}} = \frac{j}{\rho_0 \omega} \left(\frac{p_2 - p_1}{d} \right). \tag{8}$$

These estimated quantities, p^{TRAD} and $\boldsymbol{u}^{\mathrm{TRAD}}$ can exhibit bias errors when d becomes large relative to the acoustic wavelength. Bias errors associated with p^{TRAD} and $\boldsymbol{u}^{\mathrm{TRAD}}$ lead to bias errors in $\boldsymbol{I}^{\mathrm{TRAD}}$, $\boldsymbol{J}^{\mathrm{TRAD}}$, and z^{TRAD}. The traditional method bias errors are discussed in Secs. III–V for the ideal planar, monopolar, and dipolar fields. These sections refer to the spatial Nyquist frequency, which is a limiting case where the microphone spacing equals half an acoustic wavelength. This occurs at $kd = \pi$ when the wave is incident along the probe axis.

B. PAGE method

To increase the reliable bandwidth of both pressure and particle velocity estimates, the PAGE method was developed.[8] The PAGE method estimates the complex pressure using phase and amplitude as,

$$p^{\mathrm{PAGE}} = \widehat{P} e^{-j\widehat{\phi}}, \tag{9}$$

where an overhat indicates an estimated quantity. Here, $\widehat{P}$ is the estimated pressure amplitude at the center of the probe, computed as the mean of the pressure amplitudes of the two microphones,

$$\widehat{P} = \frac{1}{2}(P_1 + P_2). \quad (10)$$

The center phase estimate $\widehat{\phi}$ is a relative phase, and in practice it can be replaced with zero as it has no effect on energy-based quantities. Mann *et al.*[29] showed that when the complex pressure is expressed in terms of phase and amplitude, Euler's equation for particle velocity takes on a different form. Thomas *et al.*[8] used this formulation in the PAGE method to estimate particle velocity as

$$\boldsymbol{u}^{\mathrm{PAGE}} = \frac{e^{-j\widehat{\phi}}}{\rho_0 \omega}\left(\widehat{P}\widehat{\nabla\phi} + j\widehat{\nabla P}\right). \quad (11)$$

The gradients of pressure amplitude and phase are estimated as

$$\widehat{\nabla P} = \frac{P_2 - P_1}{d}, \quad (12)$$

$$\widehat{\nabla \phi} = \frac{\phi_2 - \phi_1}{d}. \quad (13)$$

When finding phase in practice, phase differences between the microphones are wrapped for $kd > \pi$, such that estimates of $\nabla\phi$ are incorrect. To find an accurate estimate of $\nabla\phi$, an unwrapping algorithm can be used on the phase difference as a function of frequency. Unwrapping can usually be successfully applied when the source is broadband with a smoothly varying phase and there is sufficient coherence between the microphones.[22,30] The PAGE bias error equations reported in Secs. III–V of this paper assume successful unwrapping. Without unwrapping, the bias errors would be different for $kd > \pi$.

The expressions for p^{PAGE} and $\boldsymbol{u}^{\mathrm{PAGE}}$ lead to expressions for $\boldsymbol{I}^{\mathrm{PAGE}}$, $\boldsymbol{J}^{\mathrm{PAGE}}$, and z^{PAGE},

$$\boldsymbol{I}^{\mathrm{PAGE}} = \frac{\widehat{P}^2\widehat{\nabla\phi}}{2\rho_0\omega}, \quad (14)$$

$$\boldsymbol{J}^{\mathrm{PAGE}} = -\frac{\widehat{P}\widehat{\nabla P}}{2\rho_0\omega}, \quad (15)$$

$$z^{\mathrm{PAGE}} = \frac{\widehat{P}\rho_0\omega}{\left[\widehat{P}\widehat{\nabla\phi} + j\widehat{\nabla P}\right]\cdot\widehat{\boldsymbol{e}}} = \frac{\widehat{P}^2}{2\boldsymbol{I}_c^*\cdot\widehat{\boldsymbol{e}}}. \quad (16)$$

The expression for active intensity in Eq. (14) has been reported previously by Mechel,[31] although with the development of the PAGE method, this formulation has progressed to a measurement tool. The estimate of specific acoustic impedance in Eq. (16) is in a particular direction, denoted by the unit vector $\widehat{\boldsymbol{e}}$. The remainder of this paper compares performance of the traditional and PAGE methods for a plane wave, a monopole source, and a dipole source. This systematic evaluation of the bias errors for these propagating wave fields provides an analytical foundation for the PAGE method that can guide future application and development.

III. PLANE WAVE

This section reports bias errors for a plane wave. A plane wave is an ideal field with p and u in phase, similar to the far-field behavior of many acoustic sources. Errors in traditional and PAGE estimates of p and $\boldsymbol{u}$ lead to errors in $\boldsymbol{I}$ and z, all of which are reported. Since a plane wave has zero reactive intensity, $\boldsymbol{J}$ is not discussed in this section.

A. Pressure

The complex pressure of a plane wave can be represented as $p = Ae^{-jkx}$, where A is the amplitude, x is the distance from the origin, and $k = \omega/c$ is the acoustic wave number. The traditional method estimates the pressure at the center of the intensity probe by averaging the real and imaginary parts of the frequency-dependent complex pressure, which vary in space. Fahy[7] evaluates the traditional method bias error by considering the error ratio of estimated pressure over actual pressure [see his Eq. (5.40a)],

$$\frac{p^{\mathrm{TRAD}}}{p} = \cos(kd/2). \quad (17)$$

The traditional method bias error level, $L_{\epsilon,p} = 20\log_{10}(|p^{\mathrm{TRAD}}/p|)$, is plotted as a function of kd in Fig. 2(a). The phase of the error ratio is shown in Fig. 2(b). The error is nearly zero for small values of kd, but the error increases as kd approaches the spatial Nyquist limit of π. Previous authors have given differing criteria for acceptable error, although in this work we will use Fahy's criterion of less than 5% error.[7] The error in pressure is less than 5% (about 0.4 dB) for $kd < 0.64$.

