Evolution of statistical properties for a nonlinearly propagating sinusoid

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Abstract: The nonlinear propagation of a pure sinusoid is considered using time domain statistics. The probability density function, standard deviation, skewness, kurtosis, and crest factor are computed for both the amplitude and amplitude time derivatives as a function of distance. The amplitude statistics vary only in the postshock realm, while the amplitude derivative statistics vary rapidly in the preshock realm. The statistical analysis also suggests that the sawtooth onset distance can be considered to be earlier than previously realized.

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1. Introduction

Nonlinear effects in acoustic propagation are caused by both convection and local temperature changes in the rarefaction and condensation portions of the wave, which increase or decrease the effective sound speed. This, in turn, causes a sinusoidal wave to become steepened and, in the absence of sufficient absorption, eventually form into a sawtooth wave with weak shocks. Fubini[1] derived an analytical solution for the finite-amplitude sinusoid in terms of an infinite summation of harmonics valid in the preshock realm. Blackstock[2] expanded this derivation to go from the preshock region, through the transition region and into the sawtooth realm, where the Fay solution[3] holds.

Although less common than time or frequency domain analyses, the statistics of finite-amplitude waves have been used to characterize their nonlinear behavior. Webster and Blackstock[4] performed a theoretical analysis of nonlinear distortion on amplitude density revealing that the amplitude density does not significantly change in the preshock realm. The amplitude density is the probability of occurrence of given amplitudes within the waveform and is defined by Webster and Blackstock in terms of velocity as

$$\tilde{P}(u) = \lim_{\Delta u \to 0} \frac{\sum_{i=1}^{N} \Delta t_i}{T \Delta u}, \tag{1}$$

where T is the sample length and N is the number of intervals Δt in which the signal is between the velocities u and $u + \Delta u$.

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Sakagami *et al.*[5] also used changes in the probability distribution in a study of nonlinear propagation in a duct and observed a shift in the distribution of Gaussian noise toward a more uniform distribution. McInerny[6] employed higher-order time domain statistics including skewness, kurtosis and crest factor to examine nonlinearity in measured rocket noise. Later McInerny and Olcmen[7] used a simple finite-difference estimate to approximate the pressure time-rate-of-change in a rocket noise waveform and showed that it was more sensitivity to nonlinearity than the pressure waveform statistics. Additionally Gee *et al.*[8] showed that the perception of jet noise crackle is related to amplitude time derivative statistics.

To complement the previously developed theories[1–3] for the propagating finite-amplitude sinusoid, its statistical evolution is presented in this Letter. The results are provided as a function of σ, which is the ratio of the distance and the shock formation distance for an initial sinusoid. Specifically, the normalized pressure amplitude density, standard deviation, skewness, kurtosis, and crest factor are shown to have little variation up to approximately $\sigma = 1.3$, after which there is significant deviation for all measures except skewness. The statistical evolution of the time derivative of the waveform is also shown in the preshock region, revealing rapid variations for $\sigma < 1$. These results also show that from a statistical standpoint, the onset of the sawtooth distance, which is traditionally set at $\sigma = 3$,[2] can be considered to onset prior to $\sigma = 3$.

2. Statistics of amplitude

The transition of a sinusoid to a sawtooth wave is described as a Fourier series by the well-known Blackstock bridging function.[2] The two limiting cases of the bridging function have more compact analytical solutions, but rarely are discussed in terms of the amplitude distribution. Figure 1 shows a unit sine wave and sawtooth wave with unit period along with their respective amplitude density functions. A sinusoid has a bimodal distribution with peaks where the waveform slope changes sign and a trough at the region of nearly constant slope. The sawtooth wave has a uniform distribution.

The Blackstock bridging function was used to create a sinusoid propagating from $\sigma = 0$ to 7 to investigate how the amplitude density evolves from sine wave to sawtooth. For the special case of a pure sinusoid plane wave, $\sigma = 1$ signifies that the waveform has steepened sufficiently so that a discontinuity has formed. The Fourier series harmonic amplitudes are defined according to

$$B_n = \frac{2}{n\pi} P_{\mathrm{sh}} + \frac{2}{n\pi\sigma} \int_{\Phi_{sh}}^{\pi} \cos[n(\Phi - \sigma \sin \Phi)] d\Phi, \qquad (2)$$

where P_{sh} is the pressure shock amplitude, n is an integer describing the harmonic number, and Φ is the phase. The shock amplitude is found by solving the transcendental equation

