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Unifying the inertia and Riemann curvature tensors through geometric algebra

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We follow a common thread to express linear transformations of vectors and bivectors from different fields of physics in a unified way. The tensorial representations are coordinate independent and assume a compact form using Clifford products. As specific examples, we present (a) the inertia tensor as a vector-to-vector as well as a bivector-to-bivector linear transformation; (b) the Newtonian tidal acceleration; and (c) the Riemann tensor corresponding to a Schwarzschild black hole as a bivector-to-bivector tensorial transformation. The resulting expressions have a remarkable similarity when expressed in terms of geometric products.
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I. INTRODUCTION

Geometric algebras (also called Clifford algebras) are used to endow physical spaces with a useful algebraic structure. By analyzing the physical system within this context, we can find alternative interpretations of the underlying physics.[1,2] These can simplify computational problems in addition to giving us much more compact and clean notation. In most cases, the final results can be expressed in a coordinate-free way.

An algebra is constructed by endowing a linear space with an additional binary operation called the *product* of the algebra. Although this product is usually non-commutative, it is distributive with respect to the linear space addition, and it is assumed to be associative for our case. With these rules, the idea of matrix multiplication immediately comes to mind. It will actually be useful to keep this picture in mind, as long as we conceive of the algebra's sum and product in an abstract way. An additional and essential condition for the algebra is closure with respect to its product, i.e., the complete algebra must contain all possible products of its elements. Again, in our matrix multiplication reference, this would imply choosing square matrices of fixed size: a product of two $n \times n$ matrices is again an $n \times n$ matrix, in addition to the fact that a linear combination of matrices is again a matrix.

Geometric algebras constitute a specific instance of associative algebras. The constraint imposed on their structure allows us to give concrete geometric interpretations to both the elements and the operations within the algebra.[1,3] In a sense, this is the natural extension of the Cartesian conception of identifying geometry and algebra, and unifying them into a single structure. The geometric building blocks are points, vectors, oriented surfaces, and oriented volumes. The algebraic part relates them in a constructive way and allows us to unify both concepts and equations from different fields of physics.

Our aim in this work is to apply these tools to four specific examples: the inertia tensor interpreted as (i) a second rank tensor and (ii) a fourth rank tensor; (iii) the second rank tidal acceleration tensor; and (iv) the fourth rank Weyl tensor. The main object is to find a common form of expressing all these linear transformations in a coordinate-free way by taking advantage of the Clifford product as well as its intrinsic geometric interpretation. Second rank tensors appear as vector to vector mappings, while the specific fourth rank tensors in our examples map bivectors to bivectors. A further simplification in all these examples arises from the use of symmetry: we consider the case of an axisymmetric rigid body to calculate explicitly the inertia tensor, and we assume spherical symmetry for both the non-relativistic tidal acceleration and the curvature tensor for the Schwarzschild black hole examples.

Section II introduces the main concepts of geometric algebras, as well as the notation that we will need in this work. In particular, we define the algebra associated to the Euclidean three-dimensional space, known as the Pauli algebra. Section III deals with the inertia tensor for a rigid body. The example of a simple rod is presented and the inertia tensor is written in terms of the geometric product as a vector-valued linear transformation of vectors. This is followed by the case of a general axisymmetric body with equivalent expressions. Section IV derives the tidal acceleration for the Newtonian case and the corresponding tensor is shown to have a similar form. In Section V, we present a parallel development in terms of mappings between bivectors for the case of the inertia tensor first and for the Weyl conformal tensor as a generalization of the tidal acceleration. Section VI includes some conclusions regarding the unifying power of geometric algebras in physics.

II. PAULI ALGEBRA

A. Geometric product of vectors

We first want to build up the geometric algebra starting from a physical vector space $\mathbb{V}$ regarded as an underlying part of the larger linear space of the algebra $\mathcal{G}$. We also need to admit a metric defined by the usual dot product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{V}$

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta, \tag{1}$$

where a and b are the magnitudes of the vectors and θ is the angle between them. This operation forces us to include the real numbers $\mathbb{R}$ as a linear subspace of $\mathcal{G}$, providing the Clifford algebra with a *graded* structure where the scalars have grade 0 and the physical vector space $\mathbb{V}$ has grade 1. We next find the elements of grade 2, called *bivectors*, by forming the "wedge" (or exterior, or Grassmann[4] product $\wedge$) of two vectors, thus encoding the plane defined by them. Given that two collinear vectors do not form a plane, $\mathbf{a} \wedge \mathbf{a} = 0$. Taking advantage of the distributivity rule,

$$(\mathbf{a}+\mathbf{b}) \wedge (\mathbf{a}+\mathbf{b}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0, \tag{2}$$

we obtain the antisymmetry property of the wedge product

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}. \tag{3}$$

Furthermore, the area of the parallelogram formed by the two vectors is $ab|\sin\theta|$, so the bivector represents an oriented surface (see Fig. 1).