In contrast, the PAGE formulation given in Eq. (9) estimates the correct pressure amplitude at all frequencies, which is also shown in Fig. 2(a). As shown in Fig. 2(b), the phase is correct above $kd = \pi$ only if unwrapping is applied, but since only pressure magnitude is used for $\boldsymbol{I}^{\mathrm{PAGE}}$ and z^{PAGE}, phase error has no effect on these estimates.

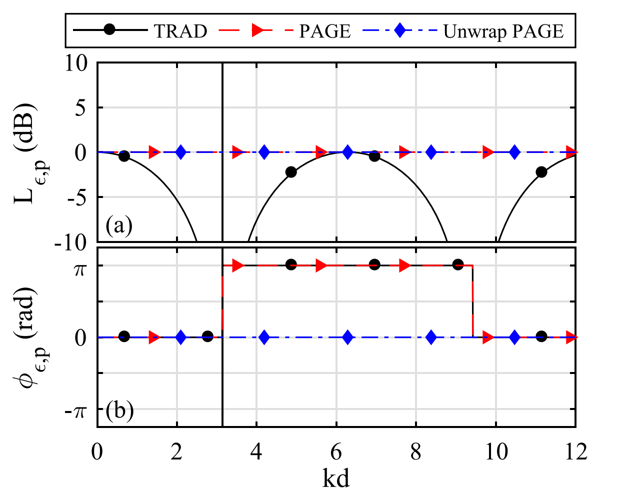

FIG. 2. (Color online) Bias errors in estimates of center pressure (a) amplitude and (b) phase of a plane wave as a function of kd using the traditional method, p^{TRAD}, and the PAGE method, p^{PAGE}, without and with phase unwrapping. The solid, vertical line is the spatial Nyquist frequency.

B. Particle velocity

For the plane wave case, the analytical particle velocity differs from the pressure by a factor of $\rho_0 c$,

$$\boldsymbol{u} = \frac{A}{\rho_0 c} e^{-jkx} \widehat{\boldsymbol{x}}, \tag{18}$$

where $\widehat{\boldsymbol{x}}$ is the unit vector in the direction of propagation. Fahy[7] [Eq. (5.40b)] gives the traditional method error ratio as

$$\frac{\boldsymbol{u}^{\mathrm{TRAD}}}{\boldsymbol{u}} = \operatorname{sinc}(kd/2). \tag{19}$$

The error (one minus the error ratio) is low for small kd and grows more slowly than the pressure error given in Eq. (17). The error level, $L_{\epsilon,u} = 20\log_{10}(|\boldsymbol{u}^{\mathrm{TRAD}}/\boldsymbol{u}|)$, is plotted in Fig. 3(a) and is less than 5% (0.4 dB) for $kd < 1.1$.

The PAGE method estimate of particle velocity given in Eq. (11) has zero error up to $kd = \pi$. Without phase unwrapping, errors in the estimate of $\nabla\phi$ lead to bias errors in amplitude for $kd > \pi$ and phase for $kd > 2\pi$. For cases where phase unwrapping may be applied, however, $\boldsymbol{u}^{\mathrm{PAGE}}$ has zero error for $kd > \pi$.

C. Active intensity

Bias errors in both the estimates of p and $\boldsymbol{u}$ contribute to bias errors in $\boldsymbol{I}$. Analytically, the active intensity of a plane wave of amplitude A is $\boldsymbol{I} = A^2/2\rho_0 c$. By combining the bias errors in Eqs. (17) and (19) and simplifying, the traditional method bias error for active intensity is found to be

$$\frac{\boldsymbol{I}^{\mathrm{TRAD}}}{\boldsymbol{I}} = \operatorname{sinc}(kd). \tag{20}$$

Figure 4 shows the error level $L_{\epsilon,I} = 10\log_{10}(|\boldsymbol{I}^{\mathrm{TRAD}}/\boldsymbol{I}|)$ as well as the phase error. For the traditional method, the error in active intensity is less than 5% (0.2 dB) for $kd < 0.55$—half the range for particle velocity. The direction of $\boldsymbol{I}^{\mathrm{TRAD}}$ is correct until $kd = \pi$. $\boldsymbol{I}^{\mathrm{PAGE}}$ has zero error up to $kd = \pi$ because there was zero error in p and $\boldsymbol{u}$. If the phase is unwrapped, there is zero error for any kd.

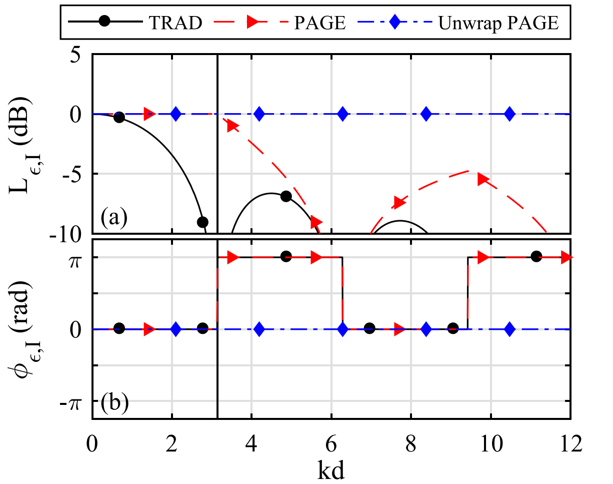

FIG. 4. (Color online) Bias errors in estimates of active intensity (a) magnitude and (b) direction of a plane wave as a function of kd using the traditional method, $\boldsymbol{I}^{\mathrm{TRAD}}$, and the PAGE method, $\boldsymbol{I}^{\mathrm{PAGE}}$, without and with phase unwrapping.

