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Generalized entropy formulation of dissipative magnetohydrodynamics

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A variational-principle formulation of dissipative magnetohydrodynamics (DMHD) including thermal conduction and viscous dissipation as well as resistive decay is presented. The functional to be minimized is an extension of the generalized entropy-production (GEP) rate first discussed by I. Prigogine [*Nonequilibrium Thermodynamics, Variational Techniques, and Stability,* edited by R. J. Donnelly, R. Herman, and I. Prigogine (University of Chicago Press, Chicago, 1966)]. Minimization of this functional at each instant of time results in the proper evolutionary behavior of the MHD fields while the correct boundary conditions are maintained. Steady-state solutions are obtained as a special case of the GEP functional minimization, which is fully consistent with earlier entropy formulations for the steady state. The method is illustrated with an explicit application to a simple, one-dimensional model of a reversed-field pinch.

I. INTRODUCTION

In this paper we present and analyze a variational formulation of dissipative magnetohydrodynamics (DMHD), a physical model often used for a plasma or conducting fluid. Central to any variational problem is a functional depending on the fields of interest; the functional in this work may be identified as a generalized entropy-production (GEP) rate. This identification is based on the properties of the functional and its reduction in simple, steady-state cases to familiar expressions for the entropy-production rate.

Our work builds on the "local potential" or "generalized entropy-production" formulation of Prigogine and co-workers.[1-8] In Sec. II we briefly review this work and construct a GEP integral for DMHD.

Computationally, the variational method here has convenient properties common to variational techniques, such as automatic inclusion of boundary conditions. It seems particularly well suited to deal with nonlinearities introduced by field-dependent transport coefficients. The method is simple, straightforward to apply, and in our examples, explicitly incorporates time dependence and deals with the physical fields directly. For a detailed example of the application of a local potential different from the examples of this paper, see Ref. 9.

A recent paper by Hameiri and Bhattacharjee[10] develops similar entropy-production integrals for magnetic fields in a resistive medium. These authors confine their discussion, however, only to the magnetic field and also consider only the steady state. They argue in the examples they consider that the steady state is as much a state of minimum entropy production rate as it is a state of minimum magnetic energy, but that the principle of minimizing the rate of entropy production has a justifiable dynamical basis and interpretation that makes it preferable. In comparable situations and restricted to the steady state, our GEP integrals reduce to identical or very similar forms. However, in this work we consider explicitly entropy production due to temperature gradients and viscous dissipation. Further, as developed here, minimizing a suitable generalized entropy-production rate is a principle which guides the evolution of the system in time, rather than just defining the steady state.

In Sec. III we outline an explicit computation of a simplified model of a plasma in a reversed-field pinch (RFP) using the GEP formulation. The explicit computation illustrates the applicability of the method to general plasma problems. In this particular application, the method reduces numerically to a weighted Galerkin approximation. Many important geometrical and physical effects have been left to future studies where the focus is on quantitative comparisons with experiment as opposed to the present study which is aimed at conceptual questions.

II. THE GENERALIZED ENTROPY-PRODUCTION FORMULATION

A. Physical interpretation and the nonstandard variation

The general formulation of dissipative evolution equations in terms of the minimization of a generalized entropy-production rate or the "local potential" was first discussed by Prigogine in Ref. 1 and most thoroughly developed in Ref. 2. Applications and discussion of the GEP method are included in Refs. 3–9. We wish to review here the physical interpretation of this method and then apply it to the equations of DMHD. In this section we construct a GEP functional and variational procedure that produces the correct partial differential equations for a DMHD system.