Clifford's stroke of genius[5] converted Schwartz's inequality (for both the dot and wedge products) to an equality by defining the geometric product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{V}$ as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{4}$$

and then building up the geometric algebra by demanding closure. This geometric product combines zero-grade scalars with second-grade bivectors with a resulting magnitude[6]

$$\|\mathbf{ab}\| = ab. \tag{5}$$

Thus, geometric algebras constrain the symmetric part of the product of two vectors to correspond to their dot product, as is evident in Eq. (4). The antisymmetric part of this product is associative, making the geometric product itself associative.[1,2]

In order to close the algebra, we need to keep incorporating new multivectors of higher grade. The wedge product of two vectors gives a bivector; bivectors can now be wedged with another vector to produce a trivector, and so on. These additional structures represent oriented volumes (and hypervolumes), as illustrated in Fig. 1 and will eventually close the algebra in a finite number of steps due to the antisymmetry property, Eq. (3). With this geometric interpretation, the wedge product turns out to be associative.[1,2,4] Because of their geometric liaison, multivectors are also very useful for interpreting the behaviors of many familiar physical quantities as we will show below.

B. Geometric algebra in 3-d

Our main example is the Clifford algebra $\mathcal{G}_3$ generated by the three-dimensional Euclidean space $\mathbb{V} = \mathbb{R}^3$. This Pauli algebra[7] is eight dimensional and consists of linear combinations of multivectors of grades 0 to 3, i.e., scalars, 3-d vectors, 3-d bivectors, and 1-d trivectors (also called pseudoscalars). The basis element of the real line $\mathbb{R}$ is the number 1. For $\mathbb{R}^3$, we choose an orthonormal basis of unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The advantage of using orthonormality is that we can rely on the more versatile Clifford product in order to construct the subsequent multivector bases. For instance, the bivector basis element $\mathbf{e}_1 \wedge \mathbf{e}_2$ turns out to be the same as the product $\mathbf{e}_1\mathbf{e}_2$ in this case. The eight basis elements of the Pauli algebra are classified by grades in Table I.

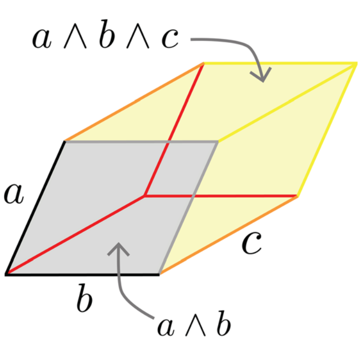

Fig. 1. The wedge product of two vectors $\mathbf{a}$, $\mathbf{b}$ is an oriented area, while the wedge of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is an oriented volume.

Notice that the three resulting unit bivectors square to -1 instead of 1. This follows from their antisymmetry; for example,

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1, \tag{6}$$

and hence $(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -(\mathbf{e}_1\mathbf{e}_1)(\mathbf{e}_2\mathbf{e}_2) = -1$. The same is true for the pseudo-scalar (unit trivector)

$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1. \tag{7}$$

It can also be appreciated from Fig. 1 that the unit trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ represents the (right handed) oriented unit cube. At the same time, we can use Eq. (7), together with the fact that the unit trivector commutes with all the basis elements, to identify it with the imaginary unit i (in an algebraic sense). This is the actual meaning of the third column in Table I.

In summary, we can take i as the basis element of the trivectors, and $\{i\mathbf{e}_k\}$ as the basis of the bivectors, for the Pauli algebra. In other words, every vector $\mathbf{a} \in \mathbb{R}^3$ has a corresponding dual bivector $\mathcal{A} = i\mathbf{a}$, and *vice versa*, $\mathbf{a} = -i\mathcal{A}$. This duality associates a vector $\mathbf{a}$ normal to the surface defined by the bivector $\mathcal{A}$ in a natural way (see Appendix).

For the present example $\mathcal{G}_3$, the duality property is expressed as

$$\mathbf{a} \wedge \mathbf{b} = i\mathbf{a} \times \mathbf{b}, \tag{8}$$

in terms of the unit pseudo-scalar i. Two main features distinguish Grassmann's wedge product from Gibbs's cross product:

(a) the $\wedge$ is well defined for any number of dimensions (as well as for pseudo-Euclidean spaces) and
(b) the $\wedge$ is associative while the $\times$ fulfills Jacobi's identity.

For the particular case of the Pauli algebra, we can thus rewrite the defining Eq. (4) as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b}, \tag{9}$$

in terms of the usual dot and cross products between two three-dimensional vectors.

C. Problem for students

Using the duality results for the Pauli algebra stated in the Appendix show that for any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$,

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = i\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \tag{10}$$

and give the geometric interpretation of this result.

Table I. The graded algebraic structure.

Grade	Basis	"Complex form"	Space
0	1	1	$\mathbb{R}$
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbb{R}^3$
2	$\mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_3\mathbf{e}_1$	$i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3$	$i\mathbb{R}^3$
3	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	i	$i\mathbb{R}$

III. THE INERTIA TENSOR

A. Angular velocity and angular momentum pseudo-vectors

The classical mechanics formula for the angular momentum vector in terms of the mass m, the position vector $\mathbf{r}$, and the velocity $\mathbf{v}$ is $\mathbf{L} = m\,\mathbf{r} \times \mathbf{v}$, with corresponding dual bivector $\mathcal{L} = i\mathbf{L}$. For any particle rotating with angular velocity ω, the tangential velocity is $\mathbf{v} = \omega \times \mathbf{r}$. Consider a rigid body rotating about some axis (see Fig. 2). Each particle will have the same angular velocity ω. Using Eq. (8), the total angular momentum of the rigid body[8] can be written in terms of the Clifford product in Eq. (9) by summing over all the particles $k = 1 \ldots N$

$$\mathbf{L} = \sum_k^N m_k \mathbf{r}_k \times (\omega \times \mathbf{r}_k) = -i \sum_k^N m_k \mathbf{r}_k \wedge (\omega \times \mathbf{r}_k)$$
$$= -i \sum_k^N m_k \mathbf{r}_k (\omega \times \mathbf{r}_k) = \sum_k^N m_k \mathbf{r}_k (\mathbf{r}_k \wedge \omega). \tag{11}$$

In the continuum limit, this becomes

$$\mathbf{L} = \int \mathbf{r}(\mathbf{r} \wedge \omega)\, dm \tag{12}$$

in terms of the mass distribution of the rigid body.