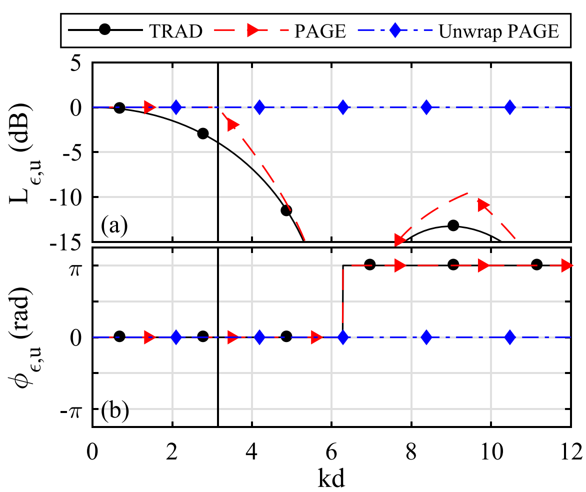

FIG. 3. (Color online) Bias errors in estimates of particle velocity (a) amplitude and (b) phase of a plane wave as a function of kd using the traditional method, $\boldsymbol{u}^{\mathrm{TRAD}}$, and the PAGE method, $\boldsymbol{u}^{\mathrm{PAGE}}$, without and with phase unwrapping. The solid, vertical line is the spatial Nyquist frequency.

D. Specific acoustic impedance

The specific acoustic impedance of a plane wave is $z = \rho_0 c$. The bias errors in z depend on the bias errors in estimates of p and $\boldsymbol{u}$, and result in an expression equivalent to one given by Champoux and L'espérance[19] [see Eq. (12) in that paper],

$$\frac{z^{\mathrm{TRAD}}}{z} = \frac{\cos(kd/2)}{\operatorname{sinc}(kd/2)}. \tag{21}$$

The error level $L_{\epsilon,z} = 20\log_{10}(|z^{\mathrm{TRAD}}/z|)$ and the phase error are shown in Fig. 5. For the traditional method, the error in specific acoustic impedance is less than 5% (0.4 dB) for

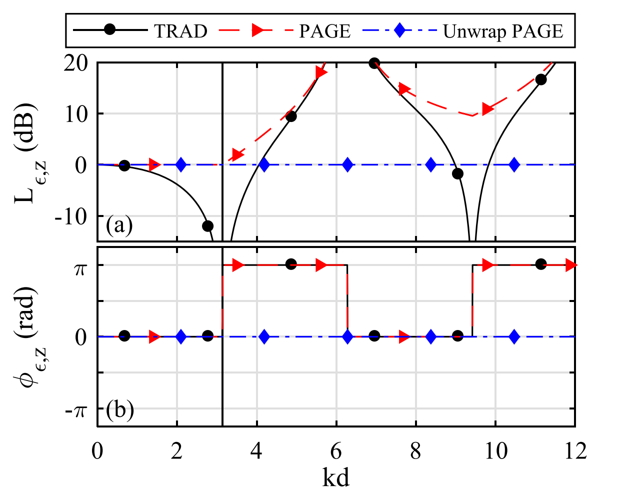

FIG. 5. (Color online) Bias errors in estimates of specific acoustic impedance (a) amplitude and (b) phase of a plane wave as a function of kd using the traditional method, z^{TRAD}, and the PAGE method, z^{PAGE}, without and with phase unwrapping.

$kd < 0.77$, whereas the phase remains correct until spatial aliasing occurs at $kd = \pi$. Since the PAGE method with phase unwrapping has no errors in pressure or particle velocity, there are no errors in z^{PAGE}, even above $kd = \pi$.

In summary for the plane wave case, traditional estimates of acoustic quantities have increasing error as kd increases. Table I shows the maximum value of kd that has less than 5% error for traditional estimation of each quantity. The approximate bandwidth for active intensity is limited to $kd < 0.55$, and for specific acoustic impedance to $kd < 0.77$, both of which are well below the spatial Nyquist frequency of $kd = \pi$. In contrast, the PAGE estimates are accurate up to $kd = \pi$, and if unwrapping is successfully applied, there are no bias errors at any kd. The absence of bias errors for the planar case is significant because many propagating sound fields can be approximated as planar at distances sufficiently far from the source. This was verified experimentally in previous work[22] where the bandwidth of active intensity calculations using the PAGE method was extended at least an order of magnitude past the traditional method. For very high frequencies, the method broke down due to insufficient coherence between the microphones.

IV. MONOPOLE SOURCE

Examining the traditional and PAGE bias errors in a monopole-radiated field shows how these methods perform in both near and far-field environments. Unlike the plane-wave case, where there is no near field, the bias errors for the monopole case depend on both the size of the probe, d, and the distance from the source, r.

A. Pressure

The analytical expression for the complex pressure a distance r from a monopole with amplitude A is $p = Ae^{-jkr}/r$. The traditional method error ratio for center pressure, based on Eq. (7), is

$$\frac{p^{\text{TRAD}}}{p} = \frac{1}{1-\beta^2/4}\left[\cos(kd/2) + \frac{j\beta}{2}\sin(kd/2)\right], \tag{22}$$

where

$$\beta = \frac{kd}{kr}. \tag{23}$$

The PAGE method formulation given in Eq. (9) results in an estimated-to-analytical ratio for the monopole field of

TABLE I. The maximum value of kd for each quantity that results in less than 5% error in a plane-wave field using traditional processing. PAGE processing has no error up to $kd = \pi$ for each quantity, and is accurate beyond that if unwrapping is successfully applied.

