Solving one-dimensional acoustic systems using the impedance translation theorem and equivalent circuits: A graduate level homework assignment

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JASA ARTICLE

Solving one-dimensional acoustic systems using the impedance translation theorem and equivalent circuits: A graduate level homework assignment[a]

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ABSTRACT:
The natural frequency resonances and sound radiation from one-dimensional acoustic systems are of great interest in the study of musical instruments, human vocal tract effects on speech, automotive exhaust pipes, duct systems for temperature control in buildings, and more. The impedance translation theorem is an approach that may be used to solve for the input impedance and therefore the resonance frequencies of one-dimensional systems. Equivalent circuits offer another approach to solving one-dimensional systems, though with equivalent circuits you can also solve for the response at any location in the system, including the radiated sound pressure. At Brigham Young University, there are two graduate level courses that teach these two techniques. One of the most challenging and memorable homework assignments from these courses is based on using one of these techniques to analyze a particular acoustic system and compare its response with the real thing. This paper discusses the basics of these two techniques and applies them to an analysis of phonemes produced by altering the human vocal tract. Details about the homework assignments are also given. © *2021 Acoustical Society of America*. https://doi.org/10.1121/10.0008932

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I. INTRODUCTION

The impedance translation theorem and equivalent circuits are theoretical approaches that may be used to solve one-dimensional acoustic systems. Impedance translation allows for the determination of the impedance at one location due to the impedance at some distance away from that location. This process can be repeated several times to translate the impedance of one end of a system to the other end of the system, even if properties of the system change along the way. Branches for this one-dimensional system can also be modeled with the impedance translation theorem by translating along these one-dimensional branches and then doing a parallel impedance summation at the junction. Equivalent circuits may be used to represent these same types of one-dimensional systems. An equivalent circuit is an electrical circuit that provides an equivalent model of the physical interactions of quantities like resistances, inertances, acoustical compliances, and waveguides with some combination of series and parallel electrical elements such as resistors, inductors, capacitors, and generic impedance elements. The circuit may be solved with traditional loop and nodal analysis (i.e., Kirchhoff's circuit laws). The major advantage of an equivalent circuit approach is that it is straightforward to couple electrical, mechanical, and acoustical domains in a single electrical circuit and quantities such as velocity, pressure, voltage, and current may be solved for anywhere in the circuit by hand using linear algebra techniques or by using a software circuit solver.

Theoretical approaches to analyzing one-dimensional systems by hand with the impedance translation theorem and equivalent circuits are often taught in graduate acoustics courses. However, both of these techniques may also be evaluated with a computational approach, particularly for one-dimensional systems with varying cross-sections. The computational approach to solving these systems should strengthen students' ability to use this type of approach in their future research endeavors and employment. In 2016 the American Association of Physics Teachers (AAPT) recommendations were provided for incorporating computational experience into undergraduate physics programs.[1] In addition to having a course that specifically focused on teaching computational approaches to problems, some have suggested that computational exercises be added to many courses within the curriculum.[2,3] Many physics (and engineering) programs are now including computational exercises into their undergraduate curricula and thus graduate students in physics and engineering programs should have the background needed to tackle graduate level computational exercises.

At Brigham Young University, two graduate level acoustics courses are offered that cover impedance translation and equivalent circuits. In the first course, the impedance translation theorem is derived for systems with transverse waves on a string with arbitrary boundary conditions or impedance changes along the string. Students are taught how to derive a transcendental equation for a given

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system that allows them to determine the natural frequencies of the system as seen at the system's input end. Later, the impedance translation theorem is applied to longitudinal waves in one-dimensional fluid systems (e.g., pipes/ducts). It is at this point in the course that a computational homework assignment is given to analyze a particular one-dimensional acoustic system, such as a human vocal tract. Over one third of the second course covers equivalent circuit analysis of one-dimensional, electro-mechano-acoustic systems. A computational homework assignment in this course has students use the equivalent circuit modeling approach to analyze an acoustic system, again such as the human vocal tract, and analyze the sound radiation from this system.

Thus, one homework assignment is assigned in each of these two courses to provide a computational experience, using the impedance translation theorem and then equivalent circuits, and students typically develop their programming in MATLAB. An effort is made in the classroom to help students get started on these assignments and typically at least one week is given to the students to work on this homework project. The homework assignment asks the students to start with a simplified model of the system by assuming the cross section is constant all along its length (like a constant cross section pipe). Then the assignment asks the students to incorporate a varying cross section along the length of the system by breaking up the system into shorter pipe segments that each have a constant cross section but each adjacent cross section segment may have larger or smaller cross-sections. Systems such as the human vocal tract, a vuvuzela, and a trombone provide nice systems in which the sound is predominately propagating in one dimension and no side branches need to be considered. Each of these systems can be divided up into cylindrical cross section segments with a fixed length or of differing lengths (the majority of the length of the trombone is a cylindrical tube of constant cross section and could be represented with one segment for this portion and then several short length segments to model the curvature of the trombone's bell). A flute could be modeled as a cylindrical cross section system with one-dimensional wave propagation and side branches. The side branches introduce some additional complications, such as multiple locations for sound radiation (introducing sound sources that interfere with one another and introducing mutual impedance effects) and parallel impedance addition when using the impedance translation theorem. Analyzing systems with a varying cross section requires students to develop loops in their code to handle the cross-sectional changes along the system's length.

This paper will center its discussion of this homework on analyzing the human vocal tract with these approaches. First, some background information on human vocal production will be provided that is typical of what the students are given as part of this assignment since vocal production is not a core topic covered in either of these classes. Then an introduction to the impedance translation theorem will be given along with instructions on how to use it to solve for the input impedance to an acoustic system. An introduction to equivalent circuits will be given along with instructions on how to use this approach to solve the same vocal tract system for the input impedance and for the radiated pressure from the system. Finally, some example results will be presented and discussed using each approach. A method to translate acoustic pressures through segments will be presented so that the impedance translation approach radiation results may be compared to those using the equivalent circuit approach. The hope is that this paper can help instructors develop a similar assignment and potentially even give this paper to their students to help them get started.

II. HUMAN VOCAL TRACT

Since students who take these two graduate level courses are typically not versed in human speech production, a simplified introduction is given in the homework assignment description and provided below.