The inertia tensor plays the role of the mass (tensor) for rotational motion:[9] the angular momentum vector $\mathbf{L}$ is obtained as the (scalar) product of the inertia tensor $\mathcal{I}$ with the angular velocity vector ω. In the language of Misner, Thorne, and Wheeler,[10] $\mathcal{I}$ is a "one-slot" machine sending vectors to vectors, so its matrix representation has two indices. In other words, it is a linear mapping $\mathcal{I} : \mathbb{R}_3 \to \mathbb{R}_3$, and hence $\mathbf{L} = \mathcal{I}(\omega)$.[8]

Both the angular velocity ω and the angular momentum $\mathbf{L}$ transform as pseudo-vectors (or axial vectors) with respect to spatial reflections and inversions. Thus, a better description of them is given in terms of their duals, the bivectors $\Omega = i\omega$, and $\mathcal{L} = i\mathbf{L}$, representing the corresponding planes (see Appendix). This relationship will be exploited in Sec. V A.

B. Moment of inertia: Example

Equation (12) defines the inertia tensor as a linear function, i.e., given any vector $\mathbf{A}$, the image vector $\mathbf{B}$ is given by

$$\mathbf{B} = \mathcal{I}(\mathbf{A}) = \int \mathbf{r}\,(\mathbf{r} \wedge \mathbf{A})\, dm, \tag{13}$$

and the matrix elements I_{kl} with respect to the given orthonormal basis $\{\mathbf{e}_k\}$ can be extracted as the projections

$$I_{kl} = \mathbf{e}_k \cdot \mathcal{I}(\mathbf{e}_l). \tag{14}$$

This 3×3 matrix $\{I_{kl}\}$ is symmetric and includes the moments and products of inertia with respect to the original basis.

Let us now look at a concrete simple example. Using Eq. (12), it is straightforward to find the inertia tensor for a rotating rod and write it in a coordinate-free way. Consider a thin rod of length a extending from $-a/2$ to $a/2$ and rotating about an arbitrary axis passing through its center (see Fig. 3). Choosing s as the integration variable, $dm = m\, ds/a$, and $\mathbf{r} = s\hat{\mathbf{n}}$ in terms of the unit vector $\hat{\mathbf{n}}$ along the rod, we have

$$\mathcal{I}(\omega) = \int_{-a/2}^{a/2} s\hat{\mathbf{n}}\,(s\hat{\mathbf{n}} \wedge \omega)\frac{m\, ds}{a} = \frac{ma^2}{12}\hat{\mathbf{n}}(\hat{\mathbf{n}} \wedge \omega). \tag{15}$$

This result can be rewritten in terms of geometric products only, using $\hat{\mathbf{n}} \wedge \omega = (\hat{\mathbf{n}}\omega - \omega\hat{\mathbf{n}})/2$ from Eq. (4). This more symmetric form,

$$\mathcal{I}(\mathbf{A}) = \frac{ma^2}{24}(\mathbf{A} - \hat{\mathbf{n}}\mathbf{A}\hat{\mathbf{n}}) \tag{16}$$

is actually the *leit motif* of the present work. Indeed the second term, $\mathbf{A}' = -\hat{\mathbf{n}}\mathbf{A}\hat{\mathbf{n}}$, has a simple geometric interpretation:[2,8] $\mathbf{A}'$ corresponds to the vector $\mathbf{A}$ reflected with respect to the plane $i\hat{\mathbf{n}}$.

C. Axially symmetric case

Let us next consider the more general case of an axially symmetric body rotating about an arbitrary axis ω. Given that the inertia tensor is symmetric, it can be diagonalized with corresponding orthogonal eigenvectors. The eigenvalues $\{I_1, I_2, I_3\}$ are real numbers and represent the principal moments of inertia.[9] Define $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ as the respective unit vectors along the principal axes of the rigid body, and assume that $\mathbf{f}_3$ is the symmetry axis, so that the two moments of inertia associated with the plane $i\mathbf{f}_3$ are equal, i.e., $I_1 = I_2$.

In the body-fixed basis (see Fig. 4), the inertia tensor can be written in terms of the components of ω as

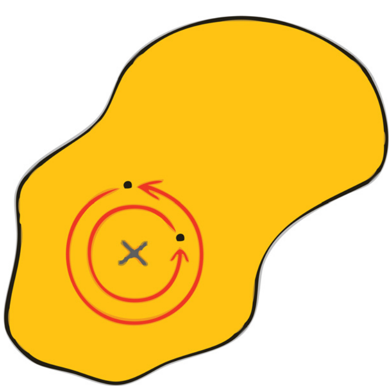

Fig. 2. All points in a rigid body rotate about the rotation axis (here, indicated by an $\times$) at the same angular velocity, ω.

