Comment on plate modal wavenumber transforms in *Sound and Structural Vibration* [Academic Press (1987, 2007)] (L)

Micah R. Shepherd and Stephen A. Hambric

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Comment on plate modal wavenumber transforms in *Sound and Structural Vibration* [Academic Press (1987, 2007)] (L)

Micah R. Shepherd[a)] and Stephen A. Hambric
Graduate Program in Acoustics, Applied Research Laboratory, The Pennsylvania State University, P.O. Box 30, State College, Pennsylvania 16804

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The wavenumber transform for rectangular, simply supported, isotropic thin plates has been rederived to correct a technical error found in the text *Sound and Structural Vibration* (Academic Press, 1985/2007) by Fahy/Fahy and Gardonio. The text states that the modal wavenumber corresponds to the peak of the wavenumber spectrum. While this is approximately true for higher-order modes, it does not hold for lower-order modes due to coupling between positive and negative wavenumber energy. The modal wavenumber is shown to be related to the zeros in the wavenumber spectrum by an integer multiple of 2π normalized by the plate length.
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I. INTRODUCTION

Wavenumber analysis is used in a wide range of structural-acoustic problems. However, there exists a misconception that the wavenumber transform of a simply supported, rectangular, thin plate mode shape peaks at the modal wavenumber. This misconception originates from the paragraph following Eq. (3.101) (pp. 181) of the Fahy and Gardonio text *Sound and Structural Vibration*[1] [Eq. (2.54), pp. 77 in the 1st Edition]. The figures corresponding to this discussion (Figs. 3.20–3.22, 3.27/36–38,44) therefore have incorrect x-axis labels. The misinterpretation of Eq. 3.101, though mostly of academic interest, has been cited by researchers to provide insight into more complex problems (see e.g. Graham,[2] Torres and Boullosa,[3] and Clark and Fuller[4]). This letter reexamines the wavenumber spectrum to correctly interpret its physical meaning. The modal wavenumber is thereby shown to be related to the zeros in the spectrum by an integer multiple of 2π normalized by the plate length.

II. WAVENUMBER SPECTRUM DERIVATION

A simply supported thin rectangular plate of dimensions $L_x \times L_y$ has flexural mode shapes described by

$$\Psi_{mn}(x,y) = \sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right), \tag{1}$$

where m and n are positive, non-zero integers representing the mode order in the x and y directions, respectively. The wavenumber representation of the normal modes is a result of a two-dimensional Fourier transform in the spatial coordinates x and y:

$$S_{mn}(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{mn}(x,y)e^{-jk_xx}e^{-jk_yy}dxdy, \tag{2}$$

where k_x, k_y are the wavenumbers in the x and y directions.

The solution to Eq. (2) is separable in x and y, and for the x direction alone is

$$S_m(k_x) = \frac{k_m[(-1)^m e^{-jk_xL_x} - 1]}{k_x^2 - k_m^2}, \tag{3}$$

where $k_m = m\pi/L_x$ is the modal wavenumber.[1] The energy spectrum or wavenumber sensitivity function can be found by squaring the magnitude of Eq. (3):

$$|S_m(k_x)|^2 = \frac{2(k_m)^2(1-(-1)^m\cos(k_xL_x))}{(k_x^2-k_m^2)^2}. \tag{4}$$

The wavenumber spectrum in the y direction is derived in a similar fashion. Using a series of trigonometric identities, an alternative formulation of Eq. (4) is given as[1]

$$|S_m(k_x)|^2 = \left[\frac{2k_m}{k_x^2-k_m^2}\right]^2 \sin^2\left(\frac{k_xL_x - k_mL_x}{2}\right). \tag{5}$$

Examination of Eq. (5) reveals a numerator term that oscillates and a denominator term that resembles the denominator term of an undamped simple harmonic oscillator. The denominator then suggests that there will be an infinite peak in the spectrum at the modal wavenumber ($k_x = k_m$). However, analysis of the numerator reveals a zero when the argument of the squared sine function equals $p\pi$, where p is an integer (positive or negative). Thus when $p = 0$, which occurs at the modal wavenumber, L'Hopitals rule must be used to determine the value of $|S_m(k_m)|^2$. The sine term dominates in the limit that $k_x \to k_m$ such that $S_m^2(k_m) = L_x^2/4$ for all values of m. Equation (5) and the value of the its peak are derived and interpreted correctly in *Sound and Structural Vibration*.

