An impedance-based formulation of frequency-domain nonlinearity indicators in finite-amplitude sound propagation

Won-Suk Ohm, Kent L. Gee, and Taeyoung Park

Citation: The Journal of the Acoustical Society of America **148**, EL295 (2020); doi: 10.1121/10.0002030
View online: https://doi.org/10.1121/10.0002030
View Table of Contents: https://asa.scitation.org/toc/jas/148/3
Published by the Acoustical Society of America

JASA EXPRESS LETTERS

https://doi.org/10.1121/10.0002030

An impedance-based formulation of frequency-domain nonlinearity indicators in finite-amplitude sound propagation

Won-Suk Ohm,[1,a)] Kent L. Gee,[2,b)] and Taeyoung Park[1]
[1]School of Mechanical Engineering, Yonsei University, Seoul 03722, Republic of Korea
[2]Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602, USA
ohm@yonsei.ac.kr, kentgee@byu.edu, pty0948@yonsei.ac.kr

Abstract: Since the Morfey-Howell Q/S was introduced as a single-point frequency-domain nonlinearity indicator for propagation of intense broadband noise [AIAA J. **19**, 986–992 (1981)], there has been debate about its validity, utility, and interpretation. In this Letter, the generalized Burgers equation is recast in terms of specific acoustic impedance along with linear absorption and dispersion coefficients, normalized quadspectral density (Q/S), and newly proposed normalized cospectral density (C/S). The formulation leads to a rather straightforward interpretation in which Q/S and C/S, respectively, represent the additional absorption and dispersion at a locale, produced by the passage of a finite-amplitude wave. https://doi.org/10.1121/10.0002030

[Editor: Régis Marchiano] Pages: EL295–EL300

Received: 29 May 2020 **Accepted:** 9 September 2020 **Published Online:** 28 September 2020

1. Introduction

A nonlinearity indicator is a measure by which the strength of nonlinearity in a wave process is quantified. Examples of nonlinearity indicators include the Gol'dberg number,[1–4] the average steepening factor,[5,6] the derivative skewness,[7] the bispectrum,[8] and the Morfey-Howell Q/S.[9] Nonlinearity indicators have been widely used to identify cumulative finite-amplitude propagation effects in jet[10–13] and rocket[14–16] noise analysis. However, the use of these indicators has often been qualitative, which has sometimes resulted in debate regarding their validity, utility, and interpretation.[11,17]

To improve physical understanding and interpretation, Reichman *et al.*[18] reformulated the Q/S spectrum by expressing the frequency-domain Burgers equation in terms of the change in sound level with respect to range. In this context, the Q/S spectrum at a given location in the sound field is interpreted as the local spatial rate of change in level of a spectral component due to nonlinearity. In addition to absorptive and geometric spreading losses, a spectral component can experience extra loss (or gain) in level through the nonlinear energy exchange with other spectral components, the extent of which is quantified by Q/S.

In this Letter, we provide an alternative, but equally intuitive interpretation of Q/S based on the familiar concept of specific acoustic impedance. Both the prior level and the current impedance-based formulations of this indicator demonstrate that its interpretation is rather straightforward, and thus they would serve to promote the wider use of Q/S in nonlinear acoustics pedagogy and systems analysis.

2. Impedance-based formulation of nonlinearity indicators

The formulation here follows from recognizing that (a) for a progressive sound wave, impedance encapsulates how the waveform is modified locally via changes in amplitude (absorption) and phase (dispersion) of its Fourier components, and hence (b) nonlinearity, being one of waveform distortion mechanisms, could be cast within the framework of impedance, specifically in the form of effective absorption and dispersion.

