Dispersive dielectrics and time reversal: Free energies, orthogonal spectra, and parity in dissipative media

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Free energies of dissipative media are reviewed. Then we use free-energy-optimal excitation and de-excitation fields to generate a dielectric's *time-reversal spectrum*, with several properties: a) The spectrum generalizes the time-reversal parity from "even" and "odd" of conservative systems to an interval $[-1,+1]$ of "time-reversal eigenvalues" λ in dissipative media. b) It yields *eigenmodes* that are complete: any state of the medium is optimally excitable or de-excitable by them. c) These excitations are orthogonal with respect to the work function of the medium and, so, d) characterize field excitations for the given medium that, when superimposed, only do work on the medium, not on each other via the medium-field interaction mechanism. Notions of en masse potential and kinetic energy in the dissipative medium arise through even ($\lambda=+1$) and odd ($\lambda=-1$) parity, but also other energy notions via alternative parity ($|\lambda|<1$) under time reversal.

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I. INTRODUCTION

It has been known for some time that traditional real-time notions of energy in passive dielectrics are ambiguous [1–3]. Ambiguity arises when "energy" depends on details of microscopic models, rather than on properties defining dielectrics via macroscopic Maxwell equations [4]. Further, Landau and Lifshitz claimed "…in the general case of arbitrary dispersion, the electromagnetic energy cannot be rationally defined as a thermodynamic quantity. This is because the presence of dispersion in general signifies a dissipation in energy…." [5]. Lacking thermodynamic interpretations generally, they offered Brillouin's [6] narrow bandwidth, *cycle-averaged* notion for "dispersive but lossless" [7] dielectrics [5,8,9]. Despite claims, we give macroscopic real-time diagnostics in arbitrarily dissipative dielectrics interpretable as thermodynamic *free energies* [10]. Dissipation's main effect—multiple free energies—stems from a medium's *time-reversal spectrum*, as also "irreversible energies" generalizing kinetic and potential energies to irreversible dynamics.

Recently, we introduced an instantaneous generalization of Brillouin's energy, valid for arbitrary dissipative dielectrics: at each moment of a passive dielectric's excitation by a pulse, an unambiguous maximum energy is subsequently recoverable from the medium [11]. This *recoverable energy* is extracted by an optimal "future" electromagnetic field starting from the moment considered. This instantaneous notion of electromagnetic energy depends only on the current dynamic macroscopic *state* of the dissipative medium. We also showed any given state of a linear, passive dielectric is created in a unique, energetically minimal manner [12]. Both real time, macroscopic notions of energy—the maximum energy extractable from a dispersive dielectric in a given state, and the minimal *creation energy* that could be imbued in it to produce its current state—are independent of microscopic model. Recoverable and creation energies, U_+ and U_-, were recently used to describe energetic features of "slow" and "fast" light, and of optical regimes such as electromagnetically induced transparency (EIT) where traditional notions fail to have physical interpretation [13]. (See [14–20] for classical and new ideas about slow and fast light.)

Importantly the goals here are alternate to the work of [21]. There the unique and time-conserved energy constructed is the total work (density) performed by the field on itself and the medium, and nontrivial representation of the conservation law via a one-parameter bath of individually reversible auxiliary fields leads to a Lagrangian formulation of the dissipative dynamics. In turn this allows for quantization of these dynamics. In contrast here we separate the conserved total energy into "reactive" and "dissipated" components. Then we separate reactive energy into components with novel but definite parity under time reversal. Generalizations of kinetic and potential energy arise by embracing macroscopic phenomenology's irreversibility in this specific way.

We review ideas of dynamical free energy provided by the viscoelasticity community showing U_- and U_+ are the maximal and minimal free energies of a dielectric. (Subscripts + and − refer to future and past.) We then present the following results regarding irreversibility in macroscopic media.

For linear media, there is a class of "past" electromagnetic excitation fields $E_-(\tau)$ (vanishing after time $t=0$) distinguished by the following property: when the medium is excited up until $t=0$ by past field $E_-(\tau)$ in this class, the medium is subsequently de-excited optimally–with maximal energy recovered from the dielectric–by a special "future" field $E_+(\tau)$ (vanishing before $t=0$) that is simply a time-reversed, dilated version of the original past field, i.e.,

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$$E_+(\tau) = \lambda E_-(-\tau). \tag{1}$$

Such past-future field pairs $(E_-(\tau), E_+(\tau))$ expose various instantaneous energetics of linear dielectrics:

(1) Only certain values of the dilation λ arise, characterizing a given medium. Passivity dictates these *time-reversal eigenvalues* lie in the interval $[-1,1]$.

(2) The *eigenspaces* of field excitations $E_-(\tau)$ are orthogonal with respect to the medium's work function: the work $W[E_-]$ performed on the dielectric by superposition $E_- = c_1E_{-\lambda_1} + c_2E_{-\lambda_2}$ of energetically optimal past fields $E_{-\lambda_1}$ and $E_{-\lambda_2}$ with time-reversal eigenvalues $\lambda_1 \neq \lambda_2$ is the sum of the work performed in the dielectric by each separately, no constructive/destructive interference arising. That is,

$$W[c_1E_{-\lambda_1} + c_2E_{-\lambda_2}] = c_1^2 W[E_{-\lambda_1}] + c_2^2 W[E_{-\lambda_2}]. \tag{2}$$

(3) The class of fields $E_-(\tau)$ are *complete*: Any medium state is reachable by linear superposition in the class, the class forming then a *state-space basis*.

(4) This basis is *preferred*: By design, $E_+(\tau)$ is optimal at extracting energy *from* the dielectric in the state produced by $E_-(\tau)$. But, fact, $E_-(\tau)$ is itself optimal at infusing energy *into* the dielectric, the same true of superpositions of past fields satisfying Eq. (1) with various λ's.

(5) Due to Eq. (2), the maximal and minimal free energies U_- and U_+ are diagonal quadratic forms in the preferred basis.

(6) Two terms in the forms for U_+ and U_- always arise, corresponding to time-reversal eigenvalues $\lambda=-1$ and $\lambda=+1$. These are labeled *en-masse* kinetic and potential energy, since they are forms in excitations that are momentum, respectively, positionlike—i.e., odd and even—under time-reversal.

(7) The quadratic forms of U_- and U_+ are identical except for time-reversal eigenvalues with $|\lambda|<1$. The associated $|\lambda|<1$-energies allowing U_- and U_+ to be distinct give rise to "irreversible energies" and to a class of objects (optimal field excitations) sporting time-reversal parities other than sign change [22].

This paper is organized as follows. Section IIgives results about extremal free energies for a broad class of dielectrics. Section III specifies a subclass of linear, time-translationally invariant dielectrics generalizing salient properties of Lorentz media. Two *Riemann-Hilbert* problems allow for a representation of a single Lorentz medium's unique (free) energy that generalizes straightforwardly to media sporting multiple free energies. Section IV introduces the time-reversal spectrum and the notion of *state* in dielectrics. Section V computes the time-reversal spectrum of an example dielectric, using it to introduce the general relationship between the spectrum and the extremal free energies. Finally we present central results on time-reversal parities giving rise to the idea of kinetic and potential energy as well as generalizations arising from the absence of time-reversal invariance.

II. WORK IN A DIELECTRIC

In Secs. II A and II B, we develop defining properties of a dielectric, linear or not. Theorems (with proofs in appendices) describe maximum and minimum free energies and their dissipation rates. (A review of the relation between work and free energy is given in Appendix A. It summarizes well-known information in the viscoelasticity community relevant to general dissipative media.) In subsequent sections time-reversal properties characterize in a novel way the extremal free energies of linear dielectrics.

A. Work and causality

For isotropic media, the work $W[E](t)$ performed (at a specified point) by an electric field time series $E(\tau)$ on a dielectric with polarization time series $P(\tau)$ during period $(-\infty, t]$ is [23]

$$W[E](t) = \int_{-\infty}^{t} E(\tau)\dot{P}[E](\tau)d\tau. \tag{3}$$

Importantly the functional $E \mapsto P[E]$ is causal: presently [24], $E \mapsto P[E]$ is *causal* if and only if $P[E](t)$ depends entirely on the history (and *not* future) of $E(t)$. Note that causality of the polarization functional $E \mapsto P[E]$ ensures the work functional $E \mapsto W[E]$ (3) is also causal. Also Eq. (3) dictates the work functional $W[E]$ ceases to evolve after field E vanishes.

In order to invoke causality, we define truncated versions of time series: let "t-past field" $E_-^t(\tau)$ and "t-future field" $E_+^t(\tau)$ be related to $E(\tau)$ through

$$E_-^t(\tau) := E(\tau)\theta_-^t(\tau) = E(\tau)\begin{cases} 1 & \text{if } \tau < t, \\ 0 & \text{if } \tau > t. \end{cases} \tag{4}$$

and

$$E_+^t(\tau) := E(\tau)\theta_+^t(\tau) = E(\tau)\begin{cases} 0 & \text{if } \tau < t, \\ 1 & \text{if } \tau > t. \end{cases} \tag{5}$$

The value that $E_{-/+}^t(\tau)$ takes at time $\tau=t$ missing in Eqs. (4) and (5) will often not matter since the media cannot respond instantly to physical fields. But to develop maximal and minimal free energies of *linear* dielectrics, idealizations like the Dirac delta time series $E(\tau) \sim \delta(\tau)$ are considered. The *polarization current* functional $E \mapsto \dot{P}[E]$ instantly responds to this idealization, which allows the work functional $E \mapsto W[E]$ to respond instantly too. We will not use the notation of Eq. (4) for these idealizations.

With the notation of Eqs. (4) and (5), causality can be expressed

$$P[E_-^t + G_+^t](\tau) = P[E_-^t + F_+^t](\tau), \quad \tau \le t, \tag{6}$$

with F and G arbitrary physical fields. In particular

$$P[E](\tau) = P[E_-^t](\tau), \quad \tau \le t. \tag{7}$$

With P replaced by any causal functional of $E(\tau)$ (such as $W = W[E]$), Eq. (7) is the main fact used in calculations involving causality. Statements such as Eq. (7) may not make sense for field idealizations needed for the extremization problems to come, but then modifications should be clear. Equation (7) will hold for the physical class $\mathcal{E}^0$ of real, continuous, absolutely integrable field time series $E(\tau)$. ($\mathcal{E}$ will denote $\mathcal{E}^0$ less continuity.)

B. Passivity, state, and the fundamental free energies

Like electrical networks, a dielectric with work functional Eq. (3) is *passive* if and only if for every admissible field $E(\tau)$ and every time $t \in (-\infty, +\infty)$

$$W[E](t) \geq 0 \tag{8}$$

[3,25–28]. This is a specific instance of Eq. (A5). The functional of Eq. (3) allows equality in Eq. (8) for at least the zero field $E(\tau) \equiv 0$. Causality Eq. (6), passivity Eq. (8) and restriction to admissible fields allows the following ideas to make sense.

Definition 1 (Irrecoverable Energy). The irrecoverable energy (density) functional $Q_+[E]$ of a dielectric with causal and passive work functional $W[E]$ is

$$Q_+[E](t) = \inf_F W[E_-^t + F_+^t](+\infty). \tag{9}$$

As suggested above, in Eq. (9), we restrict the variable future field $F_+^t(\tau) = F(\tau)\theta_+^t(\tau)$ to membership in an admissible class, such as $\mathcal{E}_0$.

Definition 2 (Recoverable Energy). The recoverable energy (density) functional $U_+[E]$ of a dielectric with causal and passive work functional $W[E]$ is

$$U_+[E](t) = W[E](t) - Q_+[E](t). \tag{10}$$

With Definition 2 we have

$$W[E](t) = U_+[E](t) + Q_+[E](t). \tag{11}$$

Equation (11) and Definition 1 indicate we are separating into two parts the work $W[E](t)$ done on a dielectric by a field E up to and including a specific time $\tau = t$. These parts are: a) the fraction of energy $Q_+[E](t)$ that cannot be returned from the medium to the field under any circumstances via the performance of work, and b) the remaining fraction of energy $U_+[E](t)$ that can be delivered to the field by the medium via the performance of work, in the limit of optimal future steering of the medium-field excitation by a future field time series $F_+^t(\tau)$ [as in Eq. (9)]. If we think of $U_+[E](t)$ as "energy," and $-Q_+[E](t)$, which is never positive, as "heat" added to the medium, in (10) we have something like a thermodynamic first law; irrecoverable energy $Q_+[E](t) \geq 0$ is heat lost from the system to an environment. This dynamical connection with the usual first law is now only formal since we do not address what temperature means in a medium model admitting only polarization as a relevant measurable. But for such a medium model, *state* must arguably be defined as follows [29,30]:

Definition 3 (Dielectric State). The state of a dielectric $E \mapsto P[E]$ at time t is the equivalence class $\Sigma(E_-^t)$ of all admissible t-past fields E_-^t that give the same future polarization map

$$E_+^t \mapsto \theta_+^t P[E_-^t + E_+^t]. \tag{12}$$

Here, we introduce the notation $\Sigma(E_-^t)$ to emphasize the relation between a state and the fraction E_-^t of any (admissible) field time series E causally producing it by time t.