Let $\phi(\mathbf{x},t)$ denote a composite field variable of the position vector $\mathbf{x}$ and the time t. The components of ϕ may consist of temperature $T(\mathbf{x},t)$, magnetic field $\mathbf{B}(\mathbf{x},t)$, velocity $\mathbf{v}(\mathbf{x},t)$, etc., so that $\phi(\mathbf{x},t) = [T(\mathbf{x},t), \mathbf{B}(\mathbf{x},t), \mathbf{v}(\mathbf{x},t), \ldots]$. The GEP functional $\Phi(\phi,\phi_0)$ is a volume integral over terms involving the two composite fields $\phi_0(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, and is thus a function of time. The field $\phi_0(\mathbf{x},t)$, with the zero subscript, is interpreted as the average (mean) value of the fields at the position $\mathbf{x}$ and the time t. The field $\phi(\mathbf{x},t)$, without the subscript, is viewed as including variations of the fields away from the average values. The minimization of $\Phi(\phi,\phi_0)$ with respect to ϕ by the methods of variational calculus is interpreted as calculating the most probable value $\phi_*(\mathbf{x},t)$, given $\phi_0(\mathbf{x},t)$. We apply *a posteriori* the "subsidiary condition"[2] that $\phi_*(\mathbf{x},t)=\phi_0(\mathbf{x},t)$. This subsidiary condition is to be interpreted as demanding that the most probable state $\phi_*(\mathbf{x},t)$ obtained from the minimization procedure coincide with the average state $\phi_0(\mathbf{x},t)$. The partial differential equations which then result from this procedure are those equations that determine the state of the system.

The primary condition for an acceptable $\Phi(\phi,\phi_0)$ is, in fact, that this minimization procedure, with the accompanying subsidiary condition, produce the correct partial differential equations. We refer to these partial differential equations collectively as the balance equations, since they represent the balance of energy, momentum, etc. A second condition placed on $\Phi(\phi,\phi_0)$ is that $\Phi(\phi,\phi_0)>\Phi(\phi_0,\phi_0)$, for all variations $\phi(\mathbf{x},t)$ away from $\phi_0(\mathbf{x},t)$. This requirement ensures that $\Phi(\phi_0,\phi_0)$ is a global minimum of $\Phi(\phi,\phi_0)$ for arbitrary variations of $\phi_0(\mathbf{x},t)$. For our purposes we also require that $\Phi(\phi,\phi_0)$ have the units of entropy production, i.e., entropy per unit time. This is so that in simple situations the functional $\Phi(\phi_0,\phi_0)$ is the entropy production rate and the variational procedure described corresponds to a generalization of the theorem of minimum entropy production.[2] We will refer to $\Phi(\phi,\phi_0)$ as the GEP functional and $\Phi(\phi_0,\phi_0)$ as the entropy-production rate. This is an important distinction in what follows.

For purposes of comparison with the published views of other authors[1,10] we make the following remarks. (i) We specifically avoid referring to the field variations $\phi(\mathbf{x},t)$ as fluctuations. It is customary to consider fluctuations in field quantities, e.g., the magnetic field $\mathbf{B}(\mathbf{x},t)$, to take place on a much finer temporal and spatial scale than for the mean fields. In the balance equations for the mean fields the fluctuations have been averaged and/or incorporated into transport coefficients. It is not clear what partial differential equations with transport coefficients should be used to model the evolution of the fluctuations taking place on the small spatial and temporal scales. In contrast, a field variation $\delta\phi(\mathbf{x},t)=\phi(\mathbf{x},t)-\phi_0(\mathbf{x},t)$ away from the mean field, obeys a linearized version of the mean-field balance equations. The effect of fluctuations in the generation of entropy is included through the transport coefficients, or explicit fields arising through the ensemble average of fluctuating quantities. (ii) As a consequence of the first, the second point we wish to make is that the entropy-production integral for a nonequilibrium system cannot depend on the field variations, which are mathematical rather than physical. The entropy production of a system, whether in an evolving state or a steady state, can depend only on the fields in the system and not on possible variations of those fields. Consequently, we identify $\Phi(\phi_0,\phi_0)$ with the entropy-production rate and not the GEP functional $\Phi(\phi,\phi_0)$.

We are not suggesting with these remarks that oscillations (unstable or stable) are precluded. Any instabilities that may develop with characteristic temporal and spatial scales consistent with a mean-field description are fully incorporated in the GEP formulation. The evolution of instabilities with shorter time scales or finer spatial scales is not properly described by mean-field equations.