We start with the generalized Burgers equation for plane progressive waves,[19]

$$\frac{\partial p}{\partial x} + L_\tau(p) = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau}, \tag{1}$$

a)Author to whom correspondence should be addressed.
b)ORCID: 0000-0002-5768-6483.

where p is the acoustic pressure, x is the propagation distance, τ is the retarded time, ρ_0 is the density, c_0 is the small-signal sound speed, β is the coefficient of nonlinearity, and $L_\tau(p)$ is a linear, retarded time operator that describes the absorption and dispersion properties of the medium. The corresponding spectral version of Eq. (1) is

$$\frac{\partial \tilde{p}}{\partial x} + L_\omega(\tilde{p}) = \frac{jk\beta}{2\rho_0 c_0^2}\tilde{q}, \tag{2}$$

where $\tilde{p}$ and $\tilde{q}$ are the Fourier transforms of the acoustic pressure and the squared acoustic pressure, respectively, ω is the angular frequency, and $k = \omega/c_0$ is the wave number. Here, the Fourier equivalent of the absorption/dispersion operator can be written as

$$L_\omega(\tilde{p}) = (\alpha + j\delta)\tilde{p}, \tag{3}$$

where α and δ are frequency-dependent absorption and dispersion coefficients, respectively. The generalized spectral Burgers equation then becomes

$$\frac{\partial \tilde{p}}{\partial x} = -(\alpha + j\delta)\tilde{p} + \frac{jk\beta}{2\rho_0 c_0^2}\tilde{q}. \tag{4}$$

Within the second-order approximation theory,[20,21] absorption and dispersion are assumed weak on the scale of wavelength such that $\alpha/k \sim O(\tilde{\varepsilon})$ and $\delta/k \sim O(\tilde{\varepsilon})$, where $\tilde{\varepsilon}$ is a small, generic ordering parameter. Given $\tilde{p} \sim O(\tilde{\varepsilon})$, the absorption/dispersion term $-(\alpha + j\delta)\tilde{p}$ is $O(\tilde{\varepsilon}^2)$. Equation (4) is thus a consistent $O(\tilde{\varepsilon}^2)$ wave equation, in which all the terms, including $\partial\tilde{p}/\partial x$, are $O(\tilde{\varepsilon}^2)$ small. This $O(\tilde{\varepsilon}^2)$ consistency ought to be maintained while manipulating Eq. (4).

Recasting the generalized spectral Burgers equation [Eq. (4)] in terms of specific acoustic impedance

$$Z = \frac{\tilde{p}}{\tilde{u}} \tag{5}$$

requires the spectral version of the momentum equation that connects the pressure gradient $\partial\tilde{p}/\partial x$ and the particle velocity $\tilde{u}$. We begin with the following one-dimensional momentum equation for Newtonian fluids in *nonretarded* time t, which is exact up to $O(\tilde{\varepsilon}^2)$,[20]

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \left(\mu_B + \frac{4}{3}\mu\right)\frac{\partial^2 u}{\partial x^2} - \frac{\partial \mathcal{L}}{\partial x}, \tag{6}$$

where u is the particle velocity, μ_B is the bulk viscosity, μ is the shear viscosity, and $\mathcal{L} = \frac{1}{2}\rho_0 u^2 - p^2/2\rho_0 c_0^2$ is the Lagrangian density. To rewrite Eq. (6) in retarded time τ for plane progressive waves [for which $\mathcal{L} = 0$ at $O(\tilde{\varepsilon}^2)$ upon substitution of the $O(\tilde{\varepsilon})$ relation $p = \rho_0 c_0 u + O(\tilde{\varepsilon}^2)$], consider the coordinate transformation[22]

$$x_1 = \tilde{\varepsilon}x, \quad \tau = t - x/c_0, \tag{7}$$

where x_1 is the slow scale corresponding to the retarded time frame τ. Partial derivatives in the transformed coordinates (x_1, τ) are then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} = \tilde{\varepsilon}\frac{\partial}{\partial x_1} - \frac{1}{c_0}\frac{\partial}{\partial \tau}, \quad \frac{\partial^2}{\partial x^2} = \tilde{\varepsilon}^2\frac{\partial^2}{\partial x_1^2} - \tilde{\varepsilon}\frac{2}{c_0}\frac{\partial^2}{\partial x_1 \partial \tau} + \frac{1}{c_0^2}\frac{\partial^2}{\partial \tau^2}. \tag{8}$$