According to Definition 3 fields E and F produce the same state in dielectric $P[E]$ at time $\tau = t$ if and only if for every such E_+^t,

$$P[E_-^t + E_+^t](\tau) = P[F_-^t + E_+^t](\tau), \quad \tau \geq t. \tag{13}$$

For a linear dielectric Eq. (13) simplifies to

$$P[E_-^t](\tau) = P[F_-^t](\tau), \quad \tau \geq t, \tag{14}$$

which says two admissible fields E and F produce the same state in dielectric $P[E]$ at a time t if and only if the dielectric *rings* the same after the *preparations* E_-^t or F_-^t. (See [29] for the importance of the notion of preparation to that of state in thermodynamics.)

Two more dynamic energy accounting ideas arise:

Definition 4 (Creation Energy). The creation energy (density) functional $U_-[E]$ of a dielectric with causal and passive work functional $W[E]$ is

$$U_-[E](t) = \inf_{F \in \Sigma(E_-^t)} W[F](t). \tag{15}$$

Here, $\Sigma(E_-^t)$ restricts the steering fields F to those yielding in dielectric $P[E]$ the same state by time t as the original field E (or t-past field E_-^t). So *creation energy* $U_-[E](t)$ is the least work a field must perform on a dielectric by time t to produce in it the same state that original field E (or E_-^t) did by that time. U_- is patently a state function, and its definition makes it a case of $\psi_M(\sigma)$ in Eq. (A4). Appendix B (as well as references [10,31,32]) proves that U_+ is also a state function. Combining Definition 1 and Definition 2 indicates that U_+ is a case of $\psi_m(\sigma)$ in Eq. (A4).

Definition 5 (Waste Energy). The waste energy (density) functional $Q_-[E]$ of a dielectric with causal and passive work functional $W[E]$ is

$$Q_-[E](t) = W[E](t) - U_-[E](t). \tag{16}$$

From Eqs. (16) and (15) we see $Q_-[E](t)$ measures excess energy used to create a given state $\Sigma(E_-^t)$ by a given field time series E over the amount minimally required to do so.

III. SIMPLE DIELECTRICS AND THE FUNDAMENTAL RIEMANN-HILBERT PROBLEMS

A. Simple dielectrics

We now specialize to *simple dielectrics*. In addition to isotropy [Eq. (3) and discussion], causality [Eqs. (6) and (7)] and passivity Eq. (8), a simple dielectric is *linear*, *stationary*, *inertial*, *bound* and *viscous*: A dielectric $P[E]$ is *linear* if for all admissible fields E_1 and E_2

$$P[c_1 E_1 + c_2 E_2](\tau) = c_1 P[E_1](\tau) + c_2 P[E_2](\tau) \tag{17}$$

for all (temporal) constants c_1 and c_2. A dielectric $P[E]$ is *stationary* (*time translationally invariant*) if for all s

$$P[E(\tau)](t-s) = P[E(\tau - s)](t) \tag{18}$$

for all t and all admissible fields E. Here, we abuse notation by denoting a time series argument E to a functional P with its own arguments (τ or $\tau - s$), but the idea of P commuting time translation should be clear. Linearity and stationarity give the usual form

$$P[E](\tau) = \int_{-\infty}^{+\infty} ds G(s)E(\tau - s), \tag{19}$$

where G is the impulse response of the medium: $G \equiv P[\delta_0]$, with δ_0 as the Dirac delta distribution supported at $\tau=0$. By causality, G is not supported for $\tau<0$.

It is well known [33] that linear, stationary constitutive relation Eq. (19) gives theorem

$$W[E](t) = \int_{-\infty}^{t} E(\tau)\dot{P}[E](\tau)d\tau = \int_{-\infty}^{\infty} \omega \, \mathrm{Im}[\chi](\omega)|\widehat{E_{-}^{t}}(\omega)|^2 d\omega, \tag{20}$$

where the *susceptibility*

$$\chi(\omega) = \sqrt{2\pi}\hat{G}(\omega), \tag{21}$$

and where hats indicate Fourier transformation: for $F \in \mathcal{E}^0$, define the Fourier transform $\hat{F}$ of F via

$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\tau)e^{i\omega\tau}d\tau. \tag{22}$$

In Eq. (20) we used

$$\widehat{E_{-}^{t}}(-\omega) = \widehat{E_{-}^{t}}^{*}(\omega), \quad \omega \in \mathbb{R}, \tag{23}$$

which follows from definition Eq. (22) and time series E_{-}^{t} real. Since polarization impulse response G is real, this *real symmetry* also holds then for $\chi(\omega)$:

$$\chi(-\omega) = \chi^{*}(\omega), \quad \omega \in \mathbb{R}. \tag{24}$$

In Eq. (20), the Fourier transform $\widehat{E_{-}^{t}}$ of the t-past field E_{-}^{t} stems from $E \mapsto E\dot{P}[E]$ [the rate of work development as per Eq. (3)] being a causal functional of its time series argument [34]. Noting $E_{-}^{t}=E$ for $t=+\infty$ and admissible E, we see Theorem Eq. (20) generalizes the long-known fact [35] that

$$W[E](+\infty) = \int_{-\infty}^{\infty} \omega \, \mathrm{Im}[\chi](\omega)|\hat{E}(\omega)|^2 d\omega. \tag{25}$$

Equation (25) is the total work dissipated in a passive dielectric arising from (energetically admissible) excitation E.

Real symmetry Eq. (23) forces the integrand in Eq. (20) to be of the same sign as the *oscillator density* $\rho(\omega)$ defined by

$$\rho(\omega) \coloneqq \omega \, \mathrm{Im}[\chi](\omega) \coloneqq \omega \frac{\chi(\omega) - \chi(-\omega)}{2i}. \tag{26}$$

So passivity Eq. (8) in linear, stationary media is [36]

$$\rho(\omega) \geq 0, \quad \omega \in \mathbb{R}, \tag{27}$$

(pointwise [37]). $\rho(\omega)$ is clearly even and real-symmetric.

Assumptions allowing Theorem Eq. (20) include the following physical ones: restrict to a space $\mathcal{E}$ of field time series E with bounded Fourier transform $\hat{E}(\omega)$, and to dielectrics $P[E]$ for which the transform $\hat{G}$ of the polarization impulse response $G=P[\delta_0]$ is order ω^{-2} as ω tends to infinity. A sufficient condition for the latter is that the time derivatives $G^{(j)}(t)$ be absolutely integrable over $t>0$ for $j=0,1,2$, and $G(0)=0$. Similarly, the transform $\hat{E}(\omega)$ will be bounded, which includes that it will be order ω^0 as ω tends to infinity, if $E^{(j)}(t)$ is absolutely integrable for $j=0$, i.e., provided $E(t)$ is absolutely integrable.

A linear dielectric with susceptibility $\chi(\omega)=\sqrt{2\pi}\hat{G}(\omega)$ order ω^{-2} for large ω is *inertial*: it corresponds to $G(0^+) = P[\delta_0](0^+)=0$; that is, inertia prevents the polarization from changing discontinuously even in the limit that a finite impulse δ_0 of momentum is instantly imparted to it.

A dielectric with $\chi(0)$ finite is *bound*. An instructive example is the Lorentz oscillator:

$$\chi(\omega) = \frac{\omega_p^2}{(-i\omega)^2 + \gamma(-i\omega) + \omega_0^2}. \tag{28}$$

In such a model

$$\chi(0) = \frac{\omega_p^2}{\omega_0^2}, \tag{29}$$

which, with inertia, is only finite (and positive) when

$$k \coloneqq \omega_0^2/\omega_p^2 > 0, \tag{30}$$

the mechanical interpretation of k being a spring constant; the dielectric is bound when "its restoring spring does not vanish." When the spring vanishes in this model we have a (Drude) conductor, rather than a dielectric. For the coming time-reversal spectrum, the distinction between dielectric and conductor is pivotal.

For a linear, stationary, inertial, bound and passive dielectric, the asymptotics of the real-symmetric susceptibility $\chi(\omega)=\sqrt{2\pi}\hat{G}(\omega)$ near $\omega=0,\infty$ must be [38]

$$\chi(\omega) \overset{\omega\to\infty}{\sim} -\left(1 - i\frac{\gamma_\infty}{\omega} + \ldots\right)\frac{\omega_p^2}{\omega^2},$$

$$\chi(\omega) \overset{\omega\to 0}{\sim} \left(1 + i\frac{\gamma_0}{\omega_0^2}\omega + \ldots\right)\frac{\omega_p^2}{\omega_0^2}, \tag{31}$$

where passivity Eq. (27) dictates the *high- and low-frequency dissipation rates* γ_∞ and γ_0 must be nonnegative. The dielectric is *viscous* if they are both positive. If so, then

$$\frac{\rho(\omega)}{\gamma_\infty} \overset{\omega\to\infty}{\sim} \left(\frac{\omega}{\omega_p}\right)^{-2}, \quad \frac{\rho(\omega)}{\gamma_0} \overset{\omega\to 0}{\sim} \left(\frac{\omega_p\omega}{\omega_0^2}\right)^{+2}. \tag{32}$$

So our generic dielectric has density ρ with exactly second-order zeroes at $\omega=0,\infty$. It is *strictly passive* if they are ρ's only real zeroes [Eq. (27) holding with equality]. A dielectric with all the above properties is *simple*.

B. Two Riemann-Hilbert problems for simple dielectrics

With G as described above,

$$\chi(z) \coloneqq \int_{0}^{+\infty} G(\tau)e^{iz\tau}d\tau \tag{33}$$

defines a function analytic in an open set containing the closed upper-half z plane $\{z \in \mathbb{C} : \mathrm{Im}[z] \geq 0\}$ (said "analytic in a closed upper-half plane"), with all the asymptotics there already described for real $z=\omega$. By Cauchy's integral formula and the various relevant symmetries we have [38] that

$$\chi(z) = \lim_{\gamma\to 0^+} \frac{2}{\pi}\int_0^{+\infty} \frac{\rho(\omega_0)}{(-iz)^2+\gamma(-iz)+\omega_0^2} d\omega_0 \tag{34}$$

holds there, representing the susceptibility as a continuum of Lorentz oscillators with varying *resonance frequencies* ω_0 and vanishingly small (high and low) dissipation rates γ. Representation Eq. (34) suggests ρ's name: $2\rho(\omega_0)/\pi$ is the density of oscillators with resonance frequency ω_0 contributing to χ. Noting $\rho \geq 0$, this also shows $\chi(z)$ has no zeroes in a closed upper-half plane [38]. So both $\chi(z)$ and $1/\chi(z)$ are analytic in the (open) upper-half plane, although the latter grows like z^2 near infinity since the former decays such as z^{-2}.

The time-reversal spectrum of a simple dielectric will be finite dimensional and without other complications if we restrict to susceptibilities $\chi(\omega)$ that are rational, with only simple poles and zeroes off the imaginary axis. Such *rational* dielectrics have susceptibilities of the form

$$\frac{\chi(\omega)}{-\omega_p^2} = \frac{\prod_{j=1}^{N}(\omega - Z_j')(\omega + Z_j'^*)}{\prod_{j=1}^{N+1}(\omega - z_j)(\omega + z_j^*)}, \tag{35}$$

with $N=0,1,2\ldots$. We take the obvious convention that no zero is located at a supposed pole; no "residue" is zero.

It should be noted that $\rho(\omega)$ has other zeroes beyond its real (second order) ones $\omega=0,\infty$. By definition Eq. (26), and given Eq. (35),

$$\begin{aligned}\frac{\rho(\omega)}{\gamma_\infty\omega_p^2\omega^2} &= \frac{\prod_{j=1}^{N}(\omega - Z_j)(\omega + Z_j^*)}{\prod_{j=1}^{N+1}(\omega - z_j)(\omega + z_j^*)}\frac{\prod_{j=1}^{N}(\omega + Z_j)(\omega - Z_j^*)}{\prod_{j=1}^{N+1}(\omega + z_j)(\omega - z_j^*)} \\ &=: f(\omega)f(-\omega) \\ &= f(\omega)\overline{f(\omega)} \geq 0.\end{aligned} \tag{36}$$

The last representation and inequality hold for real ω. When the z_j's and Z_j''s lie in the open lower-half plane, $\chi(z)$ and $1/\chi(z)$ are analytic in a closed upper-half plane, and Eqs. (35) and (36) sport all properties of a simple dielectric. We may choose the Z_j's of (36) to be there also, since when Z_j is a zero of $\rho(\omega)$, so is $-Z_j$ (parity), $-Z_j^*$ (real-symmetry), and $+Z_j^*$ (parity *and* real symmetry).