B. GEP functional for DMHD

The fundamental balance equations for the mean fields in (single-component fluid, charge neutral) DMHD are as follows:[11]

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \quad \text{(mass balance)}\ , \tag{1}$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}=-\nabla p+\frac{1}{c}\mathbf{J}\times\mathbf{B}-\nabla\cdot(\rho\mathbf{v}\mathbf{v}+\overleftrightarrow{\Pi}) \quad \text{(momentum balance)}\ , \tag{2}$$

$$\frac{\partial}{\partial t}\left[\frac{p}{\gamma-1}\right]=-\nabla\cdot\left[\mathbf{J}_q+\frac{p\mathbf{v}}{\gamma-1}\right]-p\nabla\cdot\mathbf{v}-\overleftrightarrow{\Pi}:\nabla\mathbf{v}+\mathbf{J}\cdot\left[\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right]-S \quad \text{(energy balance)}\ . \tag{3}$$

All familiar symbols in Eqs. (1)–(3) have their usual meaning with $\mathbf{J}_q(\mathbf{x},t)$ denoting heat current density and $S(\mathbf{x},t)$ denoting sources and/or sinks of energy for the "fluid" such as bremsstrahlung radiation, line radiation, etc. A positive S corresponds to a sink.

To these equations we must also add the Maxwell equations determining the electromagnetic fields with the usual neglect of the displacement current

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}\ . \tag{4}$$

In order to apply these equations to physical problems it is necessary to make certain assumptions concerning the spatial and temporal scales and to assume certain phenomenological relations. For the present we will assume simple and familiar forms but note that other possibilities exist. The form of these equations and the functional form of the transport coefficients reflect our choice for a physical model incorporating the effect of fluctuations,

$$\mathbf{J}_q=-\kappa\nabla T \quad \text{(Fourier's law)}\ , \tag{5}$$

$$\Pi_{ij} = -2\nu d_{ij}$$

$$\text{where } d_{ij} = \tfrac{1}{2}(v_{i;j} + v_{j;i}) \quad \text{(Newton's law)} , \qquad (6)$$

$$\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} + \mathcal{E}(\mathbf{B}) = \eta J \quad \text{(Ohm's law)} . \qquad (7)$$

The notation $v_{i;j}$ denotes the covariant derivative in the jth direction of the ith covariant component of the velocity $\mathbf{v}$. The transport coefficients κ, ν, η are, respectively, the thermal conduction coefficient, viscosity coefficient, and the resistivity. In principle, we consider these transport coefficients to be functions of the fields, and in particular of the temperature. The field $\mathcal{E}(\mathbf{B})$ is the familiar "dynamo" electric field arising from the coupling of the velocity field fluctuations to fluctuations in the magnetic field. For a discussion of this term, its derivation, and possible field dependencies see Refs. 12–15. Briefly, the ensemble average $\langle \mathbf{v}\times\mathbf{B}/c \rangle$ gives rise to the second and third terms on the left-hand side of Eq. (7). Only mean fields occur explicitly in Eq. (7).

It is convenient to combine Eqs. (4) with Ohm's law, Eq. (7), to obtain the induction equation

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left[\frac{1}{c}\mathbf{v}\times\mathbf{B} + \mathcal{E} - \eta\mathbf{J}\right] \quad \text{(magnetic field balance)} . \qquad (8)$$

To construct a GEP functional we use the following four criteria: (1) Under the nonstandard variation of $\Phi(\phi,\phi_0)$ described in Sec. II A, the balance Eqs. (2), (3), and (8) must be obtained. (2) The extremum $\Phi(\phi_0,\phi_0)$ corresponds to an absolute minimum. (3) The GEP functional $\Phi(\phi,\phi_0)$ dimensionally represents the rate of entropy production. (4) The implied boundary conditions must correspond to the situation under consideration.

The GEP functional takes the form $\Phi = \Phi_B + \Phi_v + \Phi_T + \Phi_\rho$, where the two terms Φ_v and Φ_T have been essentially derived in Ref. 2 and we do not repeat the derivation here. Variations of these two terms with respect to $\mathbf{v}$ and $1/T$, respectively, yield Eqs. (2) and (3). However, the specific forms used in this paper can be easily obtained with the same techniques we now demonstrate for Eq. (8).

We begin with the density for the entropy production rate due to the Ohmic dissipation. It is given by $\eta|\mathbf{J}|^2/2T$. Since we are seeking a piece of the GEP functional for the magnetic-field balance Eq. (8), it is the magnetic-field variations which are of interest. Consequently, we consider a variation of the magnetic field, and put zero subscripts on temperature-dependent quantities to prevent any contribution from these terms to the energy balance equation arising from a variation of T,

$$\delta\left[\frac{\eta_0}{2T_0}J^2\right] = \frac{\eta_0}{T_0}\mathbf{J}\cdot\delta\mathbf{J} \quad \text{where } \delta\mathbf{J} = \frac{c}{4\pi}\nabla\times\delta\mathbf{B} .$$