Figure 1 shows a drawing of the human vocal tract system.[4] When speaking, the human vocal folds (or vocal cords) are often modeled as a closed end of a pipe and the mouth is considered to be an open end, thus it is like a closed-open pipe resonator. The human vocal tract we will consider is the air-filled system between the vocal folds and the mouth opening (when the soft palate closes off the nasal cavity). The tract begins at the vocal folds, passes through the pharynx at the back of the throat, and continues through the mouth to the opening in one's lips. The vocal tract can include the nasal cavity when the soft palate does not close it off, but for simplicity we will not consider this possible portion of the tract. Humans voice many different types of sounds (phonemes of speech) by changing the cross section of their vocal tract at various locations along its length (only vowels will be considered for this analysis). For example, the cross section of the mouth is strongly affected by adjustments made to the tongue's position within the mouth and the size of the opening between the lips. These changes, along with additional changes (such as constricting one's

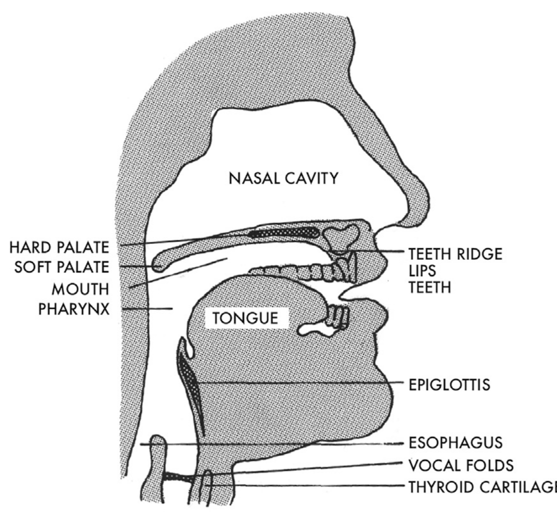

FIG. 1. Simplified schematic drawing of a human vocal tract cross section. Used with permission from Ref. 12.

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pharynx) allow humans to voice many different vowel sounds, even when they are at the same pitch.

To produce voiced speech, the lungs force air up the trachea into the larynx where the vocal folds are. The forced air pushes the vocal folds apart and then rushes through the opening between our vocal folds (this opening is called the glottis), which according to Bernoulli's principle, creates a low pressure region in the glottis resulting in an inward force on the vocal folds tending to close them. When humans voice speech, their vocal folds are also in a state of tension. Tension in the vocal folds also tends to close them. Once the vocal folds are closed, the forced air builds up pressure and pushes them back open and the cycle repeats itself (normally at a rate of around 120 Hz for men and 200 Hz for women). Tension of the vocal folds can be controlled to increase or decrease the rate of this cycle (the applied tension also changes the mass per unit length of the folds) and thus the pitch of the voice. This cycle of vocal fold vibration creates a series of input pulses or a glottal airflow waveform presented to the vocal tract that is similar to a modified sine wave with everything below zero missing (a sine wave with just the positive bumps and zero in between). Note this assumed glottal airflow only results in even harmonic partials and a modified empirical model will be given later that has both odd and even harmonic partials present. The period of this cycle corresponds to the fundamental frequency of the voice or the pitch of the voice. When speech is voiced, there are many harmonic partials of the fundamental frequency. The vocal tract has certain natural frequency resonances (so called formant frequencies) that shift up or down with changes to the tract's shape and thus change which partials are emphasized. Humans can perceive different vowel sounds based on which partials are emphasized.

Three common vowel sounds, /Λ/ (as in "f<u>u</u>n"), /i/ (as in "f<u>ee</u>t"), and /a/ (as in "h<u>o</u>t") will be considered. Story *et al.*[5] presented measurements of vocal tract cross-sections for various voiced phonemes based on magnetic resonance imaging measurements. They presented several cross-sectional views of the vocal tract of a 29 year old male subject, from which they extracted the cross-sectional area of the tract along its length. Table III in their paper gives the cross-sectional area as a function of length for several voiced phonemes at 0.397 cm intervals along the length.

To improve upon the pulsed glottal airflow model, by incorporating both odd and even harmonic partials, a sawtooth wave is used to begin with rather than a sine wave. Because the glottal airflow is not as abrupt as a sawtooth wave would suggest, a modified sawtooth wave, with N number of terms is created,

$$y(t) = \sum_{n=1}^{N} \left\{ \frac{1}{n^{0.7}} \right\} \left[-\frac{2}{n\pi} \frac{1}{(-1)^n} \sin(2\pi n f_0 t) \right], \tag{1}$$

where f_0 is the fundamental frequency of the sawtooth wave and corresponds to the rate of vocal fold vibration. The term in the square brackets is the usual term used to create a sawtooth wave, but the term in curly brackets was determined empirically to produce a waveform that more closely matches a typical glottal airflow waveform. The negative values in the function $y(t)$ are then set equal to zero. The use of $N = 15$ terms seems to be sufficient. Figure 2 displays a glottal airflow time waveform and spectrum with $f_0 = 120$ Hz obtained using this method.

III. IMPEDANCE TRANSLATION THEOREM

The impedance translation theorem can be used to determine the impedance at x in a one-dimensional system due to the impedance a distance, L, away from x. The impedance translation theorem is given by Pierce[6] and by Kinsler *et al.*[7] in their Eqs. (2.9.22), (2.9.30), (3.7.3), and (10.2.4). The general form of the impedance translation theorem in the acoustical impedance domain is

$$Z_A(x) = Z_0 \left[\frac{Z_A(x+L) + jZ_0 \tan(kL)}{Z_0 + jZ_A(x+L)\tan(kL)} \right], \tag{2}$$

where $Z_A(x)$ is the acoustical impedance you wish to solve for, $Z_A(x+L)$ is the acoustical impedance a distance, L, away from x, $Z_0 = \rho_0 c/S$ is the characteristic acoustic impedance (where ρ_0 is the mass density of the fluid, c is the speed of sound in the fluid, S is the cross-sectional area of the system between x and $x+L$), and k is the lossless acoustic wavenumber. The theorem can be used in the mechanical impedance domain by using the characteristic mechanical impedance, $\rho_0 cS$, in place of the characteristic acoustical impedance, $\rho_0 c/S$, and by translating a mechanical impedance, $Z_M(x+L)$, through a distance L to determine the mechanical impedance at x, $Z_M(x)$. Similarly, it may be used with the characteristic specific acoustic impedance, $\rho_0 c$, and specific acoustic impedances at x and at $x+L$.