With Eqs. (35) and (36) we may write

$$\begin{aligned}\rho(\omega) &= \left(\sqrt{\frac{\gamma_\infty}{\omega_p^2}} i\omega\chi_v(-\omega)\right)\left(-\sqrt{\frac{\gamma_\infty}{\omega_p^2}} i\omega\chi_v(\omega)\right) \\ &= \rho_+(-\omega)\rho_+(\omega) \\ &= \rho_-(\omega)\rho_+(\omega),\end{aligned} \tag{37}$$

where

$$\frac{\chi_v(\omega)}{-\omega_p^2} = \frac{\prod_{j=1}^{N}(\omega - Z_j)(\omega + Z_j^*)}{\prod_{j=1}^{N+1}(\omega - z_j)(\omega + z_j^*)}, \tag{38}$$

i.e., $\chi_v(\omega)$ is susceptibility $\chi(\omega)$ but with the zeroes of the latter replaced by the *causal* zeroes of $\rho(\omega)$, i.e., those in the lower-half plane. From Eq. (34) and Cauchy's theorem one finds $\rho(\omega)$ has each of the poles $\chi(\omega)$ does. So no zero in (36) is located where a pole is. Thus $\chi_v(\omega)$ *has the same singularities as* $\chi(\omega)$. Finding factors $\rho_+(\omega)$ and $\rho_-(\omega)$ in Eq. (37) which, along with their reciprocals, are analytic and with prescribed asymptotics in the upper and lower-half planes constitutes a homogeneous Riemann-Hilbert (RH) problem [39]. [Eq. (37) is readily solved numerically [11].) Factorization Eq. (37) gives

$$\begin{aligned}W[E](t) &= \int_{-\infty}^{\infty} \rho(\omega)|\widehat{E_-^t}(\omega)|^2 d\omega \\ &= \frac{\gamma}{\omega_p^2}\int_{-\infty}^{\infty} |-i\omega\chi_v(\omega)\widehat{E_-^t}(\omega)|^2 d\omega \\ &= \frac{\gamma}{\omega_p^2}\int_{-\infty}^{\infty} [\dot{P}_v(E_-^t)(\tau)]^2 d\tau \\ &= \frac{\gamma}{\omega_p^2}\int_{-\infty}^{t} [\dot{P}_v(E_-^t)(\tau)]^2 d\tau + \frac{\gamma}{\omega_p^2}\int_{t}^{\infty} [\dot{P}_v(E_-^t)(\tau)]^2 d\tau.\end{aligned} \tag{39}$$

P_v here is the *virtual* polarization. In [11] we showed

Theorem 1 (Recoverable and Irrecoverable Energy). For a simple dielectric $P[E]$, the irrecoverable energy $Q_+[E](t)$ is given by

$$Q_+(E)(t) = \frac{\gamma}{\omega_p^2}\int_{-\infty}^{t} [\dot{P}_v(E_-^t)(\tau)]^2 d\tau = \frac{\gamma}{\omega_p^2}\int_{-\infty}^{t} (\dot{P}_v(E)(\tau))^2 d\tau, \tag{40}$$

the recoverable energy $U_+[E](t)$ by

$$U_+(E)(t) = \frac{\gamma}{\omega_p^2}\int_{t}^{\infty} [\dot{P}_v(E_-^t)(\tau)]^2 d\tau. \tag{41}$$

In the last formula in Eq. (40) we took advantage of the causality of the new polarizationlike functional P_v, which is a consequence of the causal factorization of RH problem (37), i.e., $\chi_v(\omega)$'s analyticity, which is the same as $\chi(\omega)$'s. That $1/\chi_v(\omega)$ is also analytic in the upper-half plane is central to why Eqs. (40) and (41) represent the irrecoverable and recoverable energies, rather than representing waste and creation energies, which require opposite analyticity properties for the relevant reciprocal virtual susceptibility [11,12,33,40].

The monotonicity of the irrecoverable energy is clear in representation Eq. (40). In fact a necessary (and nearly sufficient) condition for the field E to be engaged in optimal energy recovery over a period is that the irrecoverable energy not increase, i.e., remain constant: A consequence of the notion of irrecoverable energy is that

$$Q_+[E](t_2) = Q_+[E](t_1), \quad t_2 \geq t_1, \tag{42}$$

if and only if, for all relevant τ, $E(\tau)=E_-^{t_1}(\tau)+E_+^{t_1}(\tau)$, with $E_+^{t_1}(\tau)$ being the optimal continuation of $E_-^{t_1}(\tau)$ after time $\tau = t_1$ (but see Appendix C). From Eq. (40), we note Eq. (42) holds for all $t_2 > t_1$ if and only if for all such t_2

$$\int_{t_1}^{t_2} \dot{P}_{\rm v}^2[E_-^{t_1} + E_+^{t_1}](\tau)d\tau = 0, \tag{43}$$

or then

$$\dot{P}_{\rm v}[E_-^{t_1} + E_+^{t_1}](\tau) = 0, \quad \tau \geq t_1, \tag{44}$$

which is to be solved for $E_+^{t_1}=E_+^{t_1}[E_-^{t_1}]$ given $E_-^{t_1}$. In the frequency domain Eq. (44) is

$$-i\omega\chi_{\rm v}(\omega)[\hat{E}_-^{t_1}(\omega) + \hat{E}_+^{t_1}(\omega)] = \hat{Z}_-(\omega), \tag{45}$$

where $\hat{E}_-^{t_1}$, $\hat{E}_+^{t_1}$ and $\hat{Z}_-$ denote the transforms of E_-, E_+, and $\dot{P}_{\rm v}[E]$. Given the space $\mathcal{E}$ of admissible fields E (adjoined with Dirac delta distributions), and the properties of polarization impulse response G, analyses like that of Eq. (33) show $\hat{E}_-^{t_1}$ is analytic and bounded in a lower-half plane, $\hat{E}_+^{t_1}$ is analytic in an upper-half plane and bounded there *away from* $\omega=0$, where it may have a simple pole, and that $\hat{Z}_-$ is analytic in a closed lower-half ω plane, but restricted to be order ω^{-1} there as ω tends to infinity. (See Appendix C) Eq. (45) is a second, *inhomogeneous* RH problem related to optimal energy recovery. Given $\hat{E}_-^{t_1}$, it can be solved *uniquely* for $\hat{E}_+^{t_1}$ and $\hat{Z}_-$ by standard techniques [41]. For rational $-i\omega\chi_{\rm v}(\omega)$, these reduce to partial fractions.

Before considering time reversal, we note a theorem that suggests its consideration.

Theorem 2 (Work and Recoverable Energy). For a simple dielectric $P[E]$,

$$\dot{P}_{\rm v}[E_+^t](\tau) = -\theta_+^t(\tau)\dot{P}_{\rm v}[E_-^t](\tau), \tag{46}$$

where E_+^t is the optimal t-future recovery field for t-past field E_-^t. Thus, the recoverable energy satisfies

$$U_+[E](t) = U_+[E_-^t](t) = W[E_+^t](+\infty). \tag{47}$$

Equation (46) follows from Eq. (44) via causality and linearity of $P_{\rm v}$. It says the entire effective polarization $P_{\rm v}$ excitation of an optimal t-future field E_+^t *in isolation*, i.e., without past field E_-^t prepended to it, is precisely the opposite of the *ringing* of the original t-past field E_-^t. The first result in Eq. (47) is just causality [see Eq. (41)], the second more notable, indicating the maximum recoverable work of a t-past field is exactly the total work an associated optimal recovery t-future field would perform on the medium in isolation. It follows from Eqs. (46), (41), and (39), along with $E_+^t=(E_+^t)_-^{+\infty}$. We now only consider distinguished time $t=0$, which is representative for a stationary dielectric.

IV. TIME-REVERSAL SPECTRUM OF A SIMPLE DIELECTRIC

A. State in rational dielectrics

In Eq. (14), two field time series $E(\tau)$ and $F(\tau)$ produce the same state in dielectric $P[E]$ by time $t=0$ if and only if $P[\Delta](\tau)=0$ for $\tau \geq 0$, where Δ denotes the difference of the two past fields $E_-(\tau)$ and $F_-(\tau)$ (not supported for $\tau>0$). From Eq. (35) and Cauchy's theorem, for $\tau \geq 0$

$$\begin{aligned} 0 &= P(\Delta)(\tau) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \chi(\omega)\hat{\Delta}(\omega)e^{-i\omega\tau}d\omega \\ &= -\sqrt{2\pi}i\sum_{j=1}^{N+1} [a_j\hat{\Delta}(z_j)e^{-iz_j\tau} + a_j^*\hat{\Delta}(-z_j^*)e^{iz_j^*\tau}]. \end{aligned} \tag{48}$$

The a_j's and a_j^*'s are residues of $\chi(\omega)$ at its poles, which are nonzero by convention. With frequencies z_j and $-z_j^*$ in Eq. (35) distinct, the $e^{-iz_j\tau}$ and $e^{iz_j^*\tau}$ time series are independent, and Eq. (48) requires each of the $\hat{\Delta}$'s vanish, i.e.,

$$\hat{E}_-(z_j) = \hat{F}_-(z_j), \quad \hat{E}_-(-z_j^*) = \hat{F}_-(-z_j^*) \tag{49}$$

for $j=1,\ldots,N+1$. So at $t=0$ the state of rational dielectric Eq. (35) excited by field $E(\tau)$ is the truncated field transform $\hat{E}_-$ at $\chi(\omega)$'s singularities; $\Sigma(E_-)$ of Definition 4 is the real-symmetric $\mathbb{C}^{2(N+1)}$-vector in Eq. (49). Since effective susceptibility $\chi_{\rm v}(\omega)$ given by Eq. (38) has the same singularities as susceptibility $\chi(\omega)$ Eq. (35),

$$\begin{aligned} 0 &= P_{\rm v}(\Delta)(\tau) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \chi_{\rm v}(\omega)\hat{\Delta}(\omega)e^{-i\omega\tau}d\omega \\ &= -\sqrt{2\pi}i\sum_{j=1}^{N+1} [\tilde{a}_j\hat{\Delta}(z_j)e^{-iz_j\tau} + \tilde{a}_j^*\hat{\Delta}(-z_j^*)e^{iz_j^*\tau}] \end{aligned} \tag{50}$$

for $\tau \geq 0$ if and only if Eq. (48). The $\tilde{a}_j$'s and $\tilde{a}_j^*$'s are the residues of $\chi_{\rm v}(\omega)$ at its poles, which are nonzero by the discussion of Eq. (38). Equations (48) and (50) show that two fields $E(\tau)$ and $F(\tau)$ produce the same state in the linear dielectric $P[E]$ by time $\tau=0$ if and only if

$$P_{\rm v}[E_-](\tau) = P_{\rm v}[F_-](\tau), \quad \tau \geq 0. \tag{51}$$

So effective polarization $P_{\rm v}$ is equivalent to polarization P in detecting state. Further, letting X denote either P or $P_{\rm v}$, either Eq. (48) or Eq. (50) gives

$$\dot{X}[E_-](\tau) = 0, \quad \tau \geq 0 \tag{52}$$

implying that, for $j=1,\ldots,N+1$,

$$\hat{E}_-(z_j) = 0, \quad \hat{E}_-(-z_j^*) = 0 \tag{53}$$

since none of the z_j's or $-z_j^*$'s is zero, which then implies

$$X[E_-](\tau) = 0, \quad \tau \geq 0. \tag{54}$$

So Eqs. (52)–(54) are equivalent and describe the *zero state*.

Later we find the zero state arising from other than the zero field $E_{-}\equiv 0$.

B. An Inner product for simple dielectrics

For simple dielectric $P[E]$, $\rho(\omega)$ is such that the form

$$\langle \hat{E},\hat{F}\rangle := \frac{\omega_p^2}{\gamma_\infty}\int_{-\infty}^{+\infty}\rho(\omega)\overline{\hat{E}(\omega)}\hat{F}(\omega)d\omega =: \int_{-\infty}^{+\infty}[\Lambda(-\omega)\hat{E}(-\omega)][\Lambda(\omega)\hat{F}(\omega)]d\omega \quad (55)$$

is a real inner product on at least the transformed space $\hat{\mathcal{E}}$ of $\mathcal{E}$ [see discussion of Eq. (27)]. On that space

$$\langle \hat{E},\hat{F}\rangle = \langle \hat{F},\hat{E}\rangle \in \mathbb{R}, \quad \|\hat{E}\|^2 := \langle \hat{E},\hat{E}\rangle \geq 0, \quad (56)$$

the last equality only when $\hat{E}\equiv 0$. In Eq. (55), we define

$$\Lambda(\omega) := -i\omega\chi_v(\omega) \quad (57)$$

as the solution factor of homogeneous RH problem Eq. (37): $\Lambda(\omega)$ is analytic in a closed upper-half plane, its only zeroes there simple ones at $\omega=0,\infty$ Eq. (38).

Via inner-product Eq. (55) and its norm, Eq. (39) states

$$W[E](0) = \frac{\gamma_\infty}{\omega_p^2}\langle \hat{E}_{-},\hat{E}_{-}\rangle = \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_{-}\|^2 \geq 0, \quad (58)$$

and Eq. (47) [with Eq. (25)] states

$$U_{+}[E](0) = U_{+}[E_{-}](0) = W[E_{+}](+\infty) = \frac{\gamma_\infty}{\omega_p^2}\langle \hat{E}_{+},\hat{E}_{+}\rangle = \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_{+}\|^2 \geq 0. \quad (59)$$

For $E=E_{-}+E_{+}$ we have

$$\langle \hat{E}_{+}+\hat{E}_{-},\hat{E}_{+}\rangle = \langle \hat{E},\hat{E}_{+}\rangle = \int_{-\infty}^{+\infty}[\Lambda(-\omega)\hat{E}(-\omega)][\Lambda(\omega)\hat{E}_{+}(\omega)]d\omega = \int_{-\infty}^{+\infty}\hat{Z}_{-}(-\omega)[\Lambda(\omega)\hat{E}_{+}(\omega)]d\omega = 0, \quad (60)$$

the last by Cauchy's theorem since $\hat{Z}_{-}(-\omega)$ *and* $\Lambda(\omega)\hat{E}_{+}(\omega)$ are analytic and order ω^{-1} at infinity in a closed upper-half ω plane. It follows that RH problem Eq. (45) gives

$$U_{+}[E](0) = -\frac{\gamma_\infty}{\omega_p^2}\langle \hat{E}_{-},\hat{E}_{+}\rangle. \quad (61)$$

This shows we can express the fundamental free energies as inner products in state space.