However, further investigation of Eq. (5) reveals that the zeros in the sensitivity function occur when

$$k_x = \pi(2p+m)/L_x, \tag{6}$$

[a)]Author to whom correspondence should be addressed. Electronic mail: mrs30@arl.psu.edu.

where again, p is a positive or negative integer with $p = 0$ case corresponding to the modal wavenumber. This illustrates that there are multiple zeros centered about the modal wavenumber for each mode and that all even-ordered modes have zeros at even multiples of π/L_x and all odd-ordered modes have zeros at odd multiples of π/L_x. Thus, the x axis label of the Fig. 3.22/38 in *Sound and Structural Vibration* is incorrect.

The values of all extrema may be located by setting the derivative of Eq. (5) to zero to obtain the transcendental equation

$$\frac{2\pi^2 L_x^3 m^2[((k_xL_x)^2 - (m\pi)^2)\sin(k_xL_x - m\pi) + 4k_xL_x(\cos(k_xL_x - m\pi) - 1)]}{((k_xL_x)^2 - (m\pi)^2)^3} = 0. \tag{7}$$

Table I displays the first three maxima for the $m = 1 - 6$ modes, showing that the peak (i.e., global maximum) does not occur at the modal wavenumbers for low wavenumbers. By setting the numerator of Eq. (7) to zero and neglecting the denominator going to zero when $k_x = k_m$, one comes to the incorrect conclusion that a maximum exists at the modal wavenumber.

Figure 1 shows the wavenumber spectrum with the numerator and denominator terms of Eq. (5) overlaid on a logarithmic scale for the first four modes of a unit length plate. The denominator peaks at the positive and negative modal wavenumbers as expected and the numerator has zeros at odd/even multiples of π depending on the mode order. The peak of the wavenumber spectrum for each mode is boxed, and the peak wavenumber approaches the modal wavenumber as the mode order increases.

Upon inspection of the $m = 1$ mode in Fig. 1, it can be seen that the denominator term does not exhibit symmetry around the modal wavenumbers. This suggests that wavenumber energy from the positive and negative modal wavenumbers interact. This positive/negative wavenumber coupling at low wavenumber effectively boosts the wavenumber spectrum between $-k_m < k_x < k_m$, the effect being strongest for the $m = 1$ mode. At low wavenumbers, the locations of the peaks in the wavenumber spectrum can be explained as being slightly lower than the modal wavenumbers due to the coupling of the negative modal wavenumbers with the positive wavenumbers and vice versa. As the mode order increases, the peak wavenumbers approach the modal wavenumbers since the coupling between positive and negative wavenumber becomes more negligible (<2% when $m = 6$).

The $m = 1$ mode is described in *Sound and Structural Vibration*[1] as being an "exception" because the peak in the sensitivity function (located at $k_x = 0$) is not at the modal wavenumber (see Table I). However, correct interpretation of the spectrum shows that the $m = 1$ mode is not an anomaly but rather has the highest coupling between positive and negative wavenumbers and therefore the greatest error between the modal wavenumber and the peak in the sensitivity function. The percent difference between the peak wavenumber and the modal wavenumber is shown for the first 6 modes in Table I.

Further analysis of Eq. (5) reveals that the modal wavenumber can be recovered by exploiting the regularity of the spacing between zeros. Using Eq. (6), the spacing between zeros can be shown to be a constant, $2\pi/L_x$, for all mode orders. Since the modal wavenumber occurs when $p = 0$, the modal wavenumber can be computed once the location

TABLE I. The modal wavenumber and the first three maxima for modes $m = 1 - 6$ with the global maximum in bold. The percent error between the modal wavenumber and the location of the global peak is also shown. The error is greatest at $m = 1$ and decreases with increasing mode order.

Mode order (m)	k_m	k_{p1}	k_{p2}	k_{p3}	% Difference
1	π/L	**0**	11.87/L	18.41/L	100.0
2	$2\pi/L$	**5.26/L**	15.09/L	21.59/L	16.3
3	$3\pi/L$	0	**8.76/L**	18.27/L	7.1
4	$4\pi/L$	3.32/L	**12.08/L**	21.44/L	3.9
5	$5\pi/L$	0	6.54/L	**15.32/L**	2.5
6	$6\pi/L$	3.22/L	9.72/L	**18.53/L**	1.7