Quantity	TRAD kd limit
p	0.64
$\boldsymbol{u}$	1.1
$\boldsymbol{I}$	0.55
z	0.77

$$\frac{p^{\text{PAGE}}}{p} = \frac{1}{1-\beta^2/4}. \tag{24}$$

Unlike the plane-wave case, the bias error in p^{PAGE} is nonzero and depends on the ratio β for the monopole source. A large value of β means the probe is close to the source relative to the microphone spacing, with a limit of $\beta = 2$. The value of β approaches zero as the microphone spacing becomes small or the distance from the source becomes large.

The monopole pressure bias errors in Eqs. (22) and (24) are equivalent to the plane wave pressure bias errors shown in Fig. 2 for the far-field case of $\beta = 0$. For nonzero values of β, error is introduced even at low values of kd. Both methods have greater than 5% error when $\beta > 0.44$, although the traditional method has additional error for large kd, as in the monopole case. If unwrapping is applied to broadband signals, the PAGE method has the correct phase past $kd = \pi$.

B. Particle velocity

The analytical expression of the acoustic particle velocity a distance r from a monopole source is

$$\boldsymbol{u} = \frac{A(-j+kr)}{\rho_0\omega}\frac{e^{-jkr}}{r^2}\hat{\boldsymbol{r}}, \tag{25}$$

where $\hat{\boldsymbol{r}}$ is the unit vector pointing away from the source. The ratio of the traditional estimate of the acoustic particle velocity [calculated using Eq. (8)] to the analytical expression is

$$\frac{\boldsymbol{u}^{\text{TRAD}}}{\boldsymbol{u}} = \frac{1}{1-\beta^2/4}\,\frac{jkr\,\text{sinc}(kd/2) + \cos(kd/2)}{(1+jkr)}. \tag{26}$$

From Eq. (11), the PAGE method error ratio is

$$\frac{\boldsymbol{u}^{\text{PAGE}}}{\boldsymbol{u}} = \frac{1}{1-\beta^2/4}. \tag{27}$$

Similar to the bias errors in the center pressure, estimates in the monopole field, $\boldsymbol{u}^{\text{TRAD}}$ and $\boldsymbol{u}^{\text{PAGE}}$ have nonzero bias errors for small kd, which depend on β. The PAGE method maintains constant bias error up to $kd = \pi$ equal to the pressure bias error, and is constant for all frequencies if unwrapping is applied.

C. Active intensity

From the expressions for pressure and particle velocity, the analytical expression for active intensity radiated from a monopole source with amplitude A is

$$\boldsymbol{I} = \frac{A^2}{2\rho_0 c r^2}\hat{\boldsymbol{r}}. \tag{28}$$

The traditional method's intensity error ratio for a monopole is similar to the plane wave [see Eq. (20)] but with an additional factor that depends on β. The ratio is reported by Thompson and Tree[10] [see their Eq. (14)] to be

$$\frac{\boldsymbol{I}^{\mathrm{TRAD}}}{\boldsymbol{I}} = \frac{1}{1-\beta^2/4}\,\mathrm{sinc}(kd). \tag{29}$$

The PAGE method error ratio calculated using Eq. (14) is

$$\frac{\boldsymbol{I}^{\mathrm{PAGE}}}{\boldsymbol{I}} = \left(\frac{1}{1-\beta^2/4}\right)^2, \tag{30}$$

which is frequency independent but does depend on β. This bias error in Eq. (30) is larger than the bias error of the traditional method in Eq. (29) at low values of kd.

Since the active intensity of the sound field from a monopole source depends on both the size of the probe, d, and the distance from the source, r, it is useful to plot the bias errors as a function of both variables. The kd versus kr plots in Fig. 6 show the bias errors in (a) $\boldsymbol{I}^{\mathrm{TRAD}}$ and (b) $\boldsymbol{I}^{\mathrm{PAGE}}$ with phase unwrapping. Lines of constant β run diagonally, over which only the frequency varies.