Substituting Eq. (8) into Eq. (6) and retaining terms up to $O(\tilde{\varepsilon}^2)$ yields

$$\tilde{\varepsilon}\frac{\partial p}{\partial x_1} = \frac{1}{c_0}\frac{\partial p}{\partial \tau} - \rho_0\frac{\partial u}{\partial \tau} + \left(\mu_B + \frac{4}{3}\mu\right)\frac{1}{c_0^2}\frac{\partial^2 u}{\partial \tau^2}. \tag{9}$$

By replacing $\tilde{\varepsilon}(\partial p/\partial x_1)$ in Eq. (9) with $\partial p/\partial x$,[22] the momentum equation in retarded time τ is obtained,

$$\frac{\partial p}{\partial x} = \frac{1}{c_0}\frac{\partial p}{\partial \tau} - \rho_0\frac{\partial u}{\partial \tau} + \left(\mu_B + \frac{4}{3}\mu\right)\frac{1}{c_0^2}\frac{\partial^2 u}{\partial \tau^2}. \tag{10}$$

Equation (10) is a consistent $O(\tilde{\varepsilon}^2)$ momentum equation for plane progressive waves, in which $\partial p/\partial x \sim O(\tilde{\varepsilon}^2)$, $\left(c_0^{-1}(\partial p/\partial \tau) - \rho_0(\partial u/\partial \tau)\right) \sim O(\tilde{\varepsilon}^2)$, and $\left(\mu_B + \frac{4}{3}\mu\right)c_0^{-2}(\partial^2 u/\partial \tau^2) \sim O(\tilde{\varepsilon}^2)$,

https://doi.org/10.1121/10.0002030

respectively. [To see this, consider that $p = \rho_0 c_0 u + O(\tilde{\varepsilon}^2)$ for plane progressive waves, and coefficients μ_B and μ are assumed $O(\tilde{\varepsilon})$.] The Fourier transform of Eq. (10) gives the spectral version of the momentum equation

$$\frac{\partial \tilde{p}}{\partial x} = jk\tilde{p} - j\omega\rho_0\tilde{u} - \left(\mu_B + \frac{4}{3}\mu\right)k^2\tilde{u}. \tag{11}$$

Manipulation of Eq. (11) leads to an expression containing the dimensionless impedance $\bar{Z} = Z/\rho_0 c_0$:

$$\begin{aligned}\frac{\partial \tilde{p}}{\partial x} &= jk\tilde{p}\left\{1 - \frac{j\omega\rho_0\tilde{u}}{jk\tilde{p}} - \frac{\left(\mu_B + \frac{4}{3}\mu\right)k^2\tilde{u}}{jk\tilde{p}}\right\}\\ &= jk\tilde{p}\left\{1 - \frac{\rho_0 c_0}{\tilde{p}/\tilde{u}} + j\frac{\left(\mu_B + \frac{4}{3}\mu\right)k}{\rho_0 c_0}\frac{\rho_0 c_0}{\tilde{p}/\tilde{u}}\right\}\\ &= jk\tilde{p}\left\{1 - \left(1 - j\frac{2\alpha_v}{k}\right)\frac{1}{\bar{Z}}\right\},\end{aligned} \tag{12}$$

where $\alpha_v = (\mu_B + \frac{4}{3}\mu)k^2/2\rho_0 c_0$ is the absorption coefficient due to viscosity.[23] Furthermore, the term within the braces in Eq. (12) can be expressed as, via binomial expansion in $\Delta\bar{Z} = \bar{Z} - 1$,

$$\begin{aligned}1 - \left(1 - j\frac{2\alpha_v}{k}\right)\frac{1}{\bar{Z}} &= 1 - \left(1 - j\frac{2\alpha_v}{k}\right)(1 + \Delta\bar{Z})^{-1}\\ &= 1 - \left(1 - j\frac{2\alpha_v}{k}\right)\left\{1 - \Delta\bar{Z} + O(\tilde{\varepsilon}^2)\right\}\\ &= \Delta\bar{Z} + j\frac{2\alpha_v}{k} + O(\tilde{\varepsilon}^2).\end{aligned} \tag{13}$$