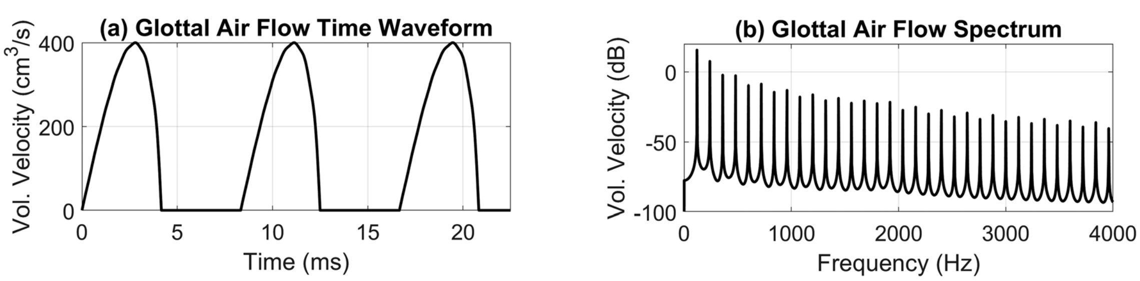

FIG. 2. (a) Pulsed glottal airflow waveform used as a source input. (b) Spectrum of the glottal airflow waveform.

The specific acoustic impedance form is useful when considering how normal incidence plane waves travel through layered media, in which case the cross-sectional area is not defined. The acoustical and mechanical impedance versions are useful when the one-dimensional system has a defined cross-sectional area. When translating through a fluid system with an abrupt change in cross-sectional area between segments, there exists a continuity of volume velocity across such a junction, not a continuity of particle velocity. Thus, when analyzing wave propagation through an acoustical system, it is critical to use the acoustical impedance version of the theorem rather than the mechanical impedance or specific acoustic impedance version of the theorem, since acoustical impedance is defined in terms of volume velocity. It should be noted that the impedance translation theorem was developed for systems operating in the steady state, assuming a time harmonic dependence, $e^{j\omega t}$. Additionally, the translation through a one-dimensional segment does not assume a lumped element approach, and instead Eq. (2) can handle resonances in the length of the segment (but not resonances in the cross section of the segment). Neither the length of each segment nor the overall length of the system must be small compared to a wavelength, but the cross-sectional dimensions must always be small compared to a wavelength.

Consider the acoustic system depicted in Fig. 3. The acoustical impedance at several locations will be discussed and determined using the impedance translation theorem to illustrate the impedance translation theorem modeling process. There is a piston at location x_0, a Helmholtz resonator side branch at location x_1, a constriction between locations x_2 and x_3, and an opening to the surrounding air at location x_4. We will determine the input acoustical impedance seen by the piston at x_0 due to the radiation impedance at x_4, according to how the system modifies it between x_0 and x_4. The radiation impedance, Z_{AR}, at x_4 can be assumed to be the same radiation impedance that would be experienced as sound is radiated from the end of a long circular pipe of radius a,[8] or by a piston at the end of a long tube (an unflanged piston),

$$Z_A(x_4) = Z_{AR} = \frac{0.247\rho_0 c(ka)^2}{S} + j\frac{0.613\rho_0 c(ka)}{S}. \tag{3}$$

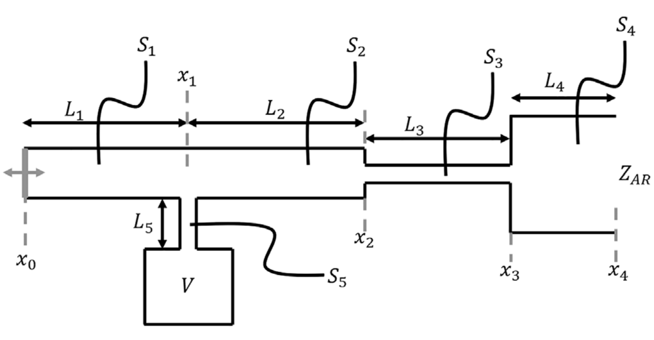

FIG. 3. Example acoustic system analyzed with impedance translation theorem and equivalent circuit techniques.

Translating Z_{AR} from x_4 to x_3 to determine the acoustical impedance experienced at x_3 looking towards the open end requires the use of Eq. (2),

$$Z_A(x_3) = Z_4\left[\frac{Z_{AR} + jZ_4\tan(kL_4)}{Z_4 + jZ_{AR}\tan(kL_4)}\right], \tag{4}$$

where Z_4 is the characteristic impedance $\rho_0 c/S_4$. $Z_A(x_3)$ may then be translated through the constriction using a similar process to obtain $Z_A(x_2)$. The impedance at x_1 due to the upper branch of the system, $Z_{AU}(x_1)$, can be determined by translating $Z_A(x_2)$ to x_1.

If we assume the lower branch of the system is simply the acoustical impedance of a Helmholtz resonator, $Z_{AL}(x_1)$, then we can determine the total impedance at x_1, $Z_A(x_1)$, from the parallel summation of $Z_{AU}(x_1)$ and $Z_{AL}(x_1)$

$$Z_A(x_1) = \frac{1}{\dfrac{1}{Z_{AU}(x_1)} + \dfrac{1}{Z_{AL}(x_1)}} = \frac{Z_{AU}(x_1)Z_{AL}(x_1)}{Z_{AU}(x_1) + Z_{AL}(x_1)}. \tag{5}$$

Finally, the input acoustical impedance, $Z_A(x_0)$, presented to the piston may be determined by translating $Z_A(x_1)$ to x_0. It is worth noting that technically, ρ_0, c, and k (the wave speed within the definition of k) can vary from segment to segment, for example, if there is a different fluid medium in each segment, or changes to the medium due to things like a varying temperature.

The resonance frequencies of the system, as seen by the source, are dependent on whether the acoustic source is modeled as a pressure source or as a volume velocity source. For the vocal tract system, it is assumed that resonance occurs when the acoustic volume velocity is maximal at the opening of the mouth for sound radiation. A higher volume velocity at the mouth leads to a higher far field pressure. Students are asked what type of source should be assumed for the acoustic system they are asked to study. The source type then determines how to find the resonance frequencies of the system from the input impedance as discussed in Sec. V.