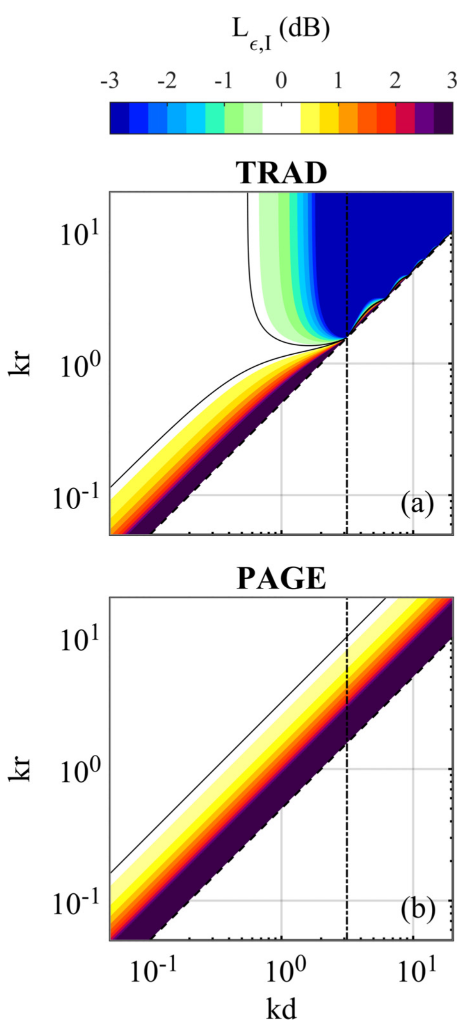

FIG. 6. (Color online) Bias errors in estimates of the magnitude of active intensity for a monopole field as a function of kd and kr: (a) $\boldsymbol{I}^{\mathrm{TRAD}}$ and (b) unwrapped $\boldsymbol{I}^{\mathrm{PAGE}}$. The vertical dashed line is the spatial Nyquist limit. To the left of this line, wrapped and unwrapped PAGE give the same results. The diagonal dashed line follows $r = d/2$. The solid black lines trace the limit of 5% error. Part (b) also acts as a plot of monopole bias errors for $\boldsymbol{J}^{\mathrm{TRAD}}$ and $\boldsymbol{J}^{\mathrm{PAGE}}$, which both equal the bias errors for $\boldsymbol{I}^{\mathrm{PAGE}}$.

Both methods have significant errors close to the source as kr approaches $kd/2$ and β approaches 2. Far from the source, both methods approach zero error as β approaches 0. Both plots in Fig. 6 have a solid, black line tracing the limit of 5% error. At low kd, the traditional method has less than 5% error when $\beta < 0.44$. The bias errors in $\boldsymbol{I}^{\mathrm{TRAD}}$ are large as kd increases, and when kr is also large, the results converge to the plane wave case with less than 5% error for $kd < 0.55$. The PAGE method maintains constant error over frequency, with less than 5% error when $\beta < 0.31$. If a 50 mm microphone spacing is used, this corresponds to a minimum distance from the center of the probe to the source of 160 mm. At low frequencies, the traditional method outperforms the PAGE method, although the difference is negligible except over a small range of near-field locations corresponding to $0.31 < \beta < 0.44$ where the traditional method is within 5% error and the PAGE method is not. Otherwise, the PAGE method, with its extended bandwidth is preferable.

D. Reactive intensity

The bias errors for the reactive intensity of the sound field from a monopole source also depend on β. The analytical expression for reactive intensity from a monopole source is

$$\boldsymbol{J} = \frac{A^2}{2\rho_0 \omega r^3}\hat{\boldsymbol{r}}. \tag{31}$$

For a two-microphone probe, expressions for $\boldsymbol{J}^{\mathrm{TRAD}}$ and $\boldsymbol{J}^{\mathrm{PAGE}}$ are equivalent regardless of the field, since both methods result in an expression involving a difference in autospectra.[20] However, since reactive intensity bias errors have not been previously reported in the literature, it is worthwhile to report them here. The reactive intensity error ratios are

$$\frac{\boldsymbol{J}^{\mathrm{TRAD}}}{\boldsymbol{J}} = \frac{\boldsymbol{J}^{\mathrm{PAGE}}}{\boldsymbol{J}} = \left(\frac{1}{1-\beta^2/4}\right)^2. \tag{32}$$

Since this is identical to the error expression for $\boldsymbol{I}^{\mathrm{PAGE}}$ given in Eq. (30), Fig. 6(b) serves as a plot of reactive intensity bias errors for both methods. Both estimates, (a) $\boldsymbol{J}^{\mathrm{TRAD}}$ and (b) $\boldsymbol{J}^{\mathrm{PAGE}}$ have less than 5% error for $\beta < 0.31$, and infinite error as r approaches $d/2$.

E. Specific acoustic impedance

The analytical expression for the specific acoustic impedance in the radial direction for a monopole field is

$$z = \frac{\rho_0 ckr}{kr - j}. \tag{33}$$

The error ratio for z^{TRAD} can be found from the ratios for p^{TRAD} and $\boldsymbol{u}^{\mathrm{TRAD}}$, given in Eqs. (22) and (26), respectively. The resulting expression matches (with some reworking and allowance for a typographical error) an expression given by Champoux and L'espérance[19] [see their Eq. (11)],

$$\frac{z^{\mathrm{TRAD}}}{z} = \beta \frac{(1+jkr)[2\cos(kd/2)+j\beta\sin(kd/2)]}{2[\beta\cos(kd/2)+j2\sin(kd/2)]}. \qquad (34)$$

For the PAGE processing method, the bias errors in p^{PAGE} [Eq. (24)] and $\boldsymbol{u}^{\mathrm{PAGE}}$ [Eq. (27)] are identical, which cancel out in the estimation of z^{PAGE}. This means that regardless of distance from the source, there is zero error in specific acoustic impedance or in any impedance-based quantities such as absorption. This is true up to $kd = \pi$, and for all frequencies where unwrapping is successfully applied. Figure 7 shows the bias errors for the traditional method as a function of kd and kr. The traditional estimate z^{TRAD} has large errors at high frequencies (near and above $kd = \pi$), whereas z^{PAGE} has no bias errors, so no plot for z^{PAGE} is provided.

To summarize the bias errors for the monopole case, the errors depend not only on kd as seen in the plane wave case, but also on the ratio β. In the case of active intensity, the PAGE method is somewhat more limited in terms of how close the probe can be to the source. However, for all the quantities reported except reactive intensity, the traditional method has increasing errors as kd approaches π, whereas the PAGE method error does not change as kd approaches π for any quantity. With unwrapping, the PAGE error in active intensity remains constant as a function of β for frequencies past $kd = \pi$, and the specific acoustic impedance estimate has zero bias error.

V. DIPOLE SOURCE

The final case considered is the field from an acoustic dipole, defined to be two out-of-phase sources with equal amplitudes and closely spaced such that their spacing is much smaller than a wavelength. A dipole creates a highly reactive near field, with a pressure term that decays as $1/r^2$, in addition to the $1/r$ term that is present for the monopole. Furthermore, the particle velocity has terms that decay as $1/r^3$, $1/r^2$, and $1/r$. Thus, there is a stronger distinction between the near and far fields and greater opportunities to observe bias errors in the intensity and specific acoustic impedance estimates.