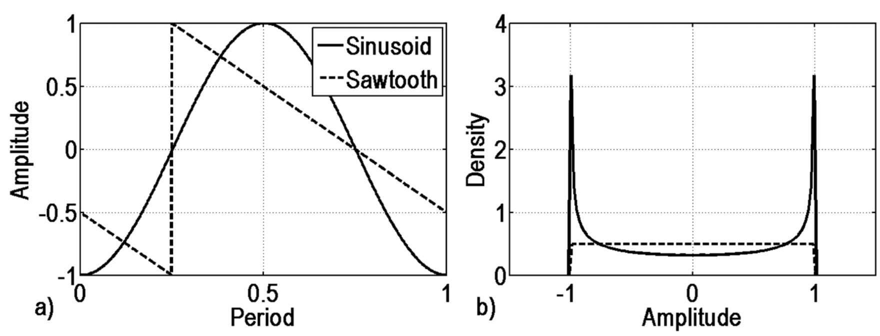

Fig. 1. (a) A single period of a unit sine (—) and sawtooth (---) wave and (b) their amplitude density functions. The sinusoid has a bimodal distribution while the sawtooth wave has a uniform distribution.

$$P_{sh} = \sin(\sigma P_{sh}), \tag{3}$$

which asymptotically approaches $\pi/(1+\sigma)$ for $\sigma \gg 1$.[9] To ensure adequate resolution at the shock front, 686 points per cycle were used in the simulations with the summation truncated at 20,580 harmonics.

Multimedia 1 shows an animation of a single period of a finite-amplitude sine wave as it distorts, forms into a weak shock, and then undergoes rapid amplitude decay. The amplitude is normalized by the initial sinusoidal amplitude and the initial waveform remains on the plot in dotted lines for comparison. The lower portion of Mm. 1 shows the corresponding estimate of the amplitude probability density function (PDF) computed using the kernel smoothing method[10] with a bandwidth of 0.025. The kernel smoothing estimate causes the front and back rise of the PDF to be finite but is necessary since the function is piecewise continuous. The smoothing bandwidth was determined by trial and error to lessen the effect of Gibbs phenomenon on the PDF for $\sigma > 1$ but still maintain the major characteristics.

Mm. 1. (Upper) A finite-amplitude sinusoid propagates from $\sigma = 0 - 7$ with the initial waveform in dotted (–). (Lower) An estimate of the amplitude probability density function (PDF) for the wave. The PDF remains relatively unchanged until $\sigma = 1.3$ after which it transitions to a uniform distribution. The width of the PDF narrows as a result of decay of the shock amplitude while the peak increases due to the decreasing negative slope between the shock fronts. This is a file of type "gif" (400 kb).

As Mm. 1 shows, the PDF remains unchanged until $\sigma = 1$, which has been shown previously by Webster and Blackstock,[3] but also remains relatively unchanged until $\sigma = 1.3$. Between approximately $\sigma = 1.3$ and 3.0, the PDF transitions quickly from being bimodal to being approximately uniform (with small disturbances due to the truncation and smoothing). Beyond $\sigma = 3.0$, the PDF shape remains uniform, which is characteristic of a perfect sawtooth wave, but the width of the PDF decreases and the peak of PDF increases (such that its integral remains unity). These two effects are due to the decay of the weak shocks and the decreasing negative slope between the shock fronts respectively.

Statistics of the amplitude density were then computed for the distorting sinusoid as a function of σ. The definitions for the standard deviation, skewness, kurtosis and crest factor are shown in Table 1 with the limiting values for a sinusoid and sawtooth. The skewness and kurtosis capture the symmetry and peakedness of the distribution, respectively. Figure 2 displays each statistical value for the finite-amplitude sinusoid as a function of normalized distance. Since the PDF estimate does not change until after $\sigma = 1$, the discussion will focus on the $\sigma > 1$ region. The standard deviation drops off slowly due to its dependence on the decreasing peak value. The roll-off most closely follows the decay of the first harmonic, shown in Fig. 2 as a dashed line. This

Table 1. Definitions of standard deviation, skewness, kurtosis, and crest factor with their respective limiting cases (sinusoid and sawtooth). $E[\cdots]$ denotes the expectation operator over n samples; x_i, the sampled value; and pk, the peak value.