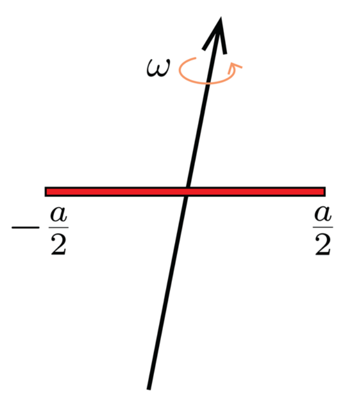

Fig. 3. A thin rod of length a rotating about its center at angular velocity ω.

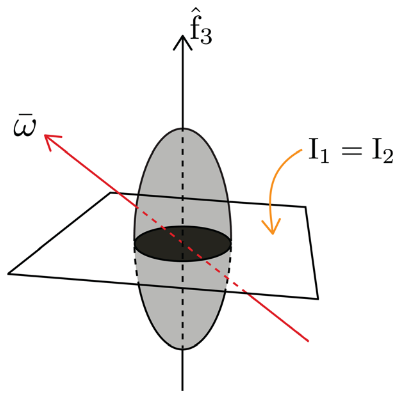

Fig. 4. An axisymmetric rigid body rotated about the arbitrary axis ω. Due to the symmetry of the system, $I_1 = I_2$.

$$\begin{aligned}\mathcal{I}(\omega) &= I_1\omega_1\mathbf{f}_1 + I_2\omega_2\mathbf{f}_2 + I_3\omega_3\mathbf{f}_3 \\ &= I_1(\omega_1\mathbf{f}_1 + \omega_2\mathbf{f}_2) + I_3\omega_3\mathbf{f}_3 \\ &= I_1\omega + (I_3 - I_1)\omega_3\mathbf{f}_3. \end{aligned} \tag{17}$$

The last term contains the component of ω along the symmetry axis and can be rewritten in terms of the geometric product

$$\omega_3 = \omega \cdot \mathbf{f}_3 = \frac{1}{2}(\mathbf{f}_3\omega + \omega\mathbf{f}_3). \tag{18}$$

Substituting back into Eq. (17), we obtain the desired form of the inertia tensor for the case of an axisymmetric body,[8]

$$\mathcal{I}(\mathbf{A}) = \frac{1}{2}(I_1 + I_3)\mathbf{A} + \frac{1}{2}(I_3 - I_1)\mathbf{f}_3\mathbf{A}\mathbf{f}_3, \tag{19}$$

expressed as a linear transformation from vectors to vectors. Written in this form, the inertia tensor for any axisymmetric rigid body appears as a simple generalization of the much simpler case of a rotating rod.

D. Problem for students

Find the inertia tensor for the case of an ellipsoidal body of dimensions a, b, c with uniform mass distribution.

IV. TIDAL FORCES: NEWTONIAN CASE

Our next example comes from a different field of physics: the theory of gravitation. As a prelude to the relativistic case, in this section, we deal with the Newtonian approximation aiming to express the tidal acceleration symmetric tensor in terms of geometric products, following the steps of Sec. III.

Einstein's equivalence principle attests that the motion of a *single* particle does not reveal any details about the space-time curvature.[11] It is the relative motion of two (or more) particles that signals the presence of gravitational attraction. For instance, if two nearby particles are falling toward an attracting gravitational source, the particle closer to the center of attraction will accelerate more. A Newtonian description gives rise to a tidal acceleration tensor involving the spatial vector. Consider two seemingly parallel paths at infinity approaching the spherical, massive body. As these paths near the body, they will begin to converge. Let the measure of this convergence be given by the three-dimensional separation vector χ. Assuming a small separation and expanding in a Taylor series, we can show that the components χ_j obey a second-order differential equation of the form[11]

$$\frac{d^2\chi_j}{dt^2} = -\sum_{k=1}^{3} a_{jk}\chi_k. \tag{20}$$

Equation (20) is known as the Newtonian deviation equation and a_{jk} are the nine components of the (symmetric) tidal acceleration tensor $\mathcal{A}$. This second-rank tensor represents a measure of acceleration for path convergence.

For a point source at the origin, the Newtonian gravitational potential is $\Phi = -M/r$, where M is the mass of the attracting body and r is the radial distance from the origin of symmetry (written in geometrized units, $G = 1$). In this case, the tidal acceleration tensor is

$$a_{jk} = \frac{\partial^2\Phi}{\partial x_j \partial x_k} = \frac{M}{r^3}(\delta_{jk} - 3n_j n_k) \tag{21}$$

in Cartesian coordinates, where δ_{jk} is Kronecker's delta, and $n_j = x_j/r$ are the components of the unit vector in the radial direction $\hat{\mathbf{r}}$.