A. Pressure

In addition to dependence on radial distance r, the dipole field varies with the angle θ from the dipole axis. The analytical expression for the complex pressure at location (r, θ) is written as[7]

$$p = A\cos(\theta)\frac{1+jkr}{r^2}e^{-jkr}, \qquad (35)$$

where A is the dipole moment source strength. Assuming the two-microphone probe axis is pointed at the center of the dipole, the angular dependence of p in Eq. (35) can be separated from the radial dependence, and the bias errors are independent of θ.

For the two-microphone probe the ratio of traditionally estimated pressure [Eq. (7)] to Eq. (35) is

$$\frac{p^{\mathrm{TRAD}}}{p} = \frac{[1+jkr+(1-jkr)\beta^2/4]\cos(kd/2)+[j\beta-kd(1-\beta^2/4)/2]\sin(kd/2)}{(1-\beta^2/4)^2(1+jkr)}. \qquad (36)$$

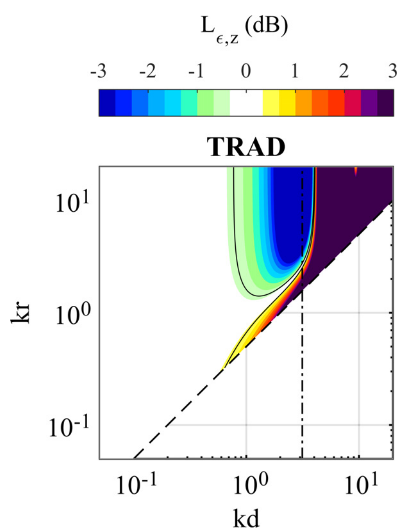

FIG. 7. (Color online) Bias errors in estimates of the amplitude of specific acoustic impedance for a monopole field as a function of kd and kr for z^{TRAD}. No plot is shown for z^{PAGE} as it has no bias errors for the monopole case. The vertical dashed line is the spatial Nyquist limit. The diagonal dashed line follows $r = d/2$. The solid black lines trace the limit of 5% error.

For the PAGE formulation [Eq. (9)], the estimated-to-analytical ratio is

$$\frac{p^{PAGE}}{p} = \frac{(kr)^2(P_1+P_2)}{2(1+jkr)}e^{j(\alpha_1+\alpha_2)/2}, \qquad (37)$$

where

$$P_1 = \frac{\sqrt{1+(kr-kd/2)^2}}{(kr-kd/2)^2}$$
$$P_2 = \frac{\sqrt{1+(kr+kd/2)^2}}{(kr+kd/2)^2}, \qquad (38)$$

which are the pressure amplitudes at microphone locations 1 and 2 for the dipole case [with the amplitude terms A and $\cos(\theta)$ omitted]. Additionally,

$$\alpha_1 = \arctan(kr - kd/2)$$
$$\alpha_2 = \arctan(kr + kd/2). \qquad (39)$$

For large values of kr, the probe is in the far field where the pressure amplitude has a $1/r$ dependence rather than a $1/r^2$ dependence. In this case, the dipole pressure bias errors in Eqs. (36) and (37) converge to Eqs. (22) and (24) for the monopole case. Otherwise, the dipole bias errors are larger than the monopole bias errors for both methods, dependent on kr. Bias errors in p^{TRAD} oscillate similar to the plane wave and the monopole cases, severely limiting the usable bandwidth. Bias errors in p^{PAGE} also change, but the error decreases as frequency increases.

B. Particle velocity

The radial component of the particle velocity is

$$\boldsymbol{u}_r = A\cos(\theta)\frac{-2j + kr(2 + jkr)}{ckr^3\rho_0}e^{-jkr}\hat{\boldsymbol{r}}. \tag{40}$$

With the probe oriented towards the center of the dipole, only the radial component of particle velocity is estimated. The angular dependence cancels out in the traditional estimated-to-analytical ratio. Using Eq. (8), this ratio is

$$\frac{\boldsymbol{u}_r^{\text{TRAD}}}{\boldsymbol{u}_r} = \frac{\left[j\beta - kd\left(1-\beta^2/4\right)/2\right]\cos(kd/2) - \left[1 + jkr + (1-jkr)\beta^2/4\right]\sin(kd/2)}{\left(1-\beta^2/4\right)^2\left[-2j + kr(2+jkr)\right]\beta/2}. \tag{41}$$

The PAGE estimated-to-analytical ratio for the particle velocity, based on Eq. (11), is

$$\frac{\boldsymbol{u}_r^{\text{PAGE}}}{\boldsymbol{u}_r} = \frac{(kr)^2\left[2j(P_2 - P_1) + (P_1 + P_2)(kd + \alpha_1 - \alpha_2)\right]}{\left[-2j + kr(2+jkr)\right]2\beta} \times e^{j(\alpha_1+\alpha_2)/2}. \tag{42}$$

As with pressure, the dipole case shows increased error for low kr and a convergence to the monopole case for high kr.

C. Active intensity

The radial component of the active intensity for the dipole is obtained from the expression for pressure in Eq. (35) and for particle velocity in Eq. (40). The active intensity may be written as

$$\boldsymbol{I}_r = \frac{k^2A^2\cos^2\theta}{2cr^2\rho_0}\hat{\boldsymbol{r}}. \tag{43}$$

Again, the angular dependence is independent of the $1/r^2$ radial dependence, such that a one-dimensional probe can obtain the radially dependent active intensity.