For the application of the impedance translation theorem to the vocal tracts used to produce the phonemes /Λ/, /i/, and /a/, typically the students are asked to start with a simplified model for /Λ/, instead of using the varying cross-sectional data given by Story *et al.*[5] The /Λ/ vocal tract can be roughly modeled as a constant cross-sectional area pipe[4] using the average cross-sectional area of the tract information given by Story *et al.*, 1.87 cm^2, and the overall length of 16.67 cm. The radiation impedance can thus be translated from the mouth up to the location of the vocal folds and the resonances of the tract may be determined from the input impedance. For a so called neutral vocal tract of constant cross-sectional area along the length, this should result in odd harmonic multiples of the fundamental resonance of the tract.

Students are then be asked to compute the input impedance of a varying cross-sectional area vocal tract. Figure 4 displays the cross-sectional data from Story *et al.* for the

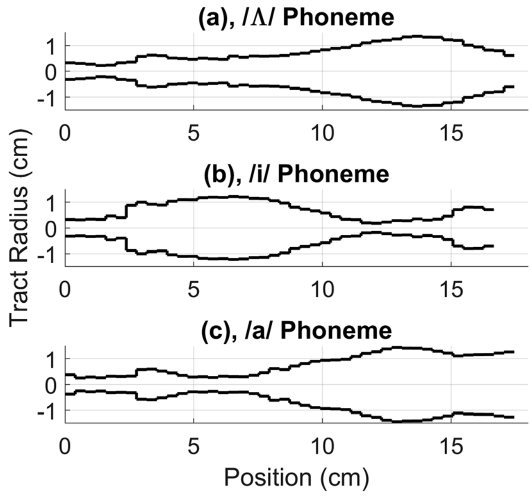

FIG. 4. Cross-sections of the human vocal tracts for the vowel phonemes /Λ/, /i/, and /a/ as given by Story *et al.* (Ref. 13).

vocal tracts configured to generate the /Λ/, /i/, and /a/ phonemes. A similar process to that described earlier in this section using Eqs. (2), (4), and (5) may then be attempted. The vocal tract should be divided up into cylindrical segments. With each translation, A and L of the segment may be varied. Students are free to use whatever computational tool they prefer to solve for the frequency-dependent input impedance but it may be wise to suggest that they use a technical programming language such as MATLAB. One issue with allowing students to use various different technical programming languages is that it becomes harder for the students to work together on their programming and difficult for a grader to attempt to locate where a student may have gone wrong in their submitted code. The suggestion can be given to compute the frequency-dependent radiation impedance at the mouth and then write a "for" loop to step through the impedance translations through each segment of the vocal tract. The advantage of a programming language like MATLAB is that the complex, frequency-dependent impedance becomes an array of numbers. Multiple arrays of numbers may be multiplied or divided, frequency value by frequency value, within a single line of code representing Eq. (2). Alternatively, two "for" loops may be needed (if matrix operations cannot be handled in a single operation like they can in MATLAB) with the inner loop computing the translated impedance for one segment at each frequency, and the outer loop stepping through each segment.

IV. EQUIVALENT CIRCUITS

Another approach to model one-dimensional wave propagation through a complicated acoustical system is to use equivalent electrical circuits to represent the interactions of electrical, mechanical, and acoustical components in the system. It should be noted that the equivalent circuit approach and the impedance translation theorem yield the same solutions but differ in the way that the problem is approached. The significant advantage of equivalent circuits is that there is a straightforward approach to obtain the pressure or the volume velocity at any location within the system, whereas obtaining these quantities with the impedance translation theorem approach at any location in the system is not typically taught. Additionally, the equivalent circuit approach more easily allows coupling of acoustical systems with mechanical and electrical system components. Here, we will only consider acoustical systems for brevity.

Segments of an acoustical system may be modeled as lumped elements in the equivalent circuit or as a waveguide to include the modeling of resonances within a given segment, by using a so-called T-network circuit. When acoustical systems are to be modeled, the acoustical-impedance equivalent circuit may often be drawn by inspection. Atmospheric pressure is represented by an electrical ground in an acoustical impedance domain circuit. Pipes of a short length L with respect to a wavelength may be represented by an inductor of inductance $\rho_0 L/S$, while cavities of volume V and of dimensions that are each small with respect to a wavelength may be represented by a grounded capacitor of capacitance $V/\left(\rho_0 c^2\right)$. Locations in the acoustical system where fluid can flow into two or more different branches of the system are modeled as nodes in the electrical circuit. The acoustic pressure is represented as an equivalent electrical voltage in the circuit, or potential quantity, while the volume velocity is represented as an equivalent electrical current in the circuit, or flow quantity. Constant-amplitude pressure sources are modeled as voltage sources and constant-amplitude volume velocity sources are modeled as current sources. An electrical resistor can model resistance to fluid flow, such as a thermoviscous acoustic resistance. Sometimes a frequency-dependent resistor or generic impedance box is needed, such as when any frequency dependence of the acoustic resistance or a complex radiation impedance needs to be modeled.

Equivalent circuits may be used to model the system depicted in Fig. 3. Figure 5(a) shows an equivalent circuit for this acoustic system modeled in the acoustical domain entirely using lumped acoustic elements. The assumed constant pressure input provided by the piston is represented as a constant voltage source. The segment of length L_1 is represented by a series inductor of acoustic mass, M_{A1}. A node is then introduced with one leg of the circuit representing the series combination of the acoustic mass, M_{A5}, and acoustic compliance, C_A, of the Helmholtz resonator (here we assume the resonator has zero internal resistance). Recall that capacitors need to be grounded. The other leg of the circuit follows the upper branch of the system. This upper branch can be modeled as three series inductors to represent the three acoustic mass segments, M_{A2}, M_{A3}, and M_{A4}. Finally, the radiation impedance is what connects the system to the atmospheric pressure so this impedance is represented by a frequency-dependent generic impedance box, connected in series to the inductors, and grounded on the other side.