C. Optimal fields under time reversal

Let a *past* field $E_{-}(\tau)$ be special in that its optimal *future* recovery field $E_{+}(\tau)$ is a (real) multiple λ of the time-reversal $E_{-}(-\tau)$ of $E_{-}(\tau)$; suppose

$$E_{+}(\tau) = \lambda E_{-}(-\tau) =: \lambda \mathrm{T}E_{-}(\tau), \quad (62)$$

where T evokes time reversal. Denote transforms of special past and optimal future fields satisfying Eq. (62) as f_λ and $f_\lambda^\dagger$: in the frequency domain write Eq. (62) as

$$f_\lambda^\dagger(\omega) = \lambda f_\lambda(-\omega) = \lambda \mathrm{T} f_\lambda(\omega). \quad (63)$$

(T acts the same in time and frequency domains with Eq. (22)]. Such special past excitations satisfy Eq. (45):

$$\Lambda(\omega)[f_\lambda(\omega)+\lambda f_\lambda(-\omega)] = Z_\lambda(\omega). \quad (64)$$

Equation (64) is *homogeneous*: except for eigenvalue λ, a solution is any multiple of a pair $(f_\lambda(\omega),Z_\lambda(\omega))$. From Eq. (45) $Z_\lambda(\omega)$ is analytic and order ω^{-1} at infinity in a closed lower-half plane, while the *analytic spectrum* $f_\lambda(\omega)$ is analytic and bounded away from $\omega=0$ in the lower-half plane. Equation (64) with these boundary conditions is a *RH eigenvalue problem*. By Eq. (44), Eq. (64) is also the eigenvalue problem

$$\dot{P}_v[E_{-}+\lambda \mathrm{T}E_{-}](\tau) = 0, \quad \tau \geq 0. \quad (65)$$

For now, any excitation $E_{-}=E_{-}^{\lambda}(\tau)$ satisfying Eq. (65) or Eq. (64) is an eigenmode of the relevant dielectric, λ an eigenvalue (under time-reversal) of the dielectric. Eigen-modes $E_{-}^{\lambda}(\tau)$ are special:

Theorem 3 (Orthogonal Excitations). In dielectric Eq. (35), excitations E_{-}^{λ} and E_{-}^{μ} for eigenvalues $\lambda \neq \mu$ are orthogonal with inner product Eq. (55), their work adding:

$$W[E_{-}^{\lambda}+E_{-}^{\mu}] = W[E_{-}^{\lambda}] + W[E_{-}^{\mu}]. \quad (66)$$

$W[E_{-}^{\lambda}]$ denotes $W[E_{-}^{\lambda}](t)=W[E_{-}^{\lambda}](+\infty)$, $t>0$. [Use Eq. (3) with $E_{-}^{\lambda}(\tau)$ not supported on $\tau>0$.]

Orthogonality is proved as follows: by Eq. (64) f_μ and f_λ satisfy

$$\Lambda(\omega)[f_\lambda(\omega)+\lambda f_\lambda(-\omega)] = Z_\lambda(\omega), \quad (67)$$

$$\Lambda(\omega)[f_\mu(\omega)+\mu f_\mu(-\omega)] = Z_\mu(\omega), \quad (68)$$

and then

$$\Lambda(-\omega)\Lambda(\omega)[f_\mu(\omega)f_\lambda(\omega)+\lambda f_\lambda(-\omega)f_\mu(\omega)] = \Lambda(-\omega)Z_\lambda(\omega)f_\mu(\omega), \quad (69)$$

$$\Lambda(-\omega)\Lambda(\omega)[f_\mu(\omega)f_\lambda(\omega)+\mu f_\mu(-\omega)f_\lambda(\omega)] = \Lambda(-\omega)Z_\mu(\omega)f_\lambda(\omega), \quad (70)$$

the integrated difference of which over real ω giving

$$(\lambda-\mu)\langle f_\lambda,f_\mu\rangle = \lambda\langle f_\lambda,f_\mu\rangle - \mu\langle f_\mu,f_\lambda\rangle = 0. \quad (71)$$

The right side of Eq. (71) is Cauchy's theorem with $\Lambda(-\omega)Z_\lambda(\omega)f_\mu(\omega)$ and $\Lambda(-\omega)Z_\mu(\omega)f_\lambda(\omega)$ analytic and order ω^{-2} in a closed lower-half plane. Equation (71) is orthogonality $\langle f_\lambda,f_\mu\rangle=0$ when $\lambda\neq\mu$. Equation (66) arises by orthogonality via Eq. (58):

$$W[E_-^{\lambda}+E_-^{\mu}]=\frac{\gamma_\infty}{\omega_p^2}\|f_\lambda+f_\mu\|^2=\frac{\gamma_\infty}{\omega_p^2}\|f_\lambda\|^2+\frac{\gamma_\infty}{\omega_p^2}\|f_\mu\|^2. \quad (72)$$

To build a *basis* of excitations able to create a state in only one way we demand $f_\lambda \neq 0$ i.e.,

$$\|f_\lambda\|^2=\langle f_\lambda,f_\lambda\rangle \neq 0 \quad (73)$$

since $\langle,\rangle$ is an inner product on $\hat{\mathcal{E}}$. By Eq. (58), Eq. (73) precludes eigenmode from doing no work. Also, without restriction Eq. (73) any real number λ is a time-reversal eigenvalue.

Equation (53) shows the field $E_-\equiv 0$ produces the zero state. Equation (73) disqualifies such trivial fields, but is not enough to ensure the uniqueness demanded of a basis: Beyond Eq. (73), we demand

$$\langle f_\lambda^\dagger,f_\lambda\rangle=\lambda\langle \mathrm{T}f_\lambda,f_\lambda\rangle \neq 0. \quad (74)$$

With Eqs. (61) and (59), Eq. (74) demands excitation E_-^{λ} generate positive recoverable energy and that optimal recovery field $E_+^{\lambda}=\lambda \mathrm{T}E_-^{\lambda}$ not vanish, which occurs if and only if eigenvalue λ is not zero when Eq. (73) is enforced [since T is unitary for Eq. (55)]. For uniqueness, the zero eigenvalue is as problematic as the zero field: violate Eq. (74) while retaining Eq. (73) by choice $\lambda=0$: Equations (64) and (65) become

$$-i\omega\chi_{\mathrm{v}}(\omega)f_0(\omega)=\Lambda(\omega)f_0(\omega)=Z_0(\omega),$$

$$\dot{P}_{\mathrm{v}}[E_-^0](\tau)=0,\quad \tau\geq 0. \quad (75)$$

The equivalence of Eqs. (52)–(54) shows Eq. (75) is

$$P_{\mathrm{v}}[E_-^0](\tau)=P[E_-^0](\tau)=0,\quad \tau\geq 0, \quad (76)$$

which says E_-^0 has prepared the zero state. Thus, allowing eigenvalue $\lambda=0$ admits each excitation E_-^0 preparing the zero state Eq. (76). The space of such fields is infinite dimensional [42]: For example, in Eq. (28) the susceptibility is

$$\chi(\omega)=\chi_{\mathrm{v}}(\omega)=-\frac{\omega_p^2}{(\omega-\omega_1)(\omega+\omega_1^*)}, \quad (77)$$

where $\omega_1=\omega_0-i\gamma$, $\omega_0,\gamma>0$. *For any integer* $n\geq 2$,

$$f_0(\omega) := \omega_p^{2n-3}\frac{(\omega-\omega_1)(\omega+\omega_1^*)}{[(\omega+\omega_1)(\omega-\omega_1^*)]^n} \quad (78)$$

is the analytic spectrum of an absolutely integrable and continuous real past field time series E_-^0 "with eigenvalue $\lambda=0$", for then

$$\widehat{P_{\mathrm{v}}(E_-^0)}(\omega)=\widehat{P(E_-^0)}(\omega)=\chi_{\mathrm{v}}(\omega)f_0(\omega)=\chi(\omega)f_0(\omega)=-\frac{\omega_p^{2n-1}}{[(\omega+\omega_1)(\omega-\omega_1^*)]^n} \quad (79)$$

is real-symmetric, analytic in a closed lower-half plane and order ω^{-2n} near infinity, satisfying the demands of $Z_0(\omega)$ made in Eq. (75). Equation (79) is an infinite list of independent admissible spectra, an infinite number of independent ways to create the zero state [43]. Excluding the zero eigenvalue forbids excitations E_-^{λ} repreparing the zero state from whence it drove the system at $\tau=-\infty$, and it is in equilibrium by span of eigenmodes uniquely: only by $E_-\equiv 0$ with Eqs. (73) and (74).

These ideas and Eq. (59) motivate Eq. (74) and also give *Theorem 4 (Zero State Recoverable Energy).* Dielectric Eq. (35)'s recoverable energy $U_+[E](t)$ vanishes at time t if and only if it is in equilibrium then.

To see this, note: From Eqs. (52)–(54)'s equivalence, the zero state is the second equation in Eq. (75), which implies the first there by uniqueness of solutions of the inhomogeneous RH problem, i.e., $E_+\equiv 0$, giving $U_+[E](0)=0$ by Eq. (59). Conversely, if $U_+[E](0)=0$, then $\langle *,*\rangle$ being an inner product and Eq. (59) gives $E_+\equiv 0$, which then gives Eq. (75) describing the zero state by Eq. (76).

To finish the example, note Eq. (79) and $n=2$ gives

$$P_{\mathrm{v}}[E_-^0](\tau)=P[E_-^0](\tau)=-\sqrt{\frac{\pi}{2}}\frac{\omega_p^3e^{\gamma\tau}}{\omega_0^3}(\omega_0\tau\cos\omega_0\tau-\sin\omega_0\tau)\theta_-^0(\tau) \quad (80)$$

which is "smooth" [44]. Equation (80) is a medium in equilibrium at $\tau=-\infty$, not so at finite times before $\tau=0$, and then back in equilibrium at and after $\tau=0$. By Eq. (78) the field E_-^0 giving rise to excitation Eq. (80) is

$$E_-^0(\tau)=\frac{\sqrt{2\pi}\omega_pe^{\gamma\tau}}{\omega_0^3}[-2\gamma^2\omega_0\tau\cos\omega_0\tau+(\omega_0^2+2\gamma^2+2\gamma\omega_0^2\tau)\sin\omega_0\tau]\theta_-^0(\tau). \quad (81)$$

D. Time reversal spectrum's properties

1. General results

Since Eq. (64) or Eq. (65) is linear, we can satisfy Eq. (73) via choice

$$\langle f_\lambda,f_\lambda\rangle=\|f_\lambda\|^2=1. \quad (82)$$

With $\lambda=0$ disallowed, Eq. (74) is the restriction

$$\langle \mathrm{T}f_\lambda,f_\lambda\rangle \neq 0. \quad (83)$$

Then integration of Eq. (69) and Cauchy's theorem give

$$\langle \mathrm{T}f_\lambda,f_\lambda\rangle+\lambda\langle f_\lambda,f_\lambda\rangle=\langle \mathrm{T}f_\lambda,f_\lambda\rangle+\lambda=0 \quad (84)$$

when $\mu=\lambda$ there, which with Eq. (83) is

$$\lambda=-\langle \mathrm{T}f_\lambda,f_\lambda\rangle \neq 0. \quad (85)$$

Since $\lambda\in\mathbb{R}$ by Eq. (56) [obviously consistent with Eq. (62)], Eq. (85) and Cauchy-Schwarz give

$$0<\lambda^2=(-\langle \mathrm{T}f_\lambda,f_\lambda\rangle)^2\leq\langle \mathrm{T}f_\lambda,\mathrm{T}f_\lambda\rangle\langle f_\lambda,f_\lambda\rangle=\langle f_\lambda,f_\lambda\rangle\langle f_\lambda,f_\lambda\rangle=1\cdot 1=1. \quad (86)$$

Recall T is unitary for Eq. (55). This produces the bounds in *Theorem 5 (The Time-reversal Spectrum)*. Dielectric Eq. (35)'s time-reversal eigenvalues lie in $[-1,1]-\{0\}$, ± 1 always occurring, and have equal algebraic and geometric multiplicity. The dimension of the eigenspaces is $2(N+1)$, that of state space Eq. (49).