Let us next rewrite this tensor $\mathcal{A}$ as a vector-to-vector mapping. Given an arbitrary 3-d vector $\mathbf{B}$ with components B_k, we can contract it with Eq. (21) to obtain the coordinate-independent form

$$\mathcal{A}(\mathbf{B}) = \frac{M}{r^3}[\mathbf{B} - 3\hat{\mathbf{r}}(\mathbf{B}\cdot\hat{\mathbf{r}})]. \tag{22}$$

This expression exhibits the dipole form familiar from electrostatics.[12] Using $\mathbf{B}\cdot\hat{\mathbf{r}} = (\mathbf{B}\hat{\mathbf{r}} + \hat{\mathbf{r}}\mathbf{B})/2$ as in Sec. III, we obtain a similar formulation as the one for the inertia tensor in Eq. (16)

$$\mathcal{A}(\mathbf{B}) = -\frac{M}{2r^3}(\mathbf{B} + 3\hat{\mathbf{r}}\mathbf{B}\hat{\mathbf{r}}). \tag{23}$$

So far, in Secs. III and IV, we have expressed the second-rank inertia and Newtonian acceleration tensors in terms of the geometric product, showing a simple symmetric form. In what follows we will deal with the nontrivial case of bivector-valued mappings of bivectors and will find very similar algebraic expressions to the ones above, but with a different geometric interpretation.

V. BIFORMS IN PHYSICS

A. Mapping bivectors to bivectors

When we considered the inertia tensor in Sec. III, we used vectors to define the rotation axes. In order to generalize rotations to higher dimensions, we need to define them with respect to a plane defined by a bivector. For the 3-d Pauli algebra, we can still use Table I and replace the pseudovector ω with the corresponding bivector

$$\Omega = i\omega, \tag{24}$$

which defines the plane perpendicular to the ω axis. This can indeed be interpreted as an imaginary vector in a pure algebraic sense.

On the other hand, we also know that the angular momentum $\mathbf{L}$ behaves as a vector with respect to rotations but not with respect to inversions or reflections. We saw in Sec. III that angular momentum can also be correctly described as a bivector. We thus define the bivector

$$\mathcal{L} = \mathbf{r} \wedge \mathbf{p} = i\mathbf{r} \times \mathbf{p} = i\mathbf{L}. \tag{25}$$

For the case of a rigid body, we have to integrate over the mass distribution (or sum over all the particles in the discrete case) as in Eq. (12)

$$\begin{aligned}\mathcal{L} &= i\int \mathbf{r}(\mathbf{r} \wedge \omega)\, dm = i\int \frac{1}{2}(r^2\omega - \mathbf{r}\omega\mathbf{r})\, dm \\ &= \int \frac{1}{2}(r^2\Omega - \mathbf{r}\Omega\mathbf{r})\, dm.\end{aligned} \tag{26}$$

Using Eq. (A3) from the Appendix, the first line in the equation above can be rewritten in terms of the angular velocity bivector $\Omega = i\omega$ as

$$\mathcal{L} = \int \mathbf{r}(\mathbf{r} \cdot \mathbf{\Omega})\, dm, = -\int \mathbf{r}(\mathbf{\Omega} \cdot r)\, dm \tag{27}$$

involving the contraction of the vector $\mathbf{r}$ with the bivector Ω, which turns out to be antisymmetric (see Appendix).

The inertia tensor now becomes a *biform*,[13] i.e., a bivector-valued linear transformation of bivectors. Thus, the inertia tensor $\mathcal{I}$ is reinterpreted as mapping the plane defined by a bivector $\mathcal{B}$ to a new plane $\mathcal{C}$

$$\mathcal{C} = \mathcal{I}(\mathcal{B}) = \int \mathbf{r}(\mathbf{r} \cdot \mathcal{B})\, dm = \int \frac{1}{2}(r^2\mathcal{B} - \mathbf{r}\mathcal{B}\mathbf{r})\, dm. \tag{28}$$

This in turn leads directly to the analogue of Eq. (19),

$$\mathcal{L} = \mathcal{I}\Omega = \frac{I_1 + I_3}{2}\Omega + \frac{I_3 - I_1}{2}\mathbf{f}_3\Omega\mathbf{f}_3, \tag{29}$$

as a mapping from bivectors to bivectors for the axisymmetric case. We would like to emphasize that although this last equation looks almost identical to Eq. (19), they are conceptually different. While the latter refers to a second rank tensor mapping vectors to vectors, Eq. (29) is a biform, i.e., a bivector-valued transformation of a bivector.

B. The Riemann curvature tensor

The following three subsections make up a series of stepping stones aiming to express the Weyl conformal tensor as a biform using geometric products.[2] In Sec. V B we give a general introduction to curvature, the Riemann curvature tensor, and its relation with the Weyl tensor.[10] In Sec. V C we concentrate on the latter's specific properties, while Sec. V D relates both curvature tensors to general relativity for the case of the Schwarzschild solution.

We usually think of a surface as being curved from the way it warps or bends in our 3-d space. Let us suppose that we are constrained to a curved 2-d surface, such as the surface of a sphere, without access to the third dimension. In this case, we would like to be able to determine whether or not our surface is curved in an *intrinsic* fashion. A way to quantify this is to calculate the curvature tensor (or Riemann tensor). It allows us to extend the notion of curvature to

(a) more than two dimensions;
(b) each point in the curved space;
(c) different directions; and
(d) Lorentzian metrics.

The Riemann tensor $\mathcal{R}$ accounts for the rotation of a vector as it travels along a closed path while always pointing in the same direction. The more the transported vector differs from the original one, the larger the curvature. On a planar surface, it is easy to set the pointing direction of any vector and keep it fixed as we travel along a circle or a square. The resulting vector points in the same direction as the original, there is no net rotation, and hence the curvature vanishes independently of the chosen direction and the chosen coordinates.