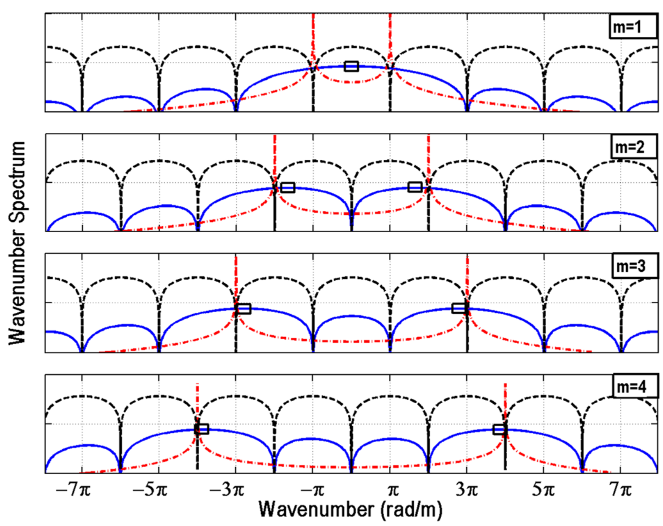

FIG. 1. (Color online) The wavenumber transform (solid) for the $m = 1 - 4$ modes with the numerator (dash) and denominator (dot-dash) terms plotted separately on a log-scale for a unit length plate. The peak in the wavenumber sensitivity function is indicated with a box. The coupling between the positive and negative wavenumbers in the denominator term accounts for the downward shift in the peak of the spectrum. As the mode order increases, the peak in the spectrum approximates the modal wavenumber ($m\pi$).

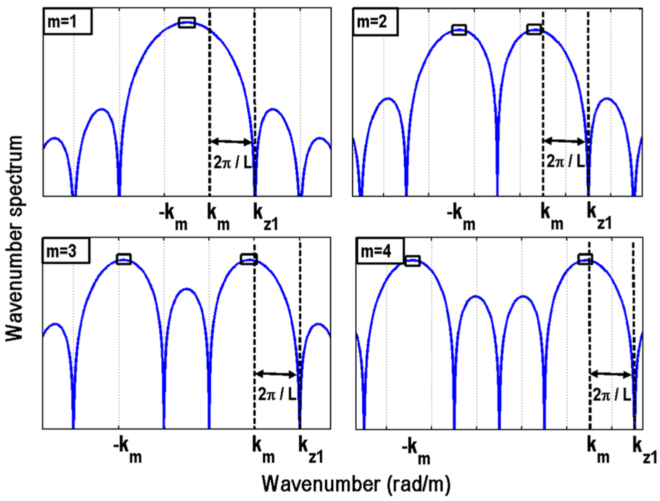

FIG. 2. (Color online) The wavenumber energy spectrum for the $m = 1 - 4$ modes with the peak in the wavenumber sensitivity function indicated with a box. The spacing between zeros ($2\pi/L$) can be used to determine the modal wavenumber by finding the first zero after the peak wavenumber and subtracting $2\pi/L$.

of the $p = 1$ zero is found (the location of the first zero after the maximum wavenumber). Mathematically, this can be written as

$$k_m = k_{z1} - 2\pi/L_x, \tag{8}$$

where k_{z1} is the wavenumber of the $p = 1$ zero. This relationship is shown graphically in Fig. 2 for mode orders $1 - 4$. The modal wavenumber can equivalently be determined by adding $2\pi/L_x$ to the wavenumber at the first zero preceding the peak ($p = -1$).

III. CONCLUSIONS

Wavenumber analysis of simply supported isotropic, thin plate flexural modes have been reexamined to correct a technical error concerning the location of the modal wavenumber in the wavenumber spectrum. The modal wavenumber has been shown to be related to the zeros in the spectrum and not equal to the peaks as explained in *Sound and Structural Vibration*[1] on pp. 181 (pp. 77). However, at high wavenumbers ($m > 5$), the percent error between the peak wavenumber and the modal wavenumber becomes small so that they are approximately equal.

It should be noted that the correction of this error has only minor implications for conceptual explanations relating to sound radiation of plates in wavenumber space (i.e., the radiation circle) and no relation to most other mathematical derivations found in Sec. 3.8/2.7.

[1]F. Fahy and P. Gardonio, *Sound and Structural Vibration,* 2nd Edition (Academic Press, Oxford, UK, 2007), pp. 181–183.

[2]W. R. Graham, "A comparison of models for the wave number-frequency spectrum of turbulent boundary layer pressures," J. Sound Vib. **206**(4), 541–565 (1997).

[3]J. A. Torres and R. Boullosa, "Radiation efficiency of a guitar top plate linked with edge or corner modes and intercell cancellation," J. Acoust. Am. **130**(1), 546–556 (2011).

[4]R. L. Clark and C. R. Fuller, "Modal sensing of efficient acoustic radiators with polyvinylidene fluoride distributed sensors in active structural acoustic control approaches," J. Acoust. Am. **91**(6), 3321–3329 (1992).