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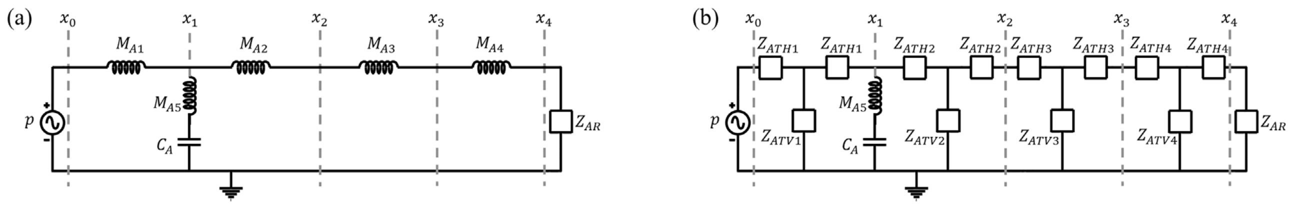

FIG. 5. Equivalent circuit models for the system depicted in Fig. 3 using (a) a purely lumped-element approach and (b) a T-network, waveguide approach.

Figure 5(b) depicts an equivalent circuit for the acoustic system in Fig. 3, except in this case waveguide T-network circuits are used to model each of the pipe segments in the top branch of the system, replacing the inductors M_{A1}, M_{A2}, M_{A3}, and M_{A4}. The two impedance boxes on the horizontal arms of a T-network are identical and have the form

$$Z_{ATH1} = j\frac{\rho_0 c}{S_1}\tan\left(\frac{kL_1}{2}\right), \tag{6}$$

while the impedance box on the vertical arm of the T-network has the form[9,10]

$$Z_{ATV1} = -j\frac{\rho_0 c}{S_1}\csc(kL_1). \tag{7}$$

The equivalent circuit in Fig. 5(a) may be used to solve for a valid, low-frequency solution where the segments of the system are still considered small in length compared to an acoustic wavelength. The equivalent circuit in Fig. 5(b) may be used to solve for valid solutions at any frequency, so long as the cross-sectional dimensions remain small compared to a wavelength and allow for one-dimensional wave propagation to be assumed.

In order to solve for the input impedance of this acoustic system, using either equivalent circuit, the impedance of the entire circuit downstream of the source may be determined using series and parallel impedance addition rules. The input impedance obtained in this manner using the circuit in Fig. 5(b) yields an identical input impedance to the one obtained if an impedance translation theorem approach was used instead. The same guidelines apply for determining the resonance frequencies seen by the source as those described in Sec. V. for constant pressure and constant volume velocity sources. For the vocal tract depicted in Fig. 4, a "for" loop can again be constructed to perform the necessary series and parallel impedance summations to determine the input impedance.

As previously mentioned, the advantage of using the equivalent circuit approach is that one can use a familiar Kirchhoff loop analysis approach to solve for the voltage at any location in the circuit and thus determine the pressure at that location. For every circuit loop, the direction of a current flowing throughout that loop is assigned (typically the clockwise direction). Then one sums up all of the voltages and voltage drops as you move around that loop. Each voltage summation equation can be arranged such that voltages produced by voltage sources (pressure sources) can be placed on the right side of the equation and the voltage drops at impedances around the loop can be lumped together multiplied by the appropriate current (volume velocity) in that loop. For each element that is included in two loops, the impedance of that element is also multiplied by the current (volume velocity) in the adjacent loop and this product is added to the voltage summation equation. These summation equations can then be translated to a matrix form. As an example, the voltage-drop summation equations for the circuit in Fig. 5(a) would be

$$\begin{aligned}&\left(j\omega M_{A1}+j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_1\\&\quad-\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_2=p,\\&-\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_1+\left(\frac{1}{j\omega C_A}+j\omega M_{A5}\right.\\&\quad\left.+j\omega M_{A2}+j\omega M_{A3}+j\omega M_{A4}+Z_{AR}\right)U_2=0.\end{aligned} \tag{8}$$

Equation (8) can be written in matrix form as

$$\begin{bmatrix}\left(j\omega M_{A1}+j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & -\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)\\ -\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & \left(\begin{array}{c}\frac{1}{j\omega C_A}+j\omega M_{A5}+j\omega M_{A2}+\cdots\\ \cdots+j\omega M_{A3}+j\omega M_{A4}+Z_{AR}\end{array}\right)\end{bmatrix}\begin{Bmatrix}U_1\\U_2\end{Bmatrix}=\begin{pmatrix}p\\0\end{pmatrix}. \tag{9}$$

The voltage-drop summation equations for the circuit in Fig. 5(b) would be

$$
\begin{aligned}
&(Z_{ATH1}+Z_{ATV1})U_1-(Z_{ATV1})U_2=p,\\
&-(Z_{ATV1})U_1+\left(Z_{ATV1}+Z_{ATH1}+j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_2-\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_3=0,\\
&-\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)U_2+\left(\frac{1}{j\omega C_A}+j\omega M_{A5}+Z_{ATH2}+Z_{ATV2}\right)U_3-(Z_{ATV2})U_4=0,\\
&-(Z_{ATV2})U_3+(Z_{ATV2}+Z_{ATH2}+Z_{ATH3}+Z_{ATV3})U_4-(Z_{ATV3})U_5=0,\\
&-(Z_{ATV3})U_4+(Z_{ATV3}+Z_{ATH3}+Z_{ATH4}+Z_{ATV4})U_5-(Z_{ATV4})U_6=0,\\
&-(Z_{ATV4})U_5+(Z_{ATV4}+Z_{ATH4}+Z_{AR})U_6=0.
\end{aligned}
\tag{10}
$$

The matrix form of Eq. (10) will not be given here for brevity.