We show equality of multiplicities and the dimension of the eigenspaces by showing they are those of a $2(N+1)$ real-symmetric matrix. We start by solving Eq. (64) for $f_\lambda^\dagger(\omega)=\lambda f_\lambda(-\omega)$ in terms of "data" $f_\lambda(\omega)$: with $\Lambda(\omega)=-i\omega\chi_v(\omega)$ of Eq. (38), we get the partial fractions expansion

$$\begin{aligned}\Lambda(\omega) &= -i\omega\chi_v(\omega)\\ &= \omega_p^2 i\omega\frac{\prod_{k=1}^{N}(\omega-Z_k)(\omega+Z_k^*)}{\prod_{k=1}^{N+1}(\omega-z_k)(\omega+z_k^*)}\\ &= \sum_{j=1}^{N+1}\left(\frac{D_j}{\omega-z_j}-\frac{D_j^*}{\omega+z_j^*}\right),\end{aligned} \tag{87}$$

with $D_j=-iz_j\tilde{a}_j$ and $D_j^*=+iz_j^*\tilde{a}_j^*$, the $\tilde{a}$'s from Eq. (50). So RH problem Eq. (64) [with $\lambda f_\lambda(-\omega)\mapsto f_\lambda^\dagger(\omega)$] becomes

$$\begin{aligned}&\Lambda(\omega)f_\lambda^\dagger(\omega)+\sum_{k=1}^{N+1}\left(\frac{D_j}{\omega-z_j}f_\lambda(z_j)-\frac{D_j^*}{\omega+z_j^*}f_\lambda(-z_j^*)\right)\\ &\quad= Z_\lambda(\omega)-\sum_{k=1}^{N+1}\left(D_j\frac{f_\lambda(\omega)-f_\lambda(z_j)}{\omega-z_j}-D_j^*\frac{f_\lambda(\omega)-f_\lambda(-z_j^*)}{\omega+z_j^*}\right).\end{aligned} \tag{88}$$

If Eq. (64) has a solution pair $(f_\lambda^\dagger(\omega),Z_\lambda(\omega))$ with the required properties, the left and right of Eq. (88) are analytic and order ω^{-1} at infinity in a closed upper, respectively, lower-half plane. Either side of Eq. (88) is then entire and vanishes at infinity. So by Liouville's theorem both sides are zero. The right being zero prescribes the required properties of $Z_\lambda(\omega)$. The left being zero gives

$$\begin{aligned}\lambda f_\lambda(-\omega) &= f_\lambda^\dagger(\omega)\\ &= -\frac{1}{\Lambda(\omega)}\sum_{j=1}^{N+1}\left[\frac{D_j}{\omega-z_j}f_\lambda(z_j)-\frac{D_j^*}{\omega+z_j^*}f_\lambda(-z_j^*)\right]\\ &= -\frac{\prod_{k=1}^{N+1}(\omega-z_k)(\omega+z_k^*)}{\omega_p^2 i(\omega-0)\prod_{k=1}^{N}(\omega-Z_k)(\omega+Z_k^*)}\\ &\quad\times\sum_{j=1}^{N+1}\left[\frac{D_j}{\omega-z_j}f_\lambda(z_j)-\frac{D_j^*}{\omega+z_j^*}f_\lambda(-z_j^*)\right],\end{aligned} \tag{89}$$

which one can check prescribes $f_\lambda^\dagger(\omega)=\lambda f_\lambda(-\omega)$ with the required properties if $f_\lambda(\omega)$ has them. So with Eq. (89) and the mentioned formula for $Z_\lambda(\omega)$ we have the unique solution pair $(f_\lambda^\dagger(\omega),Z_\lambda(\omega))$ of (64) (with $\lambda f_\lambda(-\omega)$ replaced by $f_\lambda^\dagger(\omega)$).

Equation (89) dictates $f_\lambda^\dagger(\omega)$ depends on $f_\lambda(\omega)$ only via the real symmetric, $\mathbb{C}^{2(N+1)}$ state vector

$$(f_\lambda(z_1),\ldots,f_\lambda(z_{N+1}),f_\lambda(-z_1^*),\ldots,f_\lambda(-z_{N+1}^*)). \tag{90}$$

The dependence is fully $2(N+1)$ dimensional since the coefficient $(\omega-z_j)^{-1}$'s and $(\omega+z_j^*)^{-1}$'s of the state space coordinates Eq. (90) in Eq. (89) are independent in ω.

Note Eq. (89) already allows *computation* of the time-reversal spectrum: letting $\omega=-z_k$ and $\omega=+z_k^*$, $k=1,\ldots,N+1$, gives $2(N+1)$ homogeneous equations

$$\begin{aligned}\lambda f_\lambda(z_k) &= \frac{1}{\Lambda(-z_k)}\sum_{j=1}^{N+1}\left[\frac{D_j}{z_j+z_k}f_\lambda(z_j)+\frac{D_j^*}{z_j^*-z_k}f_\lambda(-z_j^*)\right],\\ \lambda f_\lambda(-z_k^*) &= \frac{1}{\Lambda(z_k^*)}\sum_{j=1}^{N+1}\left[\frac{D_j}{z_j-z_k^*}f_\lambda(z_j)+\frac{D_j^*}{z_j^*+z_k^*}f_\lambda(-z_j^*)\right],\end{aligned} \tag{91}$$

which are complex conjugate equations with real symmetry $f_\lambda(-z_k^*)=f_\lambda^*(z_k)$ and $\lambda\in\mathbb{R}$. This suggests the eigenvalues λ *may* be those of a real-symmetric matrix. To show it must be, we do the following.

Partial fractions expansion of the r.h.s. of Eq. (89) shows the dependence of $f_\lambda^\dagger(\omega)$ on ω is via the real span of the following set of $2(N+1)$ independent, real-symmetric transforms, which are analytic and bounded away from $\omega=0$ in a closed upper-half plane:

$$\begin{aligned}\hat{\phi}_\infty(\omega) &= 1\\ \hat{\phi}_0(\omega) &= \frac{1}{-i\omega}\\ \hat{\phi}_{k,e}(\omega) &= \frac{1}{-i(\omega-Z_k)}+\frac{1}{-i(\omega+Z_k^*)},\\ \hat{\phi}_{k,o}(\omega) &= \frac{i}{-i(\omega-Z_k)}-\frac{i}{-i(\omega+Z_k^*)},\end{aligned} \tag{92}$$

Here $k=1,\ldots,N$. In the original time-reversal eigenvalue problem Eq. (64), $f_\lambda(\omega)=f_\lambda^\dagger(-\omega)/\lambda$ is the relevant component of the solution. For it the relevant basis is

$$\hat{\phi}_{\mathrm{label}}(-\omega),\quad \mathrm{label}=\infty;\ 0;\ k,e;\ k,o. \tag{93}$$

All of these but the first two lie in the transformed space $\hat{\mathcal{E}}$ (of $\mathcal{E}$) over which Eq. (55) is surely an inner product. We soon show these first two are special with regard to the energetics of simple dielectrics, but note they are not with regard to the inner product: the vanishing of Eq. (55)'s kernel $\rho(\omega)$ to second order at both $\omega=\infty,0$ allows, respectively, the first two elements of Eq. (92) or Eq. (93) also to be in the inner-product space. So, performing Gram-Schmidt on the independent set Eq. (93) with real inner product Eq. (55) gives *a* real-symmetric orthonormal basis $\{v_j\}_{j=1}^{2(N+1)}$ for the solutions $f_\lambda(\omega)$ of eigenvalue problem Eq. (91) or Eq. (64): for any solution $f_\lambda(\omega)$ of Eq. (64) there are real constants $\mathbf{C}=\{C_j\}_{j=1}^{2(N+1)}$ such that

$$f_\lambda(\omega) = \sum_{k=1}^{2(N+1)} C_k v_k(\omega), \tag{94}$$

and the eigenvalue problem Eq. (64) is

$$\sum_{k=1}^{2(N+1)} \Lambda(\omega)[C_k v_k(\omega) + \lambda C_k v_k(-\omega)] = Z_\lambda(\omega). \tag{95}$$

Multiplying both sides of (95) by $\Lambda(-\omega)v_l(\omega)$, $l = 1,2,\ldots,2(N+1)$, and integrating gives

$$\sum_{k=1}^{2(N+1)} \langle \mathrm{T}v_l, v_k\rangle C_k + \lambda C_l = 0, \quad l = 1,\ldots,2N+2 \tag{96}$$

by orthogonality and Cauchy's theorem, i.e.,

$$T\mathbf{C} = \lambda I \mathbf{C}, \tag{97}$$

where T is a real-symmetric matrix and our result follows:

$$T_{l,k} = -\langle \mathrm{T}v_l, v_k\rangle = -\langle v_k, \mathrm{T}v_l\rangle = -\langle \mathrm{T}v_k, v_l\rangle = T_{k,l}. \tag{98}$$

T is self-adjoint on inner product Eq. (55). Except for eigenvalues ± 1, all claims of Theorem 5 now hold. In proving this final claim, we show that it carries a special physical significance.

2. Kinetic and potential energies

The presence of $\hat{\phi}_0(\omega) = (-i\omega)^{-1} \notin \hat{\mathcal{E}}$ in Eq. (92) or Eq. (93) makes $a_\lambda^\dagger(\omega)$ solving Eq. (64) not analytic in a closed upper-half plane: $a_\lambda^\dagger(\omega)$ is *not* the transform of an absolutely integrable field $E \in \mathcal{E}$. The factor $-i\omega$ in $\Lambda(\omega) = -i\omega\chi_v(\omega)$, i.e., the time derivative in Eq. (44), causes this complication, forcing consideration of nonuniform sequences of absolutely integrable fields. Incompatible requirements force nonuniformity: a) By time $\tau = +\infty$ the effective polarization field $P_v[E](\tau)$ must relax to the zero state from its generally nonzero value at $\tau = 0$, but, b) as prescribed by Eq. (43), do so without incurring a time derivative. These are compatible in a limiting sense: The nonuniform sequence

$$\phi_\epsilon(\tau) = \sqrt{2\pi} e^{-\epsilon\tau}\theta_+(\tau) \underset{\epsilon\downarrow 0}{\rightarrow} \sqrt{2\pi}\theta_+(\tau) \tag{99}$$

is in $\mathcal{E}$, but with limit $\propto \theta_+(\tau)$ *not* in $\mathcal{E}$, has transforms

$$\hat{\phi}_\epsilon(\omega) = \frac{1}{-i(\omega + i\epsilon)} \underset{\epsilon\downarrow 0}{\rightarrow} \hat{\phi}_0(\omega) \tag{100}$$

with limit $\hat{\phi}_0(\omega) \notin \hat{\mathcal{E}}$, yet giving a sequence of time derivatives uniformly square integrable; in fact

$$\lim_{\epsilon\downarrow 0} \int_0^{+\infty} (\dot{\phi}_\epsilon(\tau))^2 d\tau = \lim_{\epsilon\downarrow 0} \pi\frac{\epsilon^2}{\epsilon} = 0 \tag{101}$$

as required by Eq. (43). Likewise a sequence of field time series $E \in \mathcal{E}$ with limit $\delta \notin \mathcal{E}$ exists, giving limit transform $\hat{\phi}_\infty(\omega) \propto 1 \notin \hat{\mathcal{E}}$, yet giving the required Eq. (43).

Define the potential and kinetic energy of a dielectric Eq. (35) as the type of energy recovered by (sequences of) excitations $E_-^{\lambda_0} = E_-^{+1}$ and $E_-^{\lambda_\infty} = E_-^{-1}$ with (limiting) transforms $f_\lambda(\omega)$ multiples of $\hat{\phi}_0(-\omega)$ and $\hat{\phi}_\infty(-\omega)$: in view of Eq. (64) define

$$\begin{aligned} Z_{+1}(\omega) &:= \Lambda(\omega)[\hat{\phi}_0(-\omega) + (+1)\hat{\phi}_0(\omega)] \\ &= \Lambda(\omega)\hat{\phi}_0(-\omega)[1 - (+1)1] = 0, \end{aligned} \tag{102}$$

$$\begin{aligned} Z_{-1}(\omega) &:= \Lambda(\omega)[\hat{\phi}_\infty(-\omega) + (-1)\hat{\phi}_\infty(\omega)] \\ &= \Lambda(\omega)\hat{\phi}_\infty(-\omega)[1 + (-1)1] = 0. \end{aligned} \tag{103}$$

$Z_\lambda(\omega) \equiv 0$ satisfies all demands of Eq. (64) and Theorem 5's claim that ± 1 are always eigenvalues now holds.

Equation (89) yields this same result: Choose $f_\lambda(\omega) = \hat{\phi}_\infty(\omega) = 1$ in Eq. (89) and get

$$\lambda = -\frac{1}{\Lambda(\omega)} \sum_{j=1}^{N+1} \left(\frac{D_j}{\omega - z_j} - \frac{D_j^*}{\omega + z_j^*} \right) = -\frac{1}{\Lambda(\omega)}\Lambda(\omega) = -1, \tag{104}$$

using Eq. (87), or choose $f_\lambda(\omega) = \hat{\phi}_0(\omega) = (-i\omega)^{-1}$ to get

$$\begin{aligned} \lambda\frac{1}{i\omega} &= -\frac{1}{\Lambda(\omega)} \sum_{j=1}^{N+1} \left(\frac{D_j}{\omega - z_j}\frac{1}{-iz_j} - \frac{D_j^*}{\omega + z_j^*}\frac{1}{iz_j^*} \right) \\ &= -\frac{1}{\Lambda(\omega)}\frac{\Lambda(\omega)}{-i\omega} \\ &= \frac{1}{i\omega}, \end{aligned} \tag{105}$$

i.e., $\lambda = +1$. We get Eq. (105) by noting that Eq. (87) gives partial fraction

$$\frac{\Lambda(\omega)}{-i\omega} = \frac{\Lambda(0)}{-i\omega} + \sum_{j=1}^{N+1} \left(\frac{D_j}{\omega - z_j}\frac{1}{-iz_j} - \frac{D_j^*}{\omega + z_j^*}\frac{1}{iz_j^*} \right) \tag{106}$$

where $\Lambda(0) = 0$.

V. TIME REVERSAL AND THE EXTREMAL FREE ENERGIES

After noting best computation, examples give relations between time-reversal spectra and extreme free energies.