The traditional method has an estimated-to-analytical intensity ratio of

$$\frac{\boldsymbol{I}_r^{\text{TRAD}}}{\boldsymbol{I}_r} = \frac{kd\cos(kd) - \left[1 + (kr)^2\left(1-\beta^2/4\right)\right]\sin(kd)}{kd(kr)^2\left(1-\beta^2/4\right)^2}, \tag{44}$$

which is equivalent to an expression given by Thompson and Tree[10] [see their Eq. (18)]. The PAGE processing method, calculated using Eq. (14), has a ratio of

$$\frac{\boldsymbol{I}_r^{\text{PAGE}}}{\boldsymbol{I}_r} = -\frac{kr}{4\beta}(P_1 + P_2)^2(-kd + \alpha_2 - \alpha_1). \tag{45}$$

The bias errors in the magnitude of $\boldsymbol{I}_r$ depend on both kd and kr, as displayed in Fig. 8. As r approaches d (below $kd = \pi$), the bias error in $\boldsymbol{I}_r^{\text{PAGE}}$ increases more rapidly than for $\boldsymbol{I}_r^{\text{TRAD}}$, similar to the monopole case. As kd increases, $\boldsymbol{I}_r^{\text{TRAD}}$ underestimates the magnitude of $\boldsymbol{I}_r$. Using PAGE processing, $\boldsymbol{I}_r^{\text{PAGE}}$ (with phase unwrapping) has less than 5% error at low kr if $\beta < 0.18$, and for high kr the limit is the same as the monopole case, namely, $\beta < 0.31$.

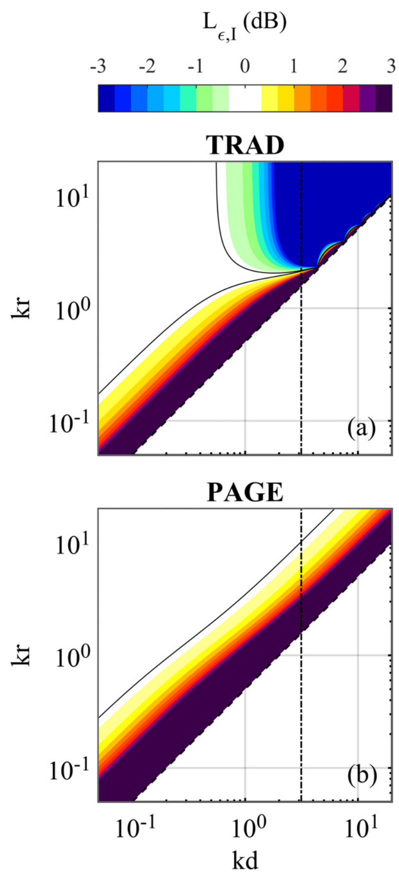

FIG. 8. (Color online) Bias errors in estimates of the magnitude of active intensity for a dipole field as a function of kd and kr: (a) $\boldsymbol{I}^{\text{TRAD}}$ and (b) unwrapped $\boldsymbol{I}^{\text{PAGE}}$. The vertical dashed line is the spatial Nyquist limit. To the left of this line, wrapped and unwrapped PAGE give the same results. The diagonal dashed line follows $r = d/2$. The solid black lines trace the limit of 5% error.

D. Reactive intensity

The analytical expression for the radial component of the reactive intensity is

$$\boldsymbol{J}_r = \frac{2+(kr)^2}{2\omega r^5 \rho_0} A^2 \cos^2\theta \hat{\boldsymbol{r}}. \quad (46)$$

The traditional and PAGE processing methods produce the same estimated-to-analytical ratio,

$$\frac{\boldsymbol{J}_r^{\mathrm{TRAD}}}{\boldsymbol{J}_r} = \frac{\boldsymbol{J}_r^{\mathrm{PAGE}}}{\boldsymbol{J}_r} = \frac{1}{(1-\beta^2/4)^4} \times \left[1 + \frac{\beta^2\left[1+(kd)^2/8-(kr)^2\right]}{2\left[2+(kr)^2\right]}\right]. \quad (47)$$

Figure 9 shows this bias error as a function of kd and kr. There is less than 5% error for low kr if $\beta < 0.20$, and for high kr it converges to the monopole case of $\beta < 0.31$.

E. Specific acoustic impedance

The last quantity to consider is the specific acoustic impedance. The analytical expression for the dipole case is

$$z = \frac{\omega\rho_0 r(1+jkr)}{-2j+kr(2+jkr)}. \quad (48)$$

The estimated-to-analytical ratio for the traditional method is

$$\frac{z^{\mathrm{TRAD}}}{z} = \frac{[-2j+kr(2+jkr)]\beta}{2(1+jkr)}\left[\frac{[1+jkr+(1-jkr)\beta^2/4]\cos(kd/2)+[j\beta-kd(1-\beta^2/4)/2]\sin(kd/2)}{[j\beta-kd(1-\beta^2/4)/2]\cos(kd/2)-[1+jkr+(1-jkr)\beta^2/4]\sin(kd/2)}\right]. \quad (49)$$

Although the PAGE method had zero error for the plane wave and monopole cases, bias errors in p^{PAGE} [Eq. (37)] and $\boldsymbol{u}^{\mathrm{PAGE}}$ [Eq. (42)] are not identical for the dipole case. Combining the errors results in the ratio

$$\frac{z^{\mathrm{PAGE}}}{z} = \frac{\beta[-2j+kr(2+jkr)](P_1+P_2)}{(1+jkr)\left[2j(P_2-P_1)+(P_1+P_2)(kd+\alpha_1-\alpha_2)\right]}. \quad (50)$$