One may then use Cramer's rule to solve for any one of the volume velocities in the circuit loops.[11,12] Cramer's rule is a linear algebra technique to solve for an independent variable from the ratio of the determinants of two matrices and can be used to solve for the current in a particular loop. If the volume velocity in the second loop of the circuit in Fig. 5(a) is desired, then the column vector of pressures replaces the second column of the impedance matrix of Eq. (9), and the determinant of this modified matrix divided by the determinant of the unmodified impedance matrix of Eq. (9) is equal to the volume velocity in the second loop

$$
U_2=\frac{\begin{vmatrix}\left(j\omega M_{A1}+j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & p\\ -\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & 0\end{vmatrix}}{\begin{vmatrix}\left(j\omega M_{A1}+j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & -\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right)\\ -\left(j\omega M_{A5}+\frac{1}{j\omega C_A}\right) & \left(\begin{array}{c}\frac{1}{j\omega C_A}+j\omega M_{A5}+j\omega M_{A2}+\cdots\\ \ldots+j\omega M_{A3}+j\omega M_{A4}+Z_{AR}\end{array}\right)\end{vmatrix}}.
\tag{11}
$$

Once the volume velocity in the loop containing Z_{AR} is known, the voltage drop (representing pressure) across Z_{AR} may be solved for and is equal to the pressure at x_4. Note that this pressure is a near-field radiated pressure, and is neither equivalent to nor directly proportional to, the far-field radiated sound pressure. When the far-field pressure from a loudspeaker in a half space is desired, it can be estimated after knowing the velocity of the loudspeaker's diaphragm.[13] The velocity of the loudspeaker's diaphragm multiplied by the area of the diaphragm is its volume velocity. Typically, one uses the diaphragm volume velocity, U_D, to estimate the radiated sound power in the near field

$$
W_{NF}=\frac{1}{2}|U_D|^2Re\{Z_{AR}\},
\tag{12}
$$

where $Re\{Z_{AR}\}$ means taking the real part of Z_{AR}, which is the radiation resistance. The assumption can be made that the propagation of sound from the near field to the far field is lossless, meaning no loss in sound power between the near field and the far field and thus $W_{NF}=W_{FF}$, the far field radiated sound power. Then assuming spherical wave radiation, the far-field radiated sound pressure, p_{FF}, at some distance, r, away from the loudspeaker can be related to the time averaged sound power, $\langle W_{FF}\rangle_T$,

$$
p_{FF}(r)=\sqrt{\frac{\langle W_{FF}\rangle_T\rho_0 c}{2\pi r^2}}.
\tag{13}
$$

The far-field radiated sound power from a directional source radiating into a half space, $\langle W_{1/2}\rangle_T$, can be related to the sound power from an omnidirectional source through the directivity factor, γ, of the directional source. A half space may be assumed if the vibrating diaphragm of a loudspeaker is assumed to radiate sound like a baffled circular piston. Thus, the on-axis, far-field radiated sound pressure from a directional source, $p_{FF}(r,0,0)$, is then

$$
p_{FF}(r,0,0)=\sqrt{\frac{\langle W_{1/2}\rangle_T\gamma\rho_0 c}{2\pi r^2}}.
\tag{14}
$$

The volume velocity through Z_{AR} of a system, like the one depicted in Fig. 5(b), can be used in an analogous manner to estimate the radiated sound pressure from an acoustic system, such as from the human mouth,

$$p_{FF}(r,0,0) = \sqrt{\frac{|U_D|^2 Re\{Z_{AR}\}\gamma\rho_0 c}{4\pi r^2}}. \tag{15}$$

Finally, for a baffled circular piston, the directivity factor is

$$\gamma = \frac{(ka)^2}{1 - \frac{J_1(2ka)}{ka}}, \tag{16}$$

where $J_1(2ka)$ is a Bessel function of order 1 with an argument $2ka$. The opening to the human mouth is, thankfully, not in an infinite baffle but this provides an assumption that can be used for a homework assignment.

In order to model vocal tract systems with equivalent circuits, such as those depicted in Fig. 4, one can construct an impedance matrix such as that shown in the square brackets of Eq. (9) and use Cramer's rule. This method requires a similarly complex computational code to be developed as was done with the impedance translation theorem approach. In some ways, the code required is more difficult to structure and there are more chances for error using Cramer's rule and equivalent circuits, partly because Cramer's rule requires the construction of two matrices (often tri-diagonal) to solve for the radiated volume velocity needed for Eq. (15). The suggestion of having students use the same programming language and many other general suggestions given in Sec. III also applies for a computational homework assignment using equivalent circuits.

V. EXAMPLE RESULTS

For the /Λ/ phoneme vocal tract, using the data provided by Story *et al.*,[5] the magnitude of the input impedance is plotted in Fig. 6(a) and the imaginary part of the input impedance (input reactance) is plotted in Fig. 6(b). The authors verified that both the impedance translation approach and the equivalent circuit approach yield the exact same results.

A. Determining resonance frequencies from input impedance

If students are asked to determine the resonance frequencies of the vocal tract by only using the input impedance information; for example, if they only use the