Here, expansion to leading order would suffice, because terms of $O(\tilde{\varepsilon}^2)$ in Eq. (13) produce $O(\tilde{\varepsilon}^3)$ terms upon substitution to Eq. (12). Therefore, the final form of the spectral momentum equation at $O(\tilde{\varepsilon}^2)$ is given by

$$\frac{\partial \tilde{p}}{\partial x} = jk\tilde{p}\left(\bar{Z} - 1 + j\frac{2\alpha_v}{k}\right). \tag{14}$$

Now substitute Eq. (14) into the generalized spectral Burgers equation [Eq. (4)] to obtain

$$jk\tilde{p}\left(\bar{Z} - 1 + j\frac{2\alpha_v}{k}\right) = -(\alpha + j\delta)\tilde{p} + \frac{jk\beta}{2\rho_0 c_0^2}\tilde{q}. \tag{15}$$

Multiplying Eq. (15) by $\tilde{p}^*$ (the complex conjugate of $\tilde{p}$) gives

$$|\tilde{p}|^2\left(\bar{Z} - 1 + j\frac{2\alpha_v}{k}\right) = j\frac{\alpha}{k}|\tilde{p}|^2 - \frac{\delta}{k}|\tilde{p}|^2 + \frac{\beta}{2\rho_0 c_0^2}\tilde{p}^*\tilde{q}. \tag{16}$$

Ensemble-averaging Eq. (16) leads to

$$S_{pp}\left(\bar{Z} - 1 + j\frac{2\alpha_v}{k}\right) = j\frac{\alpha}{k}S_{pp} - \frac{\delta}{k}S_{pp} + \frac{\beta}{2\rho_0 c_0^2}\left(jQ_{pp^2} + C_{pp^2}\right), \tag{17}$$

where $S_{pp} = E[|\tilde{p}|^2]$, $Q_{pp^2} = \mathrm{Im}(E[\tilde{p}^*\tilde{q}])$, and $C_{pp^2} = \mathrm{Re}(E[\tilde{p}^*\tilde{q}])$ are referred to as the autospectral density (or power spectral density), quadspectral density, and cospectral density, respectively,[24] and the symbol $E[\,]$ denotes ensemble averaging. Finally, dividing Eq. (17) by S_{pp} and rearranging terms yield the desired impedance form of the generalized Burgers equation

$$(\bar{Z} - 1) = j\frac{(\alpha - 2\alpha_v)}{k} - \frac{\delta}{k} + \frac{\beta}{2\rho_0 c_0^2}\left(j\frac{Q_{pp^2}}{S_{pp}} + \frac{C_{pp^2}}{S_{pp}}\right). \tag{18}$$

A few observations about Eq. (18) are noteworthy at this juncture. First, Eq. (18) is a consistent $O(\tilde{\varepsilon})$ equivalent to the $O(\tilde{\varepsilon}^2)$, generalized Burgers equation [Eq. (4)]. The reduction in order $[O(\tilde{\varepsilon}^2) \to O(\tilde{\varepsilon})]$ is simply brought by the successive multiplication and division by $\tilde{p}^*$ and S_{pp}, respectively. Second, for a typical medium in which more than one loss mechanism is present, viscous absorption α_v is much smaller than absorption due to other accompanying loss mechanisms such as molecular relaxation, at relatively low frequencies.[23,25] Because α_v makes up only a small fraction of the total absorption α, it can be ignored (i.e., $\alpha - 2\alpha_v \approx \alpha$) in Eq. (18) for a wide frequency range of practical interest. Splitting Eq. (18) into the imaginary and real parts (with α_v ignored) gives