Figure 10 shows the bias errors for specific acoustic impedance. Approaching $kd = \pi$, the traditional method has increasingly large errors. In contrast, the PAGE method has

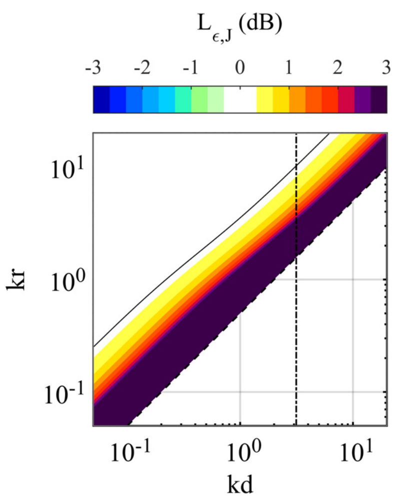

FIG. 9. (Color online) Bias errors in estimates of the magnitude of reactive intensity for a dipole field as a function of kd and kr for $\boldsymbol{J}^{\mathrm{TRAD}} = \boldsymbol{J}^{\mathrm{PAGE}}$. The vertical dashed line is the spatial Nyquist limit. The diagonal dashed line follows $r = d/2$. The solid black line traces the limit of 5% error.

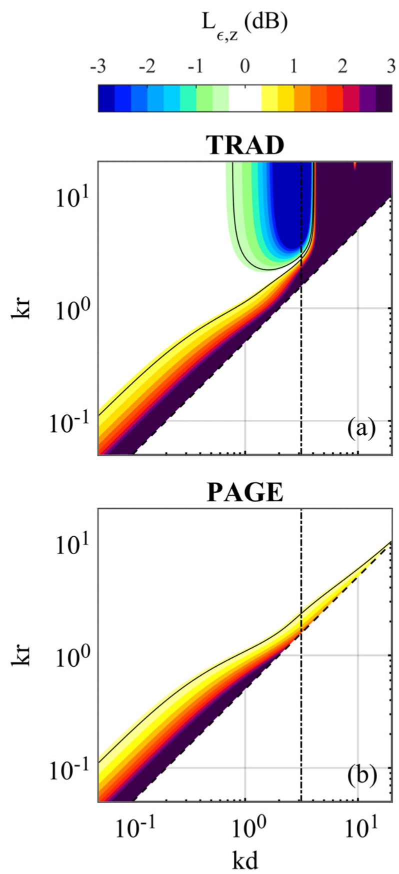

FIG. 10. (Color online) Bias errors in estimates of the amplitude of specific acoustic impedance for a dipole field as a function of kd and kr: (a) z^{TRAD} and (b) unwrapped z^{PAGE}. The vertical dashed line is the spatial Nyquist limit. To the left of this line, wrapped and unwrapped PAGE give the same results. The diagonal dashed line follows $r = d/2$. The solid black lines trace the limit of 5% error.

decreasing error above $kd = \pi$, and has less than 5% error for all values of kd if $\beta < 0.45$.

In summary, this bias error analysis for the case of the acoustic dipole provides insights into the near-field performance of the traditional and PAGE estimates of intensity and impedance. For active intensity, there is a small range of β values over which traditional estimates at low frequencies have less than 5% error and the PAGE estimates do not. However, not only is PAGE is comparable outside this narrow range but it also has the advantage of decreasing error with increasing frequency, as opposed to the traditional method, which has increasing error with increasing frequency and becomes unusable much before $kd = \pi$. The extended bandwidth of the PAGE estimates are advantageous when calculating the intensity and specific acoustic impedance from an acoustic dipole, and when unwrapping can be applied, this advantage is even more pronounced.

VI. CONCLUSIONS

Analytical bias error calculations for simple sound fields have long been a foundational part of the method of calculating acoustic intensity and specific acoustic impedance from two-microphone probes. This paper has provided a similar foundation for the phase and amplitude gradient estimator method (PAGE) by showing the bias errors for planar, monopolar, and dipolar sound fields. This bias error study has confirmed that the main advantage of the PAGE method is the bandwidth extension possible in these calculations for broadband fields. For the active intensity and specific acoustic impedance for the fields studied, the traditional method has increasing error as kd approaches π. On the other hand, the PAGE method does not have increasing error with increasing kd. Traditional estimates of reactive intensity do not exhibit error increasing with kd, and the PAGE method leaves the estimate unchanged.

As long as the probe is sufficiently far from the source (based on the source type and quantity of interest), the PAGE method is accurate in estimating acoustic intensity and specific acoustic impedance up to $kd = \pi$, a significant improvement in bandwidth over the traditional method. For broadband fields, if phase unwrapping is successfully applied, the method is accurate beyond $kd = \pi$, and is limited in bandwidth only by other sources of error such as scattering or a lack of coherence between the microphones. Because the PAGE method with unwrapping overcomes the restrictions of the spatial Nyquist limit, the microphones used for these calculations can be spaced farther apart than required by the traditional method, which in turn improves the estimates of intensity and impedance at low frequencies. Thus, the PAGE method can potentially extend the reliable bandwidth of these calculations on both the high and low end.

This bias error analysis of the PAGE method yields a foundation upon which future work can be built. An investigation into the optimal number and arrangement of microphones for multidimensional probes should be conducted along with an examination of how bias errors change when a center microphone is included in the probe. PAGE method performance in a wider range of applications, such as sound power calculations, standing wave fields and narrowband noise need to be evaluated. In addition, techniques, such as higher-order estimates of the gradient,[32] could be implemented to improve the estimates of the phase and pressure gradients for less smoothly varying fields.

ACKNOWLEDGMENTS

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