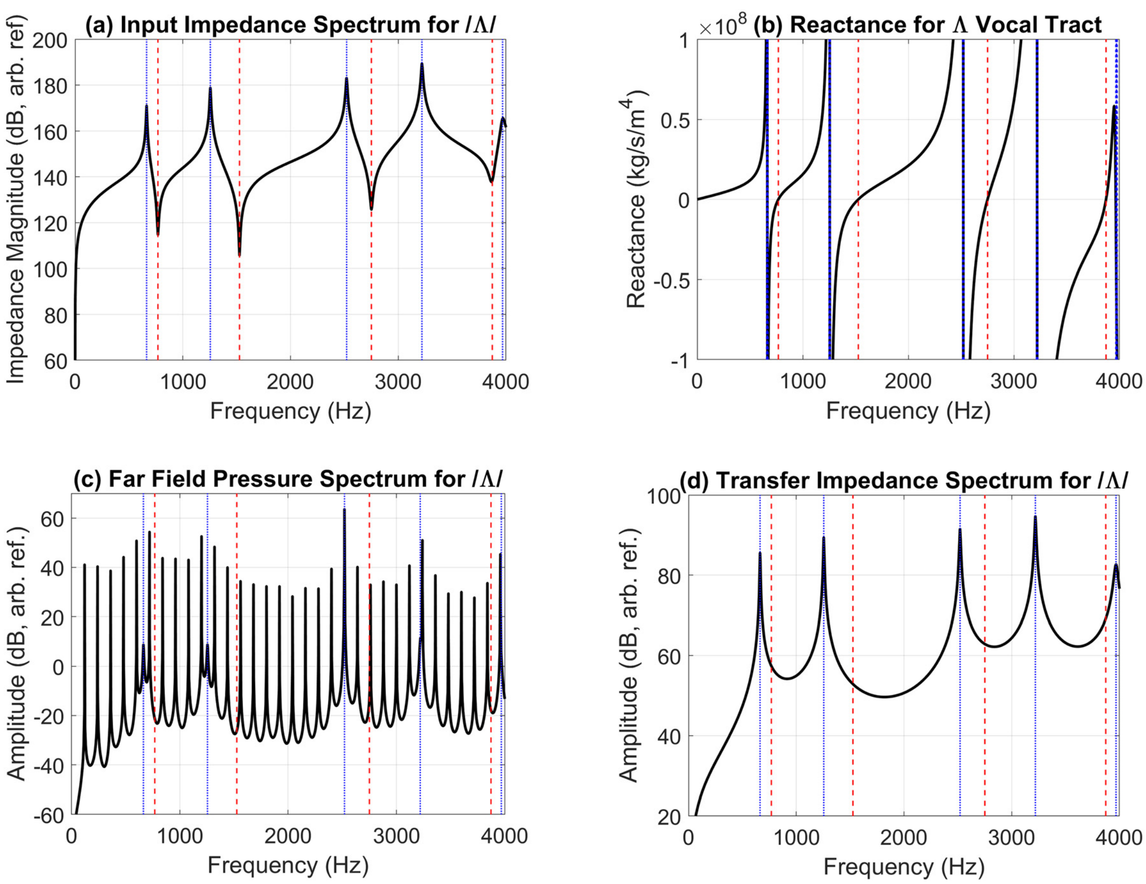

FIG. 6. (Color online) (a) Input impedance magnitude as a function of frequency for the phoneme /Λ/. (b) Reactance of the input impedance for the phoneme /Λ/. (c) Far field pressure radiation as a function of frequency for the phoneme /Λ/. (d) Transfer impedance magnitude as a function of frequency for the phoneme /Λ/. The blue dotted lines denote the peak frequencies of the input impedance and the red dashed lines correspond to positive-slope, zero crossings of the input reactance.

impedance translation approach discussed in Sec. III, what criteria should they use? Let us consider sources generating sound in a pipe-like system that has a source at one end and is open at the other end to radiate sound. Resonance will be defined here as the frequencies for which the far field radiated pressure is increased [the radiated sound power is also increased, see Eq. (14)] due to the system's response. This also corresponds to when the volume velocity of the open end of the system is maximum [see Eq. (15)].

The definition of resonance seems to vary among commonly used textbooks in acoustics. The various definitions state that resonance frequencies occur when the reactance of the input impedance goes to zero,[7] when the pressure amplitude becomes unbounded,[14] and that it always occurs at impedance minima.[15] Garrett[16] initially defines resonance as when the reactance goes to zero, but then later explores the case of a finite string with either a displacement-driven or a force-driven excitation. Importantly, Garrett makes it clear that in the displacement-driven case, in order to obtain large string velocities, i.e., resonance, the force required must go to infinity, meaning that the input impedance of a displacement-driven driver must be very large at resonance. Dudley and Strong[17] questioned why resonance frequencies are often generalized in textbooks to be defined in terms of zero reactance. They indicated that sometimes resonance frequencies are those for which the input impedance is maximum. It makes the most sense to define resonance frequencies as those in which a quantity of interest is largest, hence why here we will define resonance as when the radiated pressure is emphasized by the system.

For the vocal tract, mechanical-reed instruments, and lip-reed instruments, resonance occurs at frequencies corresponding to the peaks of the input impedance magnitude, rather than when the reactance goes to zero.[4] These instruments have sources whose motion is not as greatly impacted by the ways in which waves transit the instrument, meaning these sources have a high internal impedance and can be modeled as constant velocity sources (or constant volume velocity sources). At resonance, for such a system the source must "work very hard" (exert a lot of force) to get a large velocity response from the system at the open end, but because the source has a large internal impedance, it has an infinite ability to do so (infinite internal impedance). At resonance at the source input, the velocity is a fixed value but the pressure can be very large, thus the input impedance should be very large at resonance. For a constant pressure source, such as found in air-jet instruments, resonance frequencies for which the radiated sound pressure is emphasized by the system correspond to when the reactance of the input impedance is zero with a positive slope (and overall impedance minima). Note that these conditions hold for systems with little to no internal losses, such as the vocal tract system modeled here (no damping is included in the vocal tract).

The equivalent circuit approach may be used to determine the far field pressure as described in Sec. IV and in Eq. (15). The pressure spectrum for the phoneme /Λ/ is depicted in Fig. 6(c). It is instructive to compare the spectrum of the glottal air flow shown in Fig. 2(b) to that in Fig. 6(c). The amplitudes of the various harmonics of the fundamental frequency (harmonic partials) are modified (filtered) by the individual vocal tract. Often, partials corresponding to higher harmonics are the loudest in the resulting spectra. One may then observe that the harmonic partials that have the largest amplitudes, or at least the largest rise in amplitude relative to the partial amplitudes in Fig. 2(b), are the ones closest to the formant frequencies, which are identified as the peaks in the input impedance magnitude in Fig. 6(a); hence why we define resonance the way we do. Dividing the far field pressure by the input volume velocity yields a transfer impedance that shows more clearly how some partials are amplified (resonance) relative to other partials due to their proximity to the peaks of the transfer impedance, which correspond to the formant frequencies of the vocal tracts. The transfer impedance spectrum for the phoneme/Λ/ is depicted in Fig. 6(d).

B. Pressure translation and volume velocity translation

The impedance translation theorem allows one to translate an impedance from one end of a pipe segment to another and include resonance effects within that segment. The authors are unaware of an extension of this approach to translating a pressure from one end of a segment to another or a volume velocity from one end to another. The spatial dependence of the pressure, $p(x)$, in a one-dimensional pipe segment may be described by

$$p(x) = A\cos(kx) + B\sin(kx), \tag{17}$$

where A and B are constants determined by the boundary conditions. The particle velocity, $u(x)$, in the segment may be obtained *via* the one-dimensional, linearized Euler's equation $\rho_0(\partial u/\partial t) = -\partial p/dx$, and by assuming time harmonicity $[p(x,t) = p(x)e^{j\omega t}]$,

$$u(x) = \frac{j}{\rho_0 c}[-A\sin(kx) + B\cos(kx)]. \tag{18}$$

Thus, the spatial dependence of the acoustical impedance, $Z_A(x)$, is

$$\frac{p(x)}{u(x)S} = \frac{p(x)}{U(x)} = Z_A(x) = -j\frac{\rho_0 c}{S}\left[\frac{A\cos(kx) + B\sin(kx)}{-A\sin(kx) + B\cos(kx)}\right], \tag{19}$$

where $U(x)$ is the volume velocity. Consider a pipe that is of length, L, that spans from $x = 0$ to $x = L$. At $x = 0$, Eq. (17) yields $p(x)|_{x=0} = p(0) = A$. At $x = L$, Eq. (19) can be solved for B in terms of $p(0)$ and $Z_A(L)$,

$$B = p(0)\left[\frac{-j\dfrac{\rho_0 c}{S} + Z_A(L)\tan(kL)}{Z_A(L) + j\dfrac{\rho_0 c}{S}\tan(kL)}\right]. \tag{20}$$