$$\mathrm{Im}(\bar{Z}) = \frac{\alpha}{k} + \frac{1}{2}\beta\varepsilon\frac{Q}{S} \tag{19}$$

and

$$\mathrm{Re}(\bar{Z}) - 1 = -\frac{\delta}{k} + \frac{1}{2}\beta\varepsilon\frac{C}{S}, \tag{20}$$

where $\varepsilon = p_c/\rho_0 c_0^2 \sim O(\tilde{\varepsilon})$ is the acoustic Mach number based on the characteristic amplitude p_c of the pressure-time waveform. The two dimensionless $O(1)$ quantities Q/S and C/S are defined by

$$\frac{Q}{S} = \frac{Q_{pp^2}}{S_{pp}p_c}, \quad \frac{C}{S} = \frac{C_{pp^2}}{S_{pp}p_c}. \tag{21}$$

The choice of p_c depends on the type of waveform. For a transient pulse, the peak pressure amplitude is a convenient choice, whereas the root-mean-square pressure p_{rms} may be more appropriate for continuous random noise. With $p_c = p_{\mathrm{rms}}$, the Q/S in Eq. (21) reduces to the Morfey-Howell Q/S.[9]

3. Interpretation of *Q/S* and *C/S*

For plane progressive waves in linear acoustics, the specific acoustic impedance [Eq. (5)] can be construed as a medium property that a wave sees at any given point. For instance, a wave propagating in an ideal, lossless medium would see the specific acoustic impedance given by the characteristic impedance $\rho_0 c_0$ (or unity in dimensionless impedance),

$$\mathrm{Im}(\bar{Z}) = 0, \quad \mathrm{Re}(\bar{Z}) - 1 = 0. \tag{22}$$

Any deviation of the dimensionless impedance from unity then signifies the presence of loss mechanisms such as absorption and dispersion. If the medium is lossy, a progressive wave sees the dimensionless impedance different from unity by the amount commensurate with the strength of linear absorption and dispersion. This is immediately apparent from Eqs. (19) and (20) without the Q/S and C/S terms,

$$\mathrm{Im}(\bar{Z}) = \frac{\alpha}{k}, \quad \mathrm{Re}(\bar{Z}) - 1 = -\frac{\delta}{k}. \tag{23}$$

Note that absorption (α/k) is associated with the *imaginary* part of impedance $[\mathrm{Im}(\bar{Z})]$, which represents the overall spatial rate of change in power spectral density, or

$$\mathrm{Im}(\bar{Z}) = -\frac{1}{2S_{pp}}\frac{dS_{pp}}{d(kx)}. \tag{24}$$

[Equation (24) is obtained by combining $\tilde{p}^* d\tilde{p}/d(kx) + \mathrm{c.c.} = d(\tilde{p}\tilde{p}^*)/d(kx) = dS_{pp}/d(kx)$ and Eq. (14) with the α_v term ignored, where the symbol c.c. stands for complex conjugate.] Dispersion $(-\delta/k)$ is then related to the *real* part of impedance $[\mathrm{Re}(\bar{Z}) - 1]$, which denotes the overall spatial rate of change in phase, or

$$\mathrm{Re}(\bar{Z}) - 1 = \frac{d\phi}{d(kx)}, \tag{25}$$

where ϕ is defined by $\tilde{p} = |\tilde{p}|e^{j\phi}$ with respect to dimensionless retarded time $\omega\tau$. [To arrive at Eq. (25) combine $\tilde{p}^* d\tilde{p}/d(kx) - \mathrm{c.c.} = 2j|\tilde{p}|^2 d\phi/d(kx)$ with Eq. (14).]