A. Computing the spectrum

For any $\hat{\Phi}_k$'s listing the real-symmetric, nonorthogonal basis Eq. (93), the equation

$$\sum_{k=1}^{2(N+1)} \Lambda(\omega)[C_k\hat{\Phi}_k(\omega) + \lambda C_k\hat{\Phi}_k(-\omega)] = Z_\lambda(\omega) \tag{107}$$

arises in eigenvalue problem Eq. (64) by writing

$$f_\lambda(\omega) = \sum_{k=1}^{2(N+1)} C_k\hat{\Phi}_k(\omega). \tag{108}$$

Multiplying by $\Lambda(-\omega)\hat{\Phi}_l(\omega)$, $l = 1,2,\ldots,2(N+1)$, gives

$$\sum_{k=1}^{2(N+1)} \langle \mathrm{T}\hat{\Phi}_l, \hat{\Phi}_k \rangle C_k + \lambda \sum_{k=1}^{2(N+1)} \langle \hat{\Phi}_l, \hat{\Phi}_k \rangle C_k = 0 \qquad (109)$$

by integration and analyticity, i.e., eigenvalue problem

$$T\mathbf{C} = \lambda I \mathbf{C} \qquad (110)$$

where $T=-B^{-1}A$ and [45]

$$A_{lj} = \langle \mathrm{T}\hat{\Phi}_l, \hat{\Phi}_j \rangle = \langle \mathrm{T}\hat{\Phi}_j, \hat{\Phi}_l \rangle = A_{jl}, \qquad (111)$$

$$B_{jk} = \langle \hat{\Phi}_j, \hat{\Phi}_k \rangle = \langle \hat{\Phi}_k, \hat{\Phi}_j \rangle = B_{kj}. \qquad (112)$$

Symmetric A and B generally do not commute and $T=-B^{-1}A$ is not symmetric then: Eqs. (110)–(112) do not show the time-reversal eigenspaces is $2(N+1)$-dimensional, but give efficient computations.

B. Time-reversal representations of the extremal free energies: an example

Suppose the density ρ of a dielectric satisfies

$$\begin{aligned}\frac{\omega_p^2\rho(\omega)}{\gamma_\infty} &= \frac{\omega_p^4\omega^2(\omega^2+\Gamma^2)}{(\omega^2+\gamma^2)[(\omega^2-\omega_1^2+\gamma_1^2)^2+4\gamma_1^2\omega_1^2]} \\ &= \Lambda(-\omega)\Lambda(\omega) \\ &= \omega_p^2\omega\frac{\chi(\omega)-\chi(-\omega)}{2i}.\end{aligned} \qquad (113)$$

The medium is passive Eq. (27) with Lorentz resonance at $\omega=\pm\omega_1-i\gamma_1$ and Drude-like resonance at $\omega=-i\gamma$ for positive parameters: the medium is dielectric, not conductive. Equation (113) falls slightly outside model Eq. (36) where no poles or zeroes of $\rho(\omega)/\omega^2$ lie on the imaginary axis, but reduces to Lorentz oscillator Eq. (28) for $r:=\gamma/\Gamma=1$ and sports all other dielectric constraints. It is the simplest medium with energies other than kinetic and potential. Real symmetry, analyticity and asymptotics make RH problems Eq. (113) have unique solutions given by

$$\frac{\Lambda(\omega)}{s} = \chi_v(\omega) = \frac{(s+\Gamma)\omega_p^2}{(s+\gamma)[(s+\gamma_1)^2+\omega_1^2]} = \frac{s+\Gamma}{s+\Gamma'}\chi(\omega), \qquad (114)$$

where $s=-i\omega$ (is the Laplace frequency), and

$$\frac{\Gamma'}{\Gamma} = \frac{\gamma+2\gamma_1-\gamma_\infty}{\Gamma} = \frac{\Gamma(\gamma+2\gamma_1)+r(\gamma_1^2+\omega_1^2)}{\Gamma(\Gamma+2r\gamma_1)+(\gamma_1^2+\omega_1^2)} \overset{r\to 1}{\sim} 1. \qquad (115)$$

$\Lambda(-\omega)$'s zeroes dictate RH problem Eq. (64) has solutions $f_\lambda(\omega)$ in the real span of real-symmetric

$$\hat{\Phi}_1(\omega) = 1, \quad \hat{\Phi}_2(\omega) = \frac{1}{i\omega}, \quad \hat{\Phi}_3(\omega) = \frac{1}{i\omega+\Gamma}. \qquad (116)$$

The matrices of inner products Eqs. (111) and (112) are

$$A = \alpha\begin{pmatrix} \Gamma\Omega_\infty & 0 & \Gamma \\ 0 & -\delta & -1 \\ \Gamma & -1 & \dfrac{\beta+1}{\Omega_\infty/\Gamma} + \dfrac{\beta-1}{\delta} - \beta \end{pmatrix}, \qquad (117)$$

$$B = \alpha\begin{pmatrix} \Gamma\Omega_\infty & 0 & \Gamma \\ 0 & \delta & 1 \\ \Gamma & 1 & 1 \end{pmatrix}, \qquad (118)$$

where $\alpha > 0$ does not enter $T=-B^{-1}A$ and

$$\Gamma\Omega_\infty = \Gamma^2+\gamma_1^2+2\gamma\gamma_1+\omega_1^2, \quad \delta = \frac{\Gamma^2(\gamma+2\gamma_1)}{\gamma(\gamma_1^2+\omega_1^2)}+1,$$

$$\beta = \frac{(\Gamma-\gamma)[(\Gamma-\gamma_1)^2+\omega_1^2]}{(\Gamma+\gamma)[(\Gamma+\gamma_1)^2+\omega_1^2]}. \qquad (119)$$

Consequently

$$T = -B^{-1}A = \begin{pmatrix} -1 & 0 & -\dfrac{1+\beta}{\Omega_\infty} \\ 0 & 1 & \dfrac{1-\beta}{\delta} \\ 0 & 0 & \beta \end{pmatrix}, \qquad (120)$$

and $\beta \in (-1,1)$ is the only non-trivial time-reversal eigenvalue. Bases for the three eigenspaces of T are the columns of

$$\begin{pmatrix} 1 & 0 & -\dfrac{1}{\Omega_\infty} \\ 0 & 1 & -\dfrac{1}{\delta} \\ 0 & 0 & 1 \end{pmatrix}, \qquad (121)$$

so the associated eigenspectra are

$$f_{-1}(\omega) \propto 1\hat{\Phi}_1(\omega)+0\hat{\Phi}_2(\omega)+0\hat{\Phi}_3(\omega) = 1, \qquad (122)$$

$$f_{+1}(\omega) \propto 0\hat{\Phi}_1(\omega)+1\hat{\Phi}_2(\omega)+0\hat{\Phi}_3(\omega) = \frac{1}{i\omega}, \qquad (123)$$

$$\begin{aligned} f_\beta(\omega) &\propto -\frac{1}{\Omega_\infty}\hat{\Phi}_1(\omega) - \frac{1}{\delta}\hat{\Phi}_2(\omega) + 1\hat{\Phi}_3(\omega) \\ &= -\frac{\Gamma}{\Gamma^2+\gamma_1^2+2\gamma\gamma_1+\omega_1^2} - \left[\frac{\Gamma^2(\gamma+2\gamma_1)}{\gamma(\gamma_1^2+\omega_1^2)}+1\right]^{-1} \\ &\quad \times \frac{1}{i\omega} + \frac{1}{i\omega+\Gamma}. \end{aligned} \qquad (124)$$

We show the first of these is momentumlike, the second coordinatelike, motivating their connection to odd $(\lambda=-1)$ and even $(\lambda=+1)$ parity under time reversal. $f_\beta(\omega)$ is *not* associated with simple sign change under time reversal, but also with dilation by $\beta \in (-1,1)$. Non-trivial parity stems from macroscopic phenomenology.

According to Eqs. (58) and (66), superpositions

$$\hat{E}_- = C_{-1} f_{-1} + C_{+1} f_{+1} + C_\beta f_\beta, \tag{125}$$

give the following work done on the medium:

$$\frac{\omega_p^2 W[E_-]}{\gamma_\infty} = C_{-1}^2 + C_{+1}^2 + C_\beta^2$$
$$= \langle f_{-1}, \hat{E}_- \rangle^2 + \langle f_{+1}, \hat{E}_- \rangle^2 + \langle f_\beta, \hat{E}_- \rangle^2. \tag{126}$$

Equation (126) comes from Eq. (125) by projection onto orthonormal basis $\{f_{-1}, f_{+1}, f_\beta\}$. Inner-product Eq. (55) and normalized $f_{-1}(\omega)$ independent of ω give

$$\frac{\gamma_\infty}{\omega_p^2}\langle f_{-1}, \hat{E}_- \rangle^2 = \frac{\left[\int_{-\infty}^{+\infty} \rho(\omega) f_{-1}(-\omega) \hat{E}_-(\omega) d\omega\right]^2}{\frac{\gamma_\infty}{\omega_p^2}\cdot 1}$$
$$= \frac{\left[\int_{-\infty}^{+\infty} \rho(\omega) f_{-1}(-\omega) \hat{E}_-(\omega) d\omega\right]^2}{\int_{-\infty}^{+\infty} \rho(\omega) f_{-1}(-\omega) f_{-1}(\omega) d\omega}$$
$$= \frac{\left[\int_{-\infty}^{+\infty} \rho(\omega) \hat{E}_-(\omega) d\omega\right]^2}{\int_{-\infty}^{+\infty} \rho(\omega) d\omega}. \tag{127}$$

Similarly, Eq. (55) and normalized $f_{+1}(\omega) \propto 1/i\omega$ give

$$\frac{\gamma_\infty}{\omega_p^2}\langle f_{+1}, \hat{E}_- \rangle^2 = \frac{\left[\int_{-\infty}^{+\infty} \frac{\rho(\omega)}{-i\omega} \hat{E}_-(\omega) d\omega\right]^2}{\int_{-\infty}^{+\infty} \frac{\rho(\omega)}{\omega^2} d\omega}. \tag{128}$$

[38] gives certain *sum rules* and representations valid for all simple dielectrics. Using these in Eqs. (127) and (128), and inserting the results into Eq. (126), the work performed by field E_- with spectrum Eq. (125) on a dielectric with oscillator density Eq. (113) reduces to

$$W(E_-)(0^+) = \frac{[\dot{P}(E_-)(0^+)]^2}{2\omega_p^2} + \frac{1}{2} k [P(E_-)(0)]^2 + \frac{\gamma_\infty}{\omega_p^2}\langle f_\beta, \hat{E}_- \rangle^2, \tag{129}$$

$k = \omega_0^2/\omega_p^2 = 1/\chi(0)$ the dielectric's "spring constant" Eq. (30). In Eq. (129), the first two terms are: a) the kinetic energy of a mass ω_p^2 with momentum $\dot{P}[E_-](0^+)$, and b) the potential energy of a mass on a Hooke's law spring (of spring constant k) with *displacement* $P[E_-](0) = P[E_-](0^+)$ from equilibrium: the odd and even parities of momentum and coordinate under time-reversal are consistent with the time-reversal eigenvalues of kinetic and potential energy being $\lambda = -1$ and $\lambda = +1$. The interpretation of the third term in Eq. (129) for eigenvalue $\beta \in (-1, +1)$ Eq. (119) is more complex:

Theorem 6 (Mode Susceptibilities). If $f_\lambda(\omega)$ satisfies RH problem Eq. (64), then it also solves RH problems

$$2\sqrt{\pi}\frac{\omega_p}{\sqrt{\gamma_\infty}} \rho(\omega) f_\lambda(-\omega) = \frac{1}{\lambda}\chi_\lambda(\omega) - \chi_\lambda(-\omega), \tag{130}$$

and

$$2\sqrt{\pi}\frac{\omega_p}{\sqrt{\gamma_\infty}} \rho(\omega)[f_\lambda(\omega) + \lambda f_\lambda(-\omega)] = \left(\frac{1}{\lambda} - \lambda\right)\chi_\lambda(-\omega), \tag{131}$$

where the λ-susceptibility $\chi_\lambda(\omega)$, like $\chi(\omega)$, is real symmetric and analytic in a closed upper-half plane and at least order ω^{-1} near infinity. Consequently,

$$\frac{\gamma_\infty}{\omega_p^2}\langle f_\lambda, \hat{E}_- \rangle^2 = \frac{1}{2\lambda^2}[P_\lambda(E_-)(0^+)]^2 \tag{132}$$

with (real) λ-polarization

$$P_\lambda[E_-](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \chi_\lambda(\omega) \hat{E}_-(\omega) e^{-i\omega t} d\omega. \tag{133}$$

Equation (131) shows the *residual* $Z_\lambda(\omega)$ of RH problem Eq. (64) is

$$Z_\lambda(\omega) = \frac{\omega_p}{2\sqrt{\pi\gamma_\infty}\Lambda(-\omega)}\left(\frac{1}{\lambda} - \lambda\right)\chi_\lambda(-\omega), \tag{134}$$

which vanishes if and only if $\lambda^2 = 1$. We get Eq. (134) by comparing Eq. (131) with Eq. (64), and using $\gamma_\infty \Lambda(-\omega)\Lambda(\omega) = \omega_p^2 \rho(\omega)$. Equation (131) follows from Eq. (130) since

$$\frac{1}{\lambda}\chi_\lambda(-\omega) - \chi_\lambda(\omega) + \lambda\left[\frac{1}{\lambda}\chi_\lambda(\omega) - \chi_\lambda(-\omega)\right]$$
$$= \left(\frac{1}{\lambda} - \lambda\right)\chi_\lambda(-\omega). \tag{135}$$

To get Eq. (130), note solving the standard RH problem

$$\Phi_+(\omega) + \Phi_-(\omega) = \rho(\omega) f_\lambda(-\omega) \tag{136}$$

with $\mathcal{O}_\infty(\omega^{-1})$ [46] boundary conditions for unique real symmetric $(\Phi_+(\omega), \Phi_-(\omega))$ (analyticity noted by subscripts) given "data" $\rho(\omega) a_\lambda(-\omega)$ lets (64) be written

$$\Phi_-(-\omega) + \lambda\Phi_+(\omega) = Z_\lambda(\omega) - [\Phi_+(-\omega) + \lambda\Phi_-(\omega)]. \tag{137}$$

Each side of Eq. (137) is analytic in one or the other and then both closed half planes, vanishes at infinity, and, so, is zero. The left of Eq. (137) then allows finding Φ_- in terms of Φ_+. Inserting this into Eq. (136) gives Eq. (130) after (real) rescaling and relabeling. $\chi_\lambda \propto \Phi_+$ is then real symmetric and order ω^{-1}.