	Definition (sample)	Sinusoid	Sawtooth
Standard deviation (μ)	$\sqrt{\frac{1}{n-1}\sum_i^n (x_i - E[x])^2}$	$\frac{\sqrt{2}}{2}x_{pk}$	$\frac{\sqrt{3}}{3}x_{pk}$
Skewness	$\frac{1}{n-1}\sum_i^n \frac{(x_i - E[x])^3}{\mu^3}$	0	0
Kurtosis	$\frac{1}{n-1}\sum_i^n \frac{(x_i - E[x])^4}{\mu^4}$	$\frac{3}{2}$	$\frac{9}{5}$
Crest factor	$\frac{x_{pk}}{\mu}$	$\sqrt{2}$	$\sqrt{3}$

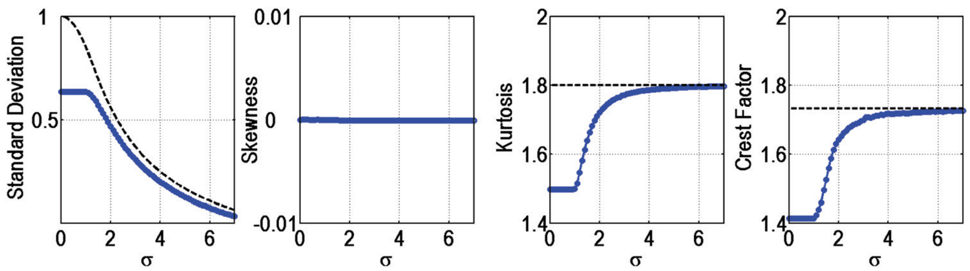

Fig. 2. (Color online) Statistical measures of a finite-amplitude sinusoid as a function of σ. All quantities are constant until the shock formation distance is reached. After shock formation, the standard deviation rolls off similar to the first harmonic amplitude (dashed line) while kurtosis and crest factor asymptote up to constant values (dashed lines) and the skewness remains nearly zero.

reveals that the RMS amplitude becomes dominated by the first harmonic amplitude as σ increases. The skewness remains effectively zero over the duration. The kurtosis increases and asymptotically approaches 9/5 (shown as a dashed line), the kurtosis value for a perfect sawtooth wave (see Table 1). The crest factor also increases and asymptotes to a value of $\sqrt{3}$.

The sawtooth region has traditionally been set at $\sigma = 3.0$ since the error between the exact solution for the shock amplitude and the large σ limit is only 3.4%.[8] However, it may be more useful to describe the sawtooth onset distance statistically by examining the percentage difference from the statistical values of a sawtooth wave. By computing the difference in statistical metrics between Eq. (3) and a sawtooth wave, the sawtooth onset region could actually be considered to occur prior to $\sigma = 3$ when maintaining a 3.0% error tolerance. Figure 3 shows the percent error in standard deviation (which is equivalent to the error in rms amplitude), kurtosis and crest factor. A box indicates the value of σ where the error becomes less than 3.0%. Because this region occurs approximately at $\sigma = 2.6$, 2.4, and 2.5 for standard deviation, kurtosis, and crest factor, respectively, the sawtooth region from a statistical perspective could be considered to onset at approximately $\sigma = 2.5$. The average error at $\sigma = 3$ is approximately 1.75%.

3. Statistics of amplitude time derivative

As shown by McInerny,[6] time waveform amplitude derivatives (i.e., slopes) are very sensitive to waveform distortion and are therefore useful in indicating nonlinearity. This is a result of the distortion being distinctly characterized by a dramatic slope increase in localized regions of the waveform. The distortion both preshock and postshock formation will be manifest in the amplitude derivative statistics, thus making it more sensitive to nonlinearity than amplitude statistics. However, the amplitude

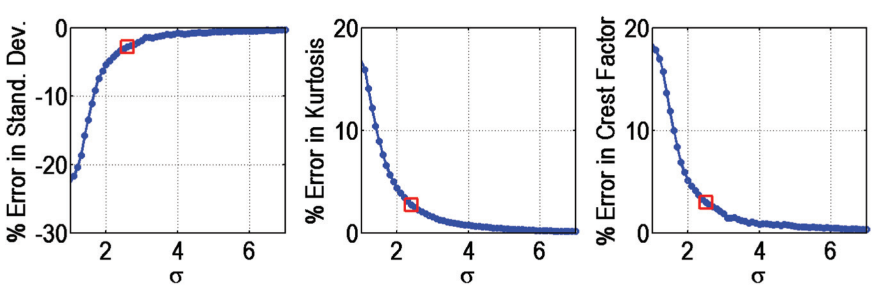

Fig. 3. (Color online) Percent error in standard deviation (rms amplitude), kurtosis, and crest factor between Eq. (3) and a sawtooth wave as a function of normalized distance. Boxes indicate the onset of less than 3% error.