Now what if the finite-amplitude effects are taken into account? The significance of Eqs. (19) and (20) is that they provide a framework within which the quantities Q/S and C/S can be interpreted as the additional change in impedance due to nonlinearity. It follows from Eqs. (19) and (20) that Q/S and C/S represent the parametrically induced change in impedance in the form of additional absorption and dispersion. Here, the passage of a finite-amplitude wave alters the apparent medium property (i.e., absorption and dispersion), the extent of which can be used to quantify the strength of nonlinearity.

https://doi.org/10.1121/10.0002030

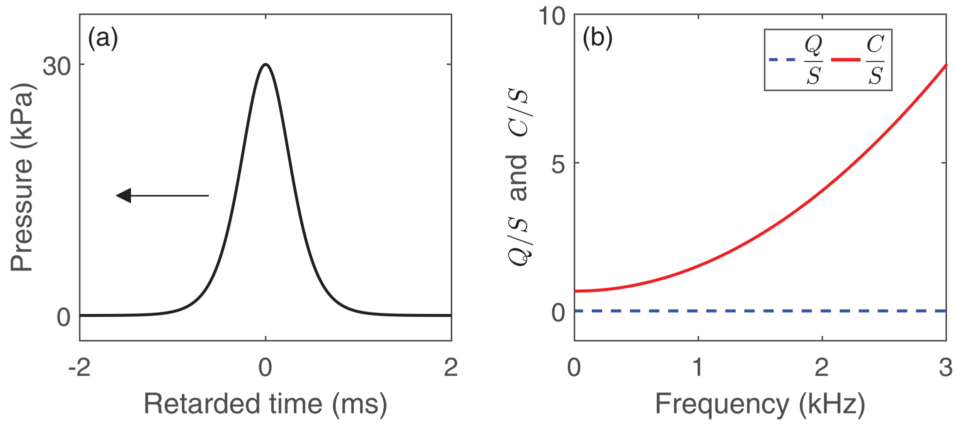

Fig. 1. (Color online) Utility of nonlinearity indicator *C*/*S*: (a) the waveform of a soliton solution to the KdV system of Ref. 28, plotted in retarded time, and (b) the corresponding *Q*/*S* and *C*/*S* computed using Eq. (21) without ensemble averaging. Note that the soliton in (a) translates to the left in retarded time with slowness $d\tau/dx = -7.77 \times 10^{-4}$ s/m (equivalent to propagation speed of 116 m/s).

A caveat: in light of Eqs. (19)–(21), one must exercise caution when using *Q*/*S* and *C*/*S* to judge "nonlinearity" of a wave. The way in which *Q*/*S* and *C*/*S* are defined [i.e., they are nondimensionalized by the characteristic amplitude p_c in Eq. (21)] renders these nonlinearity indicators dependent on the "shape" of the wave only. For example, two waves with the same waveform but at different levels, say 40 dB and 120 dB, would be equally nonlinear under *Q*/*S* and *C*/*S*. Nonlinearity due to the "size" of the wave is instead reflected by the acoustic Mach number ε as shown in Eqs. (19) and (20), the values of which for the two waves are many orders of magnitude different from each other, and so are their contributions to the parametrically induced change in impedance. Ultimately, *Q*/*S* and *C*/*S* are statements about the temporal shape of the wave, in which some waveforms, regardless of the amplitude, are more nonlinear than others.

Finally, a companion nonlinearity indicator *C*/*S* is introduced for the first time. Examination of Eqs. (19) and (20) indicates that *Q*/*S* and *C*/*S* are complementary (i.e., *C*/*S* is to dispersion as *Q*/*S* is to absorption), and together, they constitute a complete set of nonlinearity indicators for finite-amplitude waves in fluids with general absorption and dispersion laws. Note that only absorption is considered in the original derivation of the Morfey-Howell *Q*/*S*.[9]