Finally, Eqs. (132) and (133) arise from Eq. (130) and $\rho(\omega)$ and $f_\lambda(-\omega)$'s asymptotics [and inner product Eq. (55)]:

$$\frac{\sqrt{2\pi}}{\lambda}P_\lambda(E_-)(0^+) = \frac{1}{\lambda}\lim_{t\downarrow 0}\int_{-\infty}^{+\infty}\chi_\lambda(\omega)\hat{E}_-(\omega)e^{-i\omega t}d\omega + 0$$
$$= \lim_{t\downarrow 0}\int_{-\infty}^{+\infty}\left[\frac{1}{\lambda}\chi_\lambda(\omega) - \chi_\lambda(-\omega)\right]\times\hat{E}_-(\omega)e^{-i\omega t}d\omega$$
$$= \lim_{t\downarrow 0}\int_{-\infty}^{+\infty}2\frac{\sqrt{\pi}\omega_p}{\sqrt{\gamma_\infty}}\rho(\omega)f_\lambda(-\omega)\hat{E}_-(\omega)e^{-i\omega t}d\omega$$
$$= 2\frac{\sqrt{\pi}\omega_p}{\sqrt{\gamma_\infty}}\int_{-\infty}^{+\infty}\rho(\omega)f_\lambda(-\omega)\hat{E}_-(\omega)d\omega$$
$$= 2\sqrt{\pi}\frac{\sqrt{\gamma_\infty}}{\omega_p}\langle f_\lambda,\hat{E}_-\rangle. \quad (138)$$

The second equality holds by Cauchy's theorem since the integrands on either side of it are integrable for $t>0$ even when (for the left side) the order is only ω^{-1} near $\omega=\pm\infty$, and since the integrand injected in passing to the right is analytic and vanishing exponentially rapidly in a closed lower-half plane for $t>0$. When the first integrand in Eq. (138) is order ω^{-2} near $\omega=\pm\infty$, no limit is needed and $P_\lambda[E_-](0^+)=P_\lambda[E_-](0^-)=P_\lambda[E_-](0)$.

The first two terms in Eq. (129) always arise, always giving

$$\chi_{+1}(\omega) = \frac{\omega_0}{\omega_p}\chi(\omega), \quad \chi_{-1}(\omega) = -\frac{1}{\omega_p}[-i\omega\chi(\omega)]. \quad (139)$$

They are order $\mathcal{O}_\infty(\omega^{-2})$ and $\mathcal{O}_\infty(\omega^{-1})$. For the running example there is only one more result: Equation (129) is

$$W(E_-)(0^+) = \frac{[\dot{P}(E_-)(0^+)]^2}{2\omega_p^2} + \frac{1}{2}k[P(E_-)(0)]^2 + \frac{1}{2\beta^2}[P_\beta(E_-)\times(0)]^2, \quad (140)$$

where

$$P_\beta[E_-](0^+) = P_\beta[E_-](0^-) = P_\beta[E_-](0) = aP[E_-](0) - bP_v[E_-](0),$$

$$a = \frac{\Gamma\omega_0\sqrt{\gamma_\infty}}{\omega_p\sqrt{\gamma(\gamma_1^2+\omega_1^2)}}, \quad b = \frac{\sqrt{\gamma_\infty\gamma(\gamma_1^2+\omega_1^2)}}{\omega_0\omega_p}. \quad (141)$$

Note χ and χ_v are given in Eq. (114). Continuity holds since Eq. (130) gives χ_β order ω^{-2} near infinity. Specifically [47], for $s=-i\omega$

$$\chi_\beta(\omega) = \frac{\gamma_\infty^{3/2}(r^{-1}-1)}{2\sqrt{\gamma(\gamma_1^2+\omega_1^2)}}\frac{(\Gamma-\gamma_1)^2+\omega_1^2}{(\gamma+\gamma_1)^2+\omega_1^2}\times\frac{\omega_0\omega_p s}{(s+\gamma)((s+\gamma_1)^2+\omega_1^2)} = a\chi(\omega) - b\chi_v(\omega). \quad (142)$$

C. Fundamental time-reversal theorem

The example of section VB heralds this new result:

Theorem 7 (Free Energy Diagonalization). The minimal $U_+[E](0)$ and maximal $U_-[E](0)$ free energies of dielectric Eq. (35) or Eq. (36) are diagonalized by the λ-polarizations $\{P_{\lambda_j}[E](0)\}_{j=1}^{2(N+1)}$ Eqs. (133), (130), and (82):

$$U_-[E](0^+) = \sum_{j=1}^{2(N+1)}\frac{\gamma_\infty}{\omega_p^2}\langle f_{\lambda_j},\hat{E}_-\rangle^2 = \frac{\dot{P}^2[E_-](0^+)}{2\omega_p^2} + \frac{\omega_0^2P^2[E_-](0)}{2\omega_p^2} + \sum_{j=3}^{2(N+1)}\frac{P_{\lambda_j}^2[E_-](0^+)}{2\lambda_j^2}, \quad (143)$$

$$U_+[E](0^+) = \sum_{j=1}^{2(N+1)}\frac{\gamma_\infty}{\omega_p^2}\lambda_j^2\langle f_{\lambda_j},\hat{E}_-\rangle^2 = \frac{\dot{P}^2[E_-](0^+)}{2\omega_p^2} + \frac{\omega_0^2P^2[E_-](0)}{2\omega_p^2} + \sum_{j=3}^{2(N+1)}\frac{P_{\lambda_j}^2[E_-](0^+)}{2}, \quad (144)$$

where limits from above ($^+$) are not needed when $\mathcal{E}^0\subset\mathcal{E}$ is the space of admissible fields, or if any $\chi_{\lambda_j}(\omega)$ is $\mathcal{O}_\infty(\omega^{-2})$.

The $2N+2$ eigenvalues of dielectric Eq. (35) are enumerated here as $\{\lambda_j\}_{j=1}^{2(N+1)}$, with convention $\lambda_1=-1$, $\lambda_2=+1$. To show Theorem 7 first note that sequences of fields in $\mathcal{E}$ are subsumed in Eqs. (143) and (144)'s frequency-space inner products: one might naively project field transforms onto state-space basis $\{f_{\lambda_j}(\omega)\}_{j=1}^{2(N+1)}$, ignoring that there is no $E(\tau)\in\mathcal{E}$ whose transform is $1/i\omega$, and, surely, no $E(\tau)\in\mathcal{E}^0$ whose transform is 1. Forming admissible fields with spectra arbitrarily close to the spectra just described, distance measured by the norm arising from inner product Eq. (55), removes the naivete. Below we assume $t=0$ as justified above.

A past-field E_- yields a (real symmetric) state vector

$$\Sigma(E_-) = (\hat{E}_-(z_1),\ldots,\hat{E}_-(z_{N+1}),\quad \hat{E}_-(-z_1^*),\ldots,\hat{E}_-(-z_{N+1}^*)). \quad (145)$$

The basis $\{f_{\lambda_j}(\omega)\}_{j=1}^{2(N+1)}$ is orthogonal with respect to Eq. (55), hence linearly independent over Eq. (55)'s domain of integration, and then, by their analyticity, over the adjacent lower-half plane $\mathbb{C}_-$. So with real symmetry and $\hat{E}_-(\omega)$ analytic in $\mathbb{C}_-$, there are unique, real $\{C_k\}_{k=1}^{2(N+1)}$ giving

$$\hat{E}_-(p_j) = \sum_{k=1}^{2(N+1)}C_kf_{\lambda_k}(p_j), \quad j=1,\ldots,2N+2, \quad (146)$$

the p_j's $\in\mathbb{C}_-$ listing $\chi(\omega)$'s poles as in Eq. (145). For all $\omega\in\mathbb{C}_-$, define then the past field E_-'s orthogonal projection and orthogonal complement projection via

$$\hat{E}_{-\parallel}(\omega) = \sum_{j=1}^{2(N+1)}C_jf_{\lambda_j}(\omega), \quad (147)$$

$$\hat{E}_{-\perp}(\omega) = \hat{E}_-(\omega) - \hat{E}_{-\parallel}(\omega). \tag{148}$$

So for $j=1,\ldots,2(N+1)$,

$$\langle f_{\lambda_j}, \hat{E}_- \rangle = \langle f_{\lambda_j}, \hat{E}_{-\parallel} \rangle. \tag{149}$$

This follows from the fact that: a) Eqs. (147) and (146) give

$$\hat{E}_-(p_j) = \hat{E}_{-\parallel}(p_j), \tag{150}$$

for all j, i.e., $\hat{E}_-(\omega)$ and $\hat{E}_{-\parallel}(\omega)$ give the same medium state, and b) for admissible $\hat{F}_-$

$$\begin{aligned}\langle f_\lambda, \hat{F}_- \rangle &\propto \int_{-\infty}^{+\infty} \Lambda(\omega) f_\lambda(-\omega)[\Lambda(-\omega)\hat{F}_-(\omega)]d\omega \\ &= \frac{2\pi i}{\lambda}\sum_{j=1}^{N+1} [D_j f_\lambda(z_j)\Lambda(-z_j)\hat{F}_-(z_j) \\ &\quad - D_j^* f_\lambda(-z_j^*)\Lambda(z_j^*)\hat{F}_-(-z_j^*)]d\omega, \end{aligned} \tag{151}$$

which is a state function. Here, we used representation Eq. (89) of the solution $\lambda f_\lambda(-\omega)$ to RH problem Eq. (64).

Equations (149), (148), and (147) give the orthogonality

$$\langle \hat{E}_{-\parallel}, \hat{E}_{-\perp} \rangle = \sum_{j=1}^{2(N+1)} C_j \langle f_{\lambda_j}, \hat{E}_{-\perp} \rangle = 0, \tag{152}$$

and then estimate

$$\begin{aligned}\|\hat{E}_-\|^2 &= \langle \hat{E}_-, \hat{E}_- \rangle \\ &= \langle \hat{E}_{-\parallel} + \hat{E}_{-\perp}, \hat{E}_{-\parallel} + \hat{E}_{-\perp} \rangle \\ &= \|\hat{E}_{-\parallel}\|^2 + \|\hat{E}_{-\perp}\|^2 \geq \|\hat{E}_{-\parallel}\|^2. \end{aligned} \tag{153}$$

Considerations leading to Eq. (151) show Eq. (153) holds with past field $\hat{E}_-$ replaced by any $\hat{F}_- \in \Sigma(E_-)$ (yielding the same state). By Definition 4 and the fact that $\hat{E}_{-\parallel} \in \Sigma(E_-)$,

$$U_-[E](0^+) = \min_{\hat{F}_- \in \Sigma(E_-)} \frac{\gamma_\infty}{\omega_p^2}\|\hat{F}_-\|^2 \leq \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_{-\parallel}\|^2, \tag{154}$$

but now get

$$U_-[E](0^+) = \min_{\hat{F}_- \in \Sigma(E_-)} \frac{\gamma_\infty}{\omega_p^2}\|\hat{F}_-\|^2 \geq \min_{\hat{F}_- \in \Sigma(E_-)} \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_{-\parallel}\|^2 = \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_{-\parallel}\|^2 \tag{155}$$

using Eqs. (150) and (153) (with $\hat{E}_- \mapsto \hat{F}_-$). We get Eq. (143) proper by using Eq. (147) and basis $\{f_{\lambda_j}(\omega)\}_{j=1}^{2(N+1)}$'s orthonormality, giving

$$\|\hat{E}_{-\parallel}\|^2 = \sum_{j=1}^{2(N+1)} C_j^2 = \sum_{j=1}^{2(N+1)} \langle f_{\lambda_j}, \hat{E}_{-\parallel} \rangle^2, \tag{156}$$

and then using Eq. (149), which gives

$$\|\hat{E}_{-\parallel}\|^2 = \sum_{j=1}^{2(N+1)} \langle f_{\lambda_j}, \hat{E}_- \rangle^2. \tag{157}$$

The other representations in and claims about Eq. (143) follow from Eq. (132) and the considerations for Eq. (138).