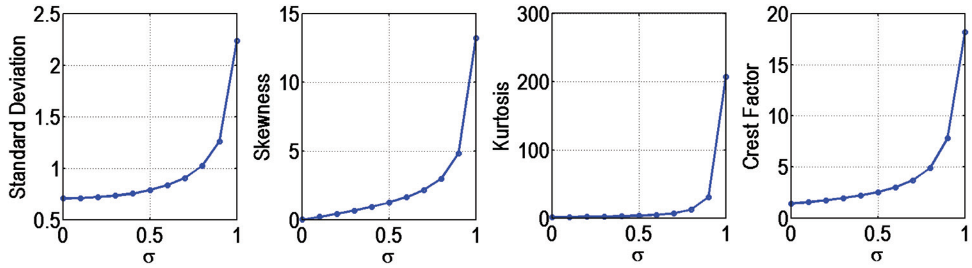

Fig. 4. (Color online) Statistical measures of the time rate of change of a finite-amplitude sinusoid as a function of σ. All quantities increase with distance, the largest increase being near $\sigma = 1$.

derivative will also be sensitive to Fourier series truncation effects, particularly when discontinuities are present. Since subsequent density estimates can be corrupted by this effect, the following time derivative analysis is restricted to the $\sigma < 1$ region.

The amplitude derivative was computed for the finite-amplitude sinusoid in Fig. 1 using a second-order finite difference scheme. Multimedia 2 shows an animation of the amplitude slope from $\sigma = 0 - 1$ with its corresponding PDF estimate. The amplitude derivative begins as a cosine wave but quickly diverges due to the slope increase caused by waveform steepening. A large peak forms at the waveform maximum where the steepening is most pronounced. The amplitude derivative at all points other than the maximum uniformly approach the value of -0.5.

Mm. 2. (Upper) The amplitude derivative for a finite-amplitude sinusoid propagating from $\sigma = 0 - 1$. The amplitude derivative distorts drastically due to the increase in slope at the amplitude maximum. A peak forms where the waveform discontinuity will develop. (Lower) The PDF estimate for the amplitude derivative. The PDF approaches a large distribution of same value (since the time change is constant for most of the waveform) with infrequent extreme outliers, corresponding to the regions of maximum steepening. This is a file of type "gif" (400 Kb).

The amplitude derivative PDF estimate at $\sigma = 0$ is the same as the original waveform PDF since sine and cosine waves have the same density over an equal period. After $\sigma > 0$, the PDF estimate changes dramatically with a large peak forming near zero due to the majority of the slope being constant valued. The derivative peaks represent such a small percentage of the waveform slope that they have little impact on the PDF.

Figure 4 displays the standard deviation, skewness, kurtosis and crest factor of the amplitude derivative up to $\sigma = 1$. All four quantities increase significantly up to $\sigma = 1$ with the kurtosis value exceeding 100. The increase is most significant near $\sigma = 1$. It is clearly evident that each quantity is very sensitive to nonlinear distortion for a sinusoid in the preshock realm. This is a result of the higher-order metrics amplifying the outliers. Skewness shows the earliest detection of nonlinearity while kurtosis shows the most significant detection since the percent change from $\sigma = 0.8$ to $\sigma = 1$ is over 700%.

4. Concluding discussion

It has been shown that statistics can be used to compliment previously developed theory on the nonlinear propagation of a sinusoid to provide additional insight into the nature of the propagation. Amplitude and amplitude derivative statistics were shown to be sensitive to nonlinear propagation in the postshock realm and the preshock realm, respectively. It was shown that the pressure amplitude PDF remains relatively unchanged up to $\sigma = 1.3$, confirming and extending previous work.[1] Additionally, a statistical method for determining the onset of the sawtooth realm shows an earlier onset than previously realized. The traditional means for determining the sawtooth onset was found by comparing the asymptotic value of the shock amplitude to the exact value.[9] Since direct comparison between the Blackstock bridging function and a sawtooth wave will include Fourier series truncation effects, a statistical approach is perhaps a more consistent methodology to account for differences in waveform shapes and amplitude. The sawtooth onset distance of approximately $\sigma = 2.5$ was shown to be within an error tolerance of 3% for the statistical metrics standard deviation, kurtosis and crest factor.

This Letter also reinforces the potential of using the derivative statistics for detecting and characterizing nonlinearity. Similar investigations could be carried out on sinusoidal or arbitrary noise waveforms propagating through lossy media to describe the preshock, postshock, and old-age regions statistically. Physical and numerical experiments could show sensitivity of statistical measures to bandwidth limitations, measurement system response, and measurement errors.

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