For dispersion-dominant systems, it is recommended that *C*/*S* be used in place of *Q*/*S* as a nonlinearity indicator. For example, consider a wave system governed by the Korteweg-de Vries (KdV) equation[26]

$$\frac{\partial p}{\partial x} = d\frac{\partial^3 p}{\partial \tau^3} + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau}, \tag{26}$$

where d is the dispersion parameter. In Eq. (26) absorption is assumed to be zero ($\alpha = 0$), and dispersion exhibits a cubic dependence on frequency ($\delta \propto \omega^3$). Wave systems with KdV-type dispersion include incompressible waves on the liquid surface[26] and sounds in bubbly liquids.[27,28] To demonstrate a potential problem with *Q*/*S* and the utility of *C*/*S* in a dispersion-dominant system, we consider the bubbly liquid of Kuznetsov *et al.*[28] with parameters $\rho_0 = 1167\, kg/m^3$, $c_0 = 107\, m/s$, $d = 2.5 \times 10^{-11}\, s^3/m$, and $\beta = 111$. When applied to a soliton solution [Fig. 1(a)] for which there is no change in power spectrum [$\mathrm{Im}(\bar{Z}) = 0$; recall Eq. (24)], *Q*/*S* ought to be identically zero [dashed line in Fig. 1(b)] in the absence of linear absorption ($\alpha/k = 0$), according to Eq. (19). It would nonetheless be wrong to suggest from $Q/S = 0$ that there is no nonlinearity. The corresponding *C*/*S*, which is a quadratic function of frequency [solid line in Fig. 1(b)], can indeed capture the interplay between nonlinearity and dispersion through Eq. (20), where nonlinearity ($C/S \propto \omega^2$) offsets the innate dispersion of the medium ($-\delta/k \propto \omega^2$) to result in a wave of permanent form traveling at a constant speed [$\mathrm{Re}(\bar{Z}) - 1 = $ constant; see Eq. (25)].

4. Conclusions

In an attempt to further demystify the Morfey-Howell nonlinearity indicator *Q*/*S*, the generalized spectral Burgers equation is recast in terms of specific acoustic impedance, which is comprised of the normalized quadspectral and cospectral densities *Q*/*S* and *C*/*S* in addition to linear absorption and dispersion coefficients. This allows an impedance-based interpretation in which *Q*/*S* and *C*/*S* signify nonlinearity-induced absorption and dispersion, respectively.

"Blessed be Fourier,"[29] for an arbitrary wave can be represented as a sum of sinusoidal waves. Then there are in effect only two ways to modify a wave at any given location: via changes in amplitude (absorption) and phase (dispersion) of each sinusoidal component. Therefore, any complex wave process, be it finite-amplitude wave motion, refraction, or diffraction, could simply be couched in extra absorption and/or dispersion, adding to the inherent medium property.

Acknowledgments

W.-S.O. gratefully acknowledges the generous support from the Department of Physics and Astronomy, Brigham Young University. The authors thank Brent O. Reichman for helpful discussions.