For claims about U_+ Eq. (144), recall Eq. (59),

$$U_+[E](0^+) = \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_+\|^2, \tag{158}$$

where $\hat{E}_+$ is the future optimal recovery field for $\hat{E}_-$, i.e., the first component of solution $(\hat{E}_+, \hat{Z}_-)$ of RH problem Eq. (45). From Eq. (89) and its discussion, we see it is a state function, and, so, as per Eq. (150), we can replace the data $\hat{E}_-$ in Eq. (45) by $\hat{E}_{-\parallel}$ without changing $\hat{E}_+$ or $U_+[E](0)$:

$$U_+[E](0^+) = U_+[\hat{E}_{-\parallel}](0^+) = \frac{\gamma_\infty}{\omega_p^2}\|\hat{E}_+[\hat{E}_{-\parallel}]\|^2, \tag{159}$$

emphasizing $\hat{E}_+$'s *linear* dependence on $\hat{E}_{-\parallel}$: By Eq. (147)

$$\begin{aligned}\hat{E}_+[\hat{E}_{-\parallel}] &= \hat{E}_+\left[\sum_{j=1}^{2(N+1)} C_j f_{\lambda_j}\right] \\ &= \sum_{j=1}^{2(N+1)} C_j \hat{E}_+[f_{\lambda_j}] \\ &= \sum_{j=1}^{2(N+1)} C_j \lambda_j \mathrm{T} f_{\lambda_j} = \sum_{j=1}^{2(N+1)} C_j f_{\lambda_j}^\dagger, \end{aligned} \tag{160}$$

since $\lambda_j \mathrm{T} f_{\lambda_j} = f_{\lambda_j}^\dagger$ is the component $\hat{E}_+[f_{\lambda_j}]$ of the solution to Eq. (64) for data f_{λ_j}. So by orthonormality,

$$\begin{aligned}U_+[E](0^+) &= \frac{\gamma_\infty}{\omega_p^2}\left\|\sum_{j=1}^{2(N+1)} C_j \lambda_j \mathrm{T} f_{\lambda_j}\right\|^2 \\ &= \sum_{j=1}^{2(N+1)} \frac{\gamma_\infty}{\omega_p^2}\lambda_j^2 C_j^2 \\ &= \sum_{j=1}^{2(N+1)} \frac{\gamma_\infty}{\omega_p^2}\lambda_j^2 \langle f_{\lambda_j}, \hat{E}_- \rangle^2. \end{aligned} \tag{161}$$

Here we used the unitarity of T with respect to Eq. (55). Equation (161) is Eq. (144), other claims there following as for $U_-[E](0)$ via Eqs. (132) and (138) of Sec. V B.

The diagonalization in Theorem 7 indicates (in agreement with Theorem 5) that kinetic and potential energies are always present in the class of rational dielectrics. The other terms in the diagonalization, corresponding to eigenvalues $\lambda \in (-1,1)$, stem from the irreversibility of dissipation and mark the existence of novel notions of energy dissipative media.

VI. SUMMARY

General results on free energy in dielectrics have been presented. We represented the maximal and minimal free energies of linear, time-translationally invariant dielectrics by means of medium eigenexcitation and de-excitation fields. These eigenmodes completely span the state space of the dielectric, and they transform in a generalized way under time reversal, i.e., by signed dilation. The dilating factor $\lambda \in [-1,+1]$ is a time-reversal eigenvalue, and is uniquely connected with the dissipativity of the medium and the multiplicity of free energies describing it. Usual notions of time-reversal parity suggest $\lambda=-1$ labels the medium's electromagnetic *en-masse* kinetic energy and $\lambda=+1$ its electromagnetic *en-masse* potential energy. Novel parities associated with time-reversal values of $|\lambda|<1$ arise from macroscopic, irreversible phenomenology, but are otherwise on an equal footing with those of kinetic and potential energies. These results give precision to the concerns originally raised by Landau, Lifshitz, and others [8,35] regarding the thermodynamic admissibility of medium-field energy in dissipative media.

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APPENDIX A: FREE ENERGY

Dynamical free energies of materials with fading memory have been studied for some time [13,48–51]. As in ordinary (near-) equilibrium thermodynamics, a dynamical free energy is any state function $\psi=\psi(\sigma)$ giving bounds on the amount of work realizable from transitions between states of the system [52]: for any path $\Pi_{\sigma_2\leftarrow\sigma_1}=\Pi_{\sigma_2\sigma_1}$ between initial and final states σ_1 and σ_2 of the system,

$$\psi(\sigma_2) - \psi(\sigma_1) \leq W(\Pi_{\sigma_2\sigma_1}). \tag{A1}$$

The right side of Eq. (A1) is the work done *on* the dielectric (*by* a field) via a path between states. If initial and final states σ_1 and σ_2 give $\psi(\sigma_1) > \psi(\sigma_2)$, rewriting Eq. (A1) as

$$0 < \psi(\sigma_1) - \psi(\sigma_2) \geq - W(\Pi_{\sigma_2\sigma_1}) \tag{A2}$$

shows the difference in the free energies is a positive upper bound on the amount of work $-W(\Pi_{\sigma_2\sigma_1})$ performable *by* the system in making the transition $\sigma_1 \to \sigma_2$.

Eq. (A1) is the main ingredient making a free energy a Lyapunov function for a non-equilibrium system [53,54]: for example, after an electric field time series $\mathbf{E}=\mathbf{E}(t)$ ceases to subsidize the free energy of a dielectric by doing work on it, Eq. (A1) indicates the state $\sigma=\sigma(t)$ evolves in time so that the free energy decreases if evolving at all, i.e.,

$$\psi[\sigma(t_2)] - \psi[\sigma(t_1)] \leq 0 \tag{A3}$$

for all $t_2 > t_1$ [with inequality in Eq. (A3) usually holding]. Finally, as in equilibrium thermodynamics a non-negative dynamical free energy can often be adjusted to vanish on, and only on, the system's equilibrium state, so that it measures deviations of the system from equilibrium. For passive dielectrics, we will see property Eq. (A3) holds with strict inequality when $\sigma(t_1)$ is not equilibrium, indicating a steady trend to equilibrium when external fields vanish.

A free energy usually gives equality in Eq. (A1) for some processes, at least between some pairs of states. This generalizes the idea of reversible process from equilibrium thermodynamics to systems with memory: they are "as reversible as possible." Reversible processes missing between some states forces the existence of multiple free energies satisfying Eq. (A1) [33,55,56]. We connect this phenomenon with novel notions of parity under time reversal. This also gives rise to notions of energy distinct from potential and kinetic, but having other important properties in common with those ideas.

Usually there are two extremal free energies Eq. (A1):

$$\psi_M(\sigma) = \min_{\Pi} W(\Pi_{\sigma\mathbf{0}}) = W(\Pi^{\min}_{\sigma\mathbf{0}}),$$
$$\psi_m(\sigma) = \max_{\Pi} - W(\Pi_{\mathbf{0}\sigma}) = - \min_{\Pi} W(\Pi_{\mathbf{0}\sigma}) = - W(\Pi^{\min}_{\mathbf{0}\sigma}), \tag{A4}$$

are the maximum (M) and minimum (m) free energies for typical dissipative systems. They are the minimum work required to create a state σ from *equilibrium* 0 and, respectively, the maximum work extractable from the state σ "by transition to equilibrium." Equilibrium 0 is often distinguished by requiring

$$W(\Pi_{\sigma\mathbf{0}}) \geq 0 \tag{A5}$$

for any final state σ and path Π, equality possible for some path if and only if $\sigma=\mathbf{0}$. It can be shown that $\psi_M(\sigma)$ and $\psi_m(\sigma)$ both satisfy Eq. (A1).

Of free energies Eq. (A1) with $\psi(\mathbf{0})=0$, ψ_M and ψ_m of Eq. (A4) are upper and lower bounds, respectively: Eq. (A1) states

$$0 - \psi(\sigma_1) = \psi(\mathbf{0}) - \psi(\sigma_1) \leq W(\Pi_{\mathbf{0}\sigma_1}). \tag{A6}$$

Choice $\Pi_{\mathbf{0}\sigma_1}=\Pi^{\min}_{\mathbf{0}\sigma_1}$, rearrangement and Eq. (A4) then give

$$\psi(\sigma_1) \geq - W(\Pi^{\min}_{\mathbf{0}\sigma_1}) = \psi_m(\sigma_1). \tag{A7}$$

Likewise, Eq. (A1) states

$$\psi(\sigma_2) - 0 = \psi(\sigma_2) - \psi(\mathbf{0}) \leq W(\Pi_{\sigma_2\mathbf{0}}). \tag{A8}$$

Choice $\Pi_{\sigma_2\mathbf{0}}=\Pi^{\min}_{\sigma_2\mathbf{0}}$, rearrangement and Eq. (A4) then give

$$\psi(\sigma_2) \leq W(\Pi^{\min}_{\sigma_2\mathbf{0}}) = \psi_M(\sigma_2). \tag{A9}$$

Thus, all free energies for the dielectric are bounded above by $\psi_M(\sigma)$ and below by $\psi_m(\sigma)$

APPENDIX B: RECOVERABLE ENERGY, A STATE FUNCTION

The work done by $E(\tau)$ during period $\tau \in [t_1,t_2]$ is

$$\Delta_{t_1}^{t_2} W[E] = \int_{t_1}^{t_2} E(\tau)\dot{P}[E](\tau)d\tau. \tag{B1}$$

Given (3), it is also

$$\Delta_{t_1}^{t_2} W[E] = W[E](t_2) - W[E](t_1). \tag{B2}$$

So note

$$\begin{aligned} U_+[E](t) &= W[E](t) - Q_+[E](t) \\ &= W[E_-^t + G_+^t](t) - \inf_F W[E_-^t + F_+^t](+\infty) \\ &= \sup_F - \{W[E_-^t + F_+^t](+\infty) - W[E_-^t + G_+^t](t)\} \\ &= \sup_F - \{W[E_-^t + F_+^t](+\infty) - W[E_-^t + F_+^t](t)\} \\ &= \sup_F - \Delta_t^\infty W[E_-^t + F_+^t] \\ &= -\inf_F \Delta_t^\infty W[E_-^t + F_+^t]. \end{aligned} \tag{B3}$$

Beyond Eq. (B2) we used $W[E](t) = W[E_-^t + G_+^t](t)$ for any admissible t-future field G_+^t, which is causality Eq. (6). Equations (B3) and (B1) give

$$\begin{aligned} U_+[E](t) &= -\inf_F \Delta_t^\infty W[E_-^t + F_+^t] \\ &= -\inf_F \int_t^\infty (E_-^t + F_+^t)(\tau)\dot{P}[E_-^t + F_+^t](\tau)d\tau \\ &= -\inf_F \int_t^\infty F(\tau)\dot{P}[E_-^t + F_+^t](\tau)d\tau. \end{aligned} \tag{B4}$$

Here we used $E_-^t + F_+^t = F$ over the integration. So then

$$\begin{aligned} U_+[E](t) &= -\inf_F \int_t^\infty F(\tau)\dot{P}[E_-^t + F_+^t](\tau)d\tau \\ &= -\inf_F \int_t^\infty F(\tau)\dot{P}[(E')_-^t + F_+^t](\tau)d\tau \\ &= U_+[E'](t) \end{aligned} \tag{B5}$$

provided, for $\tau \geq t$,

$$\dot{P}[(E')_-^t + F_+^t](\tau) = \dot{P}[E_-^t + F_+^t](\tau). \tag{B6}$$

By definition Eq. (13), Eq. (B6) holds if E_-^t and $(E')_-^t$ have produced the same state in the dielectric $P[E]$ at time t. By Eqs. (B5) and (B6), the recoverable energy $U_+[E]$ is a state function. Lorentz oscillator examples show that the irrecoverable energy $Q_+[E]$ and the work $W_+[E]$ are not always state functions.

APPENDIX C: SEQUENCES

In the main text we often pretend the infimum of the recoverable energy is a minimum, but actually we implicitly consider sequences. The need to consider sequences when restricting to the space $\mathcal{E}$ of absolutely integrable fields arises since RH problem Eq. (37) gives factors $-i\omega\chi_v(\omega)$ and $i\omega\chi_v(-\omega)$ with simple zeroes at $\omega=0$, their reciprocals having simple poles at $\omega=0$: optimal energy recovery fields have transforms with singularity structure given by this reciprocal, yet no field in $\mathcal{E}$ has a transform with pole at $\omega=0$. Similarly, factors $-i\omega\chi_v(\omega)$ and $i\omega\chi_v(-\omega)$ having simple zeroes at $\omega=\infty$ dictates that generic optimal energy recovery fields have transforms that do not vanish near infinity, yet no field in $\mathcal{E}$ has this property (by the Riemann-Lebesgue lemma [37]). Consideration of sequences in general is unavoidable (when these factors have real zeroes) because use of the space $\mathcal{E}$ is nearly unavoidable.

The location of real zeroes of factors solving Eq. (37) indicate "where" in frequency space sequences must be considered. These zeroes arise from the real zeroes of density $\rho(\omega)$, which, for a simple dielectric, are only at 0 and ∞. If the density has other real zeroes, sequences of fields in $\mathcal{E}$ with spectrum peaked at these zeroes, but whose limiting field is not itself in $\mathcal{E}$, must also be considered.

APPENDIX D: UNIQUENESS

Two solutions $(\hat{E}_+, \hat{Z}_-)$ and $(\hat{E}'_+, \hat{Z}'_-)$ of Eq. (45) satisfy

$$-i\omega\chi_v(\omega)(\hat{E}_+ - \hat{E}'_+) = \hat{Z}_- - \hat{Z}'_-. \tag{D1}$$

The left side is analytic and tending to zero in a closed upper-half plane and the right has that property in a closed lower-half plane. So both sides are entire and vanishing at infinity, and, so, are zero by Liouville's theorem.

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