For the more general case of a higher dimensional curved space, we need to follow an infinitesimal path in order to obtain the *local* curvature at each point.[14] The chosen small closed path defines a two-dimensional surface. We could, for instance, choose two coordinates u and v and form a four-sided (oriented) loop of sides Δu and Δv. Let us next choose a vector $\mathbf{X}$ and parallel-transport it along the closed loop to the rotated final vector $\mathbf{X}'$. The bivectors $\hat{\mathbf{u}} \wedge \mathbf{X}$ and $\hat{v} \wedge \mathbf{X}'$ define two different planes. The Riemann tensor maps one bivector onto the second bivector. In other words, the curvature tensor $\mathcal{R}$, in any number of dimensions $d \geq 2$, is a linear biform mapping bivectors to bivectors

$$\mathcal{R} : \text{bivectors} \rightarrow \text{bivectors}. \tag{30}$$

Hence, $\mathcal{R}$ is a fourth-rank tensor.

In the theory of general relativity, the Riemann tensor $\mathcal{R}$ can be considered as a "tidal acceleration" tensor[10] generalizing the corresponding Newtonian tensor appearing in Eq. (20) to the 4-d curved space-time.

A more restricted measure of the curvature is given by the Weyl conformal tensor $\mathcal{W}$. It measures the tidal acceleration taken along a geodesic and detects only the distortion of the shape of the body but carries no information on its change of volume. Indeed it is the only part of the curvature that exists in free space. In two and three dimensions, the Weyl tensor vanishes identically. In four or more dimensions, $\mathcal{W}$ corresponds to the traceless part of the Riemann tensor $\mathcal{R}$. The difference between these two tensors is expressed in terms of the second-rank Ricci tensor mapping 4-d vectors to 4-d vectors.[10]

In order to study the Riemann tensor and its trace in four dimensions, we would have to consider the corresponding 16-dimensional algebra instead of the more limited 8-dimensional Pauli algebra. However, given the special properties of $\mathcal{W}$, it is possible and convenient to express it in terms of a combination of vectors and bivectors of the Pauli algebra. The more complete derivation starting from a space-time split of bivectors[15] and considering biforms within the 16-dimensional Dirac algebra is given in detail elsewhere.[2] In this paper, we exploit the fact that the Pauli algebra is isomorphic to the even subalgebra of the Dirac algebra.[15]

C. The Weyl tensor

In Sec. III, we found the principal moments as the eigenvalues of the inertia tensor $\mathcal{I}$. Given that this transformation is real and symmetric, there is no question about its three eigenvalues being real. Knowing these eigenvalues allowed us to write $\mathcal{I}$ in a simple form, especially in the axisymmetric case. This is true even when we consider $\mathcal{I}$ as a biform, as in Eq. (29).

Our aim now is to generalize this procedure to the case of a curvature biform in a four-dimensional curved space. In this case, the local (tangent) space turns out to be Minkowski space, $\mathbb{M}^4 = \mathbb{R}^{1,3}$. As is well known, the $(1+3)$-d flat space-time of special relativity is Lorentzian, meaning that the "distance" Δs between two events (i.e., two points in this space) acquires a hyperbolic character

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta \mathbf{r})^2, \tag{31}$$

where $\mathbf{r}$ refers to vectors in the usual 3-d space.

We now have six basis bivectors that can be separated into two different types:

(a) three spatial bivectors corresponding to the three orthogonal spatial planes whose square is -1 and
(b) three spatio-temporal planes where one of the defining axes is time and whose square is $+1$.

Strictly speaking, these six bivectors form part of the larger Dirac algebra $\mathcal{G}_{1,3}$.[15] However, if we choose the time axis to coincide with the direction of the observer, we can identify[15,16] the set in (b) with the 3-d basis vectors in Table I, while (a) coincides with the 3-d bivectors $\{i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3\}$. From this point of view, Dirac bivectors correspond to Pauli "complex" vectors. Within this particular reference frame, it turns out to be sufficient to restrict ourselves to the $\mathcal{G}_3$ algebra in order to study Lorentz transformations and the Weyl conformal tensor. As mentioned above, the Ricci tensor and the complete Riemann tensor require, in addition, the inclusion of mappings between 4-d vectors absent in the Pauli algebra.

Incidentally, given the hyperbolic nature of Minkowski space, we can also construct "null elements" whose squares vanish (i.e., they are nilpotent), such as $\mathbf{e}_1 \pm i\mathbf{e}_3$. These turn out to play an important role in describing the paths followed by light and have been extensively used in the Newman-Penrose description of general relativity.[17]

In this paper, we shall follow a different path: instead of considering six basis elements for the bivectors in the tangent $\mathbb{M}^4$ space, we restrict ourselves to a real 3-d vector local basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and allow for "complex" coefficients. Within this frame, the biform corresponding to the Weyl conformal tensor $\mathcal{W}$ can now be interpreted as a linear mapping between complex 3-d vectors. In a way, this is a generalization of Sec. V A above where the inertia tensor appeared as a linear transformation between "imaginary" 3-d vectors. Although $\mathcal{W}$ is still symmetric as a linear transformation, it turns out to be complex symmetric so there is no longer a guarantee that the corresponding eigenvalues are real.