References and Links

[1]Z. A. Gol'dberg, "Second approximation acoustic equations and the propagation of plane waves of finite amplitude," Sov. Phys. Acoust. **2**, 346–350 (1956).
[2]Z. A. Gol'dberg, "On the propagation of plane waves of finite amplitude," Sov. Phys. Acoust. **3**, 340–347 (1957).
[3]D. T. Blackstock, "Thermoviscous attenuation of plane, periodic, finite-amplitude sound waves," J. Acoust. Soc. Am. **36**, 534–542 (1964).
[4]M. F. Hamilton, "Effective Gol'dberg number for diverging waves," J. Acoust. Soc. Am. **140**, 4419–4427 (2016).
[5]J. A. Gallagher and D. K. McLaughlin, "Experiments on the nonlinear characteristics of noise propagation from low and moderate Reynolds number supersonic jets," in the *7th Aeroacoustics Conference*, AIAA Paper No. 81-2041 (1981).
[6]M. B. Muhlestein, K. L. Gee, T. B. Neilsen, and D. C. Thomas, "Evolution of the average steepening factor for nonlinearly propagating waves," J. Acoust. Soc. Am. **137**, 640–650 (2015).
[7]B. O. Reichman, M. B. Muhlestein, K. L. Gee, T. B. Neilsen, and D. C. Thomas, "Evolution of the derivative skewness for nonlinearly propagating waves," J. Acoust. Soc. Am. **139**, 1390–1403 (2016).
[8]Y. C. Kim and E. J. Powers, "Digital bispectral analysis and its applications to nonlinear wave interactions," IEEE Trans. Plasma Sci. **7**, 120–131 (1979).
[9]C. L. Morfey and G. P. Howell, "Nonlinear propagation of aircraft noise in the atmosphere," AIAA J. **19**, 986–992 (1981).
[10]P. Mora, N. Heeb, J. Kastner, E. J. Gutmark, and K. Kailasanath, "Impact of heat on the pressure skewness and kurtosis in supersonic jets," AIAA J. **52**, 777–787 (2014).
[11]W. J. Baars, C. E. Tinney, M. S. Wochner, and M. F. Hamilton, "On cumulative nonlinear acoustic waveform distortions from high-speed jets," J. Fluid Mech. **749**, 331–366 (2014).
[12]K. G. Miller and K. L. Gee, "Model-scale jet noise analysis with a single-point, frequency-domain nonlinearity indicator," J. Acoust. Soc. Am. **143**, 3479–3492 (2018).
[13]K. L. Gee, A. A. Atchley, L. E. Falco, M. R. Shepherd, L. S. Ukeiley, B. J. Jansen, and J. M. Seiner, "Bicoherence analysis of model-scale jet noise," J. Acoust. Soc. Am. **128**, EL211–EL216 (2010).
[14]S. A. McInerny and S. M. Ölçmen, "High-intensity rocket noise: Nonlinear propagation, atmospheric absorption, and characterization," J. Acoust. Soc. Am. **117**, 578–591 (2005).
[15]S. A. McInerny, "Launch vehicle acoustics, Part 2: Statistics of the time domain data," J. Aircraft **33**, 518–523 (1996).
[16]K. L. Gee, R. J. Kenny, T. B. Neilsen, T. W. Jerome, C. M. Hobbs, and M. M. James, "Spectral and statistical analysis of noise from reusable solid rocket motors," Proc. Mtgs. Acoust. **18**, 040002 (2012).
[17]W.-S. Ohm, K. L. Gee, and B. O. Reichman, "In defense of the Morfey-Howell single-point nonlinearity indicator: An impedance-based interpretation," Proc. Mtgs. Acoust. **29**, 045003 (2016).
[18]B. O. Reichman, K. L. Gee, T. B. Neilsen, and K. G. Miller, "Quantitative analysis of a frequency-domain nonlinearity indicator," J. Acoust. Soc. Am. **139**, 2505–2513 (2016).
[19]M. F. Hamilton, Yu. A. Il'inskii, and E. A. Zabolotskaya, "Dispersion," in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 5, pp. 152–153.
[20]M. F. Hamilton and C. L. Morfey, "Model equations," in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 3, pp. 49–54.
[21]O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Plenum, New York, 1977), Chap. 2.
[22]M. F. Hamilton and C. L. Morfey, "Model equations," in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 3, pp. 56–57.
[23]D. T. Blackstock, *Fundamentals of Physical Acoustics* (Wiley, New York, 2000), Chap. 9.
[24]J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures* (Wiley, Hoboken, NJ, 2010), Chap. 5, pp. 118–122.
[25]For instance, viscous absorption α_v is less than 10% of relaxation absorption α_r up to 21 kHz in air (20 °C, 1 atm, 40% RH) and 116 kHz in seawater (13 °C, 0 m, salinity 35‰).
[26]M. J. Ablowitz, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons* (Cambridge University Press, Cambridge, 2011), Chap. 5.
[27]M. F. Hamilton, Yu. A. Il'inskii, and E. A. Zabolotskaya, "Dispersion," in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 5, pp. 167–174.
[28]V. V. Kuznetsov, V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, "Propagation of perturbations in a gas-liquid mixture," J. Fluid Mech. **85**, 85–96 (1978).
[29]D. T. Blackstock, *Fundamentals of Physical Acoustics* (Wiley, New York, 2000), Chap. 1, p. 45.