Substitution back into Eq. (17) yields an expression for $p(L)$ in terms of $p(0)$ and $Z_A(L)$

$$p(L) = p(0)\left[\cos(kL) + \frac{-j\frac{\rho_0 c}{S} + Z_A(L)\tan(kL)}{Z_A(L) + j\frac{\rho_0 c}{S}\tan(kL)}\sin(kL)\right] = p(0)\left[\frac{Z_A(L)}{Z_A(L)\cos(kL) + j\frac{\rho_0 c}{S}\sin(kL)}\right]. \quad (21)$$

This means that $p(L)$ may be determined once $p(0)$ is known and $Z_A(L)$ is known. $Z_A(L)$ is the impedance seen at $x = L$ when looking in the direction of $x > L$. Recognizing that $Z_A(L) = p(L)/U(L)$ allows $U(L)$ to be determined if $p(0)$ and $Z_A(L)$ are known,

$$U(L) = p(0)\left[\frac{1}{Z_A(L)\cos(kL) + j\frac{\rho_0 c}{S}\sin(kL)}\right]. \quad (22)$$

Expressions for $U(L)$ in terms of $U(0)$ or for $p(L)$ in terms of $U(0)$ may be determined by using the relation $Z_A(0) = p(0)/U(0)$.

Thus, once the impedance translation theorem has determined the input impedances at each segment junction, then $p(0)$ or $U(0)$ on one end of a segment may be used to translate to $p(L)$ or $U(L)$ on the other end of the segment. This process can be repeated for each segment from the vocal folds through to the pressure or volume velocity at the mouth. Then Eq. (15) may be used to compute the far field pressure. Carrying out this process for the /Λ/ phoneme vocal tract yields the exact same far field pressure spectrum shown in Fig. 6(c) and yields the same transfer impedance shown in Fig. 6(d).

C. Auralization of phonemes

The complex, far field pressure spectrum that results from Eq. (15) using either the pressure translation approach or the equivalent circuit approach may be used to create time waveforms that auralize the phonemes modeled. An inverse fast Fourier transform may be used to compute these time waveforms. Programs such as MATLAB can create a symmetric spectrum to use in the inverse fast Fourier transform. The computed frequency spectrum from 0 Hz up to the highest frequency computed is reversed in frequency space and then the complex conjugates of these frequencies are added on as the frequencies in the spectrum from the highest frequency computed, which is now effectively the Nyquist frequency, up to twice the highest frequency computed, which is now effectively the sampling frequency. The resulting time waveforms are steady state simulated sounds of the phonemes. The linked multimedia files provide examples of the sound of three simulated phonemes:

Mm. 1. Auralization of the phoneme /Λ/ produced from the far field pressure computed using either the pressure translation approach or the equivalent circuit approach with a fundamental frequency of 120 Hz. This is a file of type "wav" (172 KB).

Mm. 2. Auralization of the phoneme /i/ produced from the far field pressure computed using either the pressure translation approach or the equivalent circuit approach with a fundamental frequency of 120 Hz. This is a file of type "wav" (172 KB).

Mm. 3. Auralization of the phoneme /a/ produced from the far field pressure computed using either the pressure translation approach or the equivalent circuit approach with a fundamental frequency of 120 Hz. This is a file of type "wav" (172 KB).

D. Assignment assessment

The impedance translation assignment has been given in the first course, as mentioned in the introduction, a total of three different semesters to 18 students. A formal pedagogical study of student learning has not been conducted. However, these 18 students were recently asked for feedback on this specific assignment. None of them are current students in the course and it has been at least a year since each of these students has worked on the assignment. Thus, the students should not feel any pressure to rate the assignment higher because they are still enrolled in the course. The former students were asked "on a scale of 1–10, how much did this assignment help you develop computational skills?" (10 meaning it helped them tremendously). The average numerical response was 6.9, with two of the students giving a 3 and a 5 score because they felt they already had very strong computational skills. The students who responded with a higher number of responses indicated that the assignment forced them to develop their programming skills and ask for help. Then they were asked "on a scale of 1–10, how much did you enjoy the idea of the assignment, to model a realistic system?" (10 meaning they enjoyed it a lot). The average response was 9.1 and several students commented that they really enjoyed the opportunity to study a realistic system and apply what they had learned. Finally, they were asked "On a scale of 1–10, how successful do you think you were in this assignment?" (10 meaning they felt very successful). They responded with an average score of 7.6. Some of them commented that they feel that they could probably be more successful if they tried the assignment over again now that they have developed more programming skills in their research. Many of them commented on how hard the assignment was and some commented that the assumptions they made probably impacted their success with the assignment. Thus, the assignment was challenging, the students enjoyed it, and many said it helped them develop programming skills.

The equivalent circuit assignment has been given in the second course a total of two different semesters to 11 students, though it has been four years since the last time this assignment was given. Author B.E.A. will be regularly teaching this course again and plans to regularly give this assignment moving forward. No assessment of this assignment has been done.

VI. CONCLUSION

This paper has described the basics of the impedance translation theorem and the equivalent circuits modeling techniques. A computational exercise has been described for each of two graduate level courses that each utilize one of these techniques. The human vocal tract was given as an example system that could be solved with either approach. With either technique, the input impedance may be solved for and it was determined that, for this system, the peaks of the input impedance magnitude correspond to the resonances of the vocal tract. These resonances amplify the radiated pressure near the vocal tract resonance (formant) frequencies. A brief discussion has been given of how the definition of resonance can depend on the type of source that is assumed. A technique was presented in which the pressure or volume velocity may be translated through various segments of a system using an impedance translation type approach. Suggestions were given for how one could auralize the phonemes of a particular vocal tract.

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