Regarding the application of these ideas to general relativity, we can confine ourselves to the source-free case of Einstein's equation where the Riemann tensor $\mathcal{R}$ coincides with the Weyl tensor $\mathcal{W}$. The reason is that the difference between the two tensors involves the Ricci tensor, which vanishes in this case.[10]

The Weyl tensor $\mathcal{W}$ has the following useful properties[2,10,18] as a linear transformation:

(a) $\mathcal{W}$ is complex symmetric;
(b) $\mathcal{W}$ is traceless;
(c) $\mathcal{W}$ is self-dual, i.e., $\mathcal{W}(i\mathcal{B}) = i\mathcal{W}(\mathcal{B})$ for any complex vector $\mathcal{B}$.

In other words, $\mathcal{W}$ can be represented as a 3×3 complex symmetric, traceless matrix $\mathbf{W}$ with respect to the given 3-d basis

$$W^{kl} = W^{lk};$$
$$\mathrm{Tr}(W) = 0;$$
$$\mathcal{W}(\mathcal{B}) = \sum_{kl} W^{kl}\mathbf{e}_k\mathcal{B}\mathbf{e}_l. \tag{32}$$

In analogy to the principal moments of inertia, we can solve the eigenvalue problem,

$$\mathcal{W}(\mathcal{V}) = \lambda\mathcal{V}. \tag{33}$$

There are three complex eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ associated to each point of the curved space, with three corresponding eigenvectors $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$. Given that $\mathbf{W}$ is traceless, the sum of the three eigenvalues vanishes so only two of them are independent. Comparing with the inertia tensor in Sec. V A above, we notice that the complex eigenvectors include in general an additional real part, absent in the $\mathbb{R}^3$ case. This is indeed due to the presence of additional space-time planes in $\mathbb{M}^4$.

The particular case in which the three eigenvectors span the entire 3-d space[14,17–19] is especially important from the point of view of Einstein's general theory of relativity. In other words, there are no null eigenplanes in this case and we can choose three orthonormal (principal) vectors, $\mathcal{V}_i = \mathbf{f}_i$ with $\mathbf{f}_i^2 = 1$ for $i = 1, 2, 3$. First, let us assume that the three eigenvalues are all different. Using the identity

$$\sum_i \mathbf{f}_i\mathcal{B}\mathbf{f}_i = -\mathcal{B}, \tag{34}$$

valid for any complex vector $\mathcal{B}$, and taking into account the traceless nature of $\mathcal{W}$, we can eliminate λ_3 and expand it[13] using only two eigenvalues

$$\mathcal{W}(\mathcal{B}) = \left(\mathbf{f}_1\mathcal{B}\mathbf{f}_1 + \frac{1}{3}\mathcal{B}\right)G_1 + \left(\mathbf{f}_2\mathcal{B}\mathbf{f}_2 + \frac{1}{3}\mathcal{B}\right)G_2, \tag{35}$$

where $G_1 = 2\lambda_1 + \lambda_2$ and $G_2 = 2\lambda_2 + \lambda_1$ are functions of the space-time event depending on the curvature properties of the $1+3$-dimensional curved space.

D. The Riemann tensor for the Schwarzschild solution

In Einstein's theory of general relativity, the space-time dependent metric tensor[10,11] is usually taken as the physical object, characterizing the gravitational system by defining a measure of the space-time deformation. By the same token, the curvature tensor $\mathcal{R}$ is the crucial geometric component and it plays the role of a gravitational field strength, given that a nonvanishing curvature tensor implies the presence of gravitation.[20] The curvature tensor corresponds to the tidal acceleration tensor whose Newtonian limit is given by Eq. (20).

For the source-free case, $\mathcal{R}$ and $\mathcal{W}$ coincide. Thus, Eq. (35) above is a canonical form of the Riemann tensor for a whole class of solutions of Einstein vacuum equations.[19] In particular, if there is a geometric symmetry involved, we can follow the same procedure as in Sec. III for the inertia tensor. This will allow us to eliminate one more term in Eq. (35) and rewrite it as

$$\mathcal{W}(\mathcal{B}) = G(t, \mathbf{r})\left(\mathbf{f}_1 \mathcal{B} \mathbf{f}_1 + \frac{1}{3}\mathcal{B}\right), \tag{36}$$

where the explicit form of G is determined by the specific physical problem.

The simplest nontrivial case is that of a static, spherically symmetric space-time. The Schwarzschild solution[10,11] to Einstein's vacuum equations follows from the following (somewhat redundant) premises:

(a) no time dependence of G;
(b) spherical symmetry, $G = G(r)$;
(c) behaves locally as $\mathbb{M}^4$; and
(d) coincides asymptotically with the Newtonian case.

Following the ideas developed in Sec. III, we can now choose the unit principal vector $\mathbf{f}_1$ as corresponding to spherical symmetry, i.e., $\mathbf{f}_1 = \hat{\mathbf{r}}$ according to (b) above. The Riemann tensor for the Schwarzschild solution can be expressed as a biform in our now-familiar form[2]

$$\mathcal{R}(\mathcal{B}) = \mathcal{W}(\mathcal{B}) = -\frac{3M}{2r^3}\left(\hat{\mathbf{r}} \mathcal{B} \hat{\mathbf{r}} + \frac{1}{3}\mathcal{B}\right), \tag{37}$$

where M is the black hole mass. It differs from Eq. (23) only in the fact that $\mathcal{W}$ maps tangent planes to tangent planes in the curved space of general relativity instead of the simpler Cartesian 3-d space.

VI. DISCUSSION

The geometric interpretation of bivectors as oriented surfaces in Grassmann algebras becomes especially relevant in Minkowski 4-d space-time and associated curved spaces, where we have to distinguish between purely spatial and time-space mixed surfaces. By including a metric and the corresponding dot product in Clifford algebras, we can extend this geometric advantage to an algebraic one as well. Bivectors in $\mathbb{R}^3$ can be treated as imaginary 3-d vectors eliciting a duality relation between both sets. Adding time as a coordinate in Minkowski space $\mathbb{M}^4$ produces three more planes with a hyperbolic geometry. The corresponding unit bivectors square to 1 (instead of -1) and hence can be identified with the $\mathbb{R}^3$ basis vectors, at least as long as we choose the observer's reference frame. This space-time split[15] allows us to refer all possible $\mathbb{M}^4$ bivectors to the Pauli algebra as defined in Table I. Algebraically, we treat the linear combinations of the six independent Dirac bivector basis elements as complex 3-d vectors within $\mathcal{G}_3$. Thus, antisymmetric tensors (such as the electromagnetic field) can be rewritten as complex 3-d vectors.[15,16]

The final expressions in Secs. III and IV are written in terms of Clifford products. Given that they correspond to linear mappings of vectors, they can be expressed in terms of dot products as well, i.e., projecting the vector along the chosen fixed direction. Equations (17) and (22) have this form. In the case of biforms, i.e., tensors mapping bivectors to bivectors, this is not so simple given that the product of two bivectors is a linear combination of scalars, bivectors, and tetravectors corresponding to dot products, commutators, and wedge products.[2] So, in this respect, Eqs. (29) and (35) are true canonical representations for the fourth-rank axisymmetric inertia tensor and Weyl tensor for the cases considered in this work.

APPENDIX: PSEUDO-VECTORS AND BIVECTORS

In the usual 3-d Euclidean geometry, the cross product of two vectors turns out to be a pseudo-vector in the sense that the resulting vector does not change sign with respect to a spatial inversion, while a regular vector does. Typical examples are the angular momentum, the torque, and the magnetic field. The angular velocity (being proportional to the angular momentum in the simplest case) is also a pseudo-vector.

The concept of a pseudo-vector can be generalized to any number of dimensions (and pseudo-Euclidean spaces) by shifting our geometric perspective. Instead of looking at the directional aspect of the pseudo-vector, we can convey the same information by defining the bivector dual to the pseudo-vector. This bivector defines the plane perpendicular to the pseudo-vector. Unlike the concept of a pseudo-vector, the bivector (and its corresponding plane) is well defined in any number of dimensions. From the tensorial point of view, the bivector corresponds to a second-rank antisymmetric tensor.

For instance, in the case of the motion of a planet with constant angular momentum $\mathbf{L}$, its dual bivector $\mathcal{L} = i\mathbf{L}$ defines the plane of planetary motion through the wedge product, Eq. (25). The angular velocity bivector Ω in Eq. (24) thus describes the dynamics of the rotational motion of a plane independently of the dimension of the vector space.

The wedge product and the dot product in Eq. (4) have entirely different geometric interpretations and their algebraic properties are also different. Hence, it is necessary to consider the commutation relation of the pseudo-scalar i with respect to each of them separately. In the specific case of the Pauli algebra $\mathcal{G}_3$, i commutes with every element of the algebra with respect to the geometric product.[21] However, when we consider the dot and the wedge products on their own, this is no longer true. Indeed the duality relation of Eq. (8) can be generalized.[2] Given two 3-d vectors $\mathbf{a}$ and $\mathbf{b}$, we can consider the contraction of the vector $\mathbf{a}$ with the bivector $\mathbf{b}i$

$$\mathbf{a} \cdot (\mathbf{b}i) = (\mathbf{a} \wedge \mathbf{b})i. \tag{A1}$$

The result of this contraction (dot product) is a vector. This also shows explicitly that we cannot "pull out" i when dealing with the dot or the wedge product separately, in spite of the fact that this is a valid operation with respect to the full geometric product.

We also have that $(\mathbf{b}i) \cdot \mathbf{a} = (\mathbf{b} \wedge \mathbf{a})i = -\mathbf{a} \cdot (\mathbf{b}i)$, and in general the dot product of a vector and a bivector is antisymmetric,

$$\mathbf{a} \cdot \mathcal{B} = -\mathcal{B} \cdot \mathbf{a}, \tag{A2}$$

instead of being symmetric as in the usual case of two vectors ($\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

For the wedge product, duality reads

$$\mathbf{a} \wedge (\mathbf{b}i) = (\mathbf{a} \cdot \mathbf{b})i, \tag{A3}$$

confirming the fact that the wedge product of a vector with a bivector is a pseudo-scalar in $\mathcal{G}_3$.

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Hand Centrifuge

Internal gearing is used to rotate the two balanced specimen vials at a high angular velocity. This example is in the collection of historical scientific apparatus at Creighton University, and probably dates from the nineteen twenties. (Notes by Thomas B. Greenslade, Jr., Kenyon College; photograph by Vacek Miglus, Wesleyan University)