Using standard vector identities, one finds

$$\frac{\eta_0}{T_0}\mathbf{J}\cdot\delta\mathbf{J} = \nabla\cdot\left[\delta\mathbf{B}\times\frac{\eta_0 c}{4\pi T_0}\mathbf{J}\right] + \frac{c}{4\pi T_0}\delta\mathbf{B}\cdot\left[\nabla\times(\eta_0\mathbf{J}) + T_0\nabla\left[\frac{1}{T_0}\right]\times\eta_0\mathbf{J}\right] . \qquad (9)$$

The first term in the square brackets on the right-hand side of Eq. (9) we recognize as the last term in the induction equation, Eq. (8). The second term in the brackets is an "unwanted" term that does not appear in the induction equation and we must subtract a term in the functional to compensate. Upon integration the divergence term in Eq. (9) converts to a surface integral that determines boundary conditions when set equal to zero. With this insight as to the entropy production from the Ohmic dissipation we see that

$$\Phi_B = \int_V dv\left\{\frac{\eta_0}{2T_0}J^2 + \frac{c}{4\pi}\mathbf{B}\cdot\left[\frac{1}{T_0 c}\frac{\partial \mathbf{B}_0}{\partial t} - \frac{1}{T_0 c}\nabla\times(\mathbf{v}_0\times\mathbf{B}_0) - \frac{1}{T_0}\nabla\times\mathcal{E}_0 - \nabla\left[\frac{1}{T_0}\right]\times\eta_0\mathbf{J}_0\right]\right\} \qquad (10)$$

will serve as a GEP functional. The fourth term in the square brackets in Eq. (10) is inserted specifically for the purpose of subtracting out the unwanted term that appeared in Eq. (9) under the variation. This cancellation happens after application of the subsidiary condition. We note, however, that this term couples gradients in temperature to the magnetic field and gives a term in the GEP functional similar to the Nernst effect.[16]

The boundary condition implied by the functional of Eq. (10) [from the divergence term in Eq. (9)] is

$$\int_{\partial V}\left[\delta\mathbf{B}\times\frac{\eta_0 c}{4\pi T_0}\mathbf{J}_0\right]\cdot\hat{\mathbf{n}}\, da = 0 , \qquad (11)$$

where ∂V denotes the boundary of the integration volume V.

Corresponding to the balance Eq. (2) for the velocity field $\mathbf{v}(\mathbf{x},t)$ we can use the GEP functional

$$\Phi_v = \int_V dv\left\{\frac{\nu_0}{T_0}d^{ij}d_{ij} + \frac{1}{T_0}\mathbf{v}\cdot\left[\nabla\cdot(\rho_0\mathbf{v}_0\mathbf{v}_0) + \nabla p_0 - T_0\overleftrightarrow{\Pi}_0\cdot\nabla\left[\frac{1}{T_0}\right] - \frac{1}{c}\mathbf{J}_0\times\mathbf{B}_0 + \frac{\partial}{\partial t}(\rho_0\mathbf{v}_0)\right]\right\} , \qquad (12)$$

which implies the boundary condition

$$\int_{\partial V}\left[\frac{\delta\mathbf{v}\cdot\overleftrightarrow{\Pi}_0}{T_0}\right]\cdot\hat{\mathbf{n}}\,da=0\ . \tag{13}$$

For the balance Eq. (3) we use $1/T$ as the unknown field and have the GEP functional

$$\Phi_T=\int_V dv\left\{\frac{\kappa_0}{2}T_0^2\left|\nabla\left[\frac{1}{T}\right]\right|^2-\frac{1}{T}\left[\frac{\partial}{\partial t}\left[\frac{p_0}{\gamma-1}\right]+\nabla\cdot\left[\frac{p_0\mathbf{v}_0}{\gamma-1}\right]+p_0\nabla\cdot\mathbf{v}_0+\overleftrightarrow{\Pi}_0{:}\nabla\mathbf{v}_0-\eta_0\mathbf{J}_0^2+S_0\right]\right\}\ . \tag{14}$$

The corresponding boundary condition is

$$\int_{\partial V}\delta\left[\frac{1}{T}\right]\kappa_0T_0^2\nabla\left[\frac{1}{T}\right]\cdot\hat{\mathbf{n}}\,da=0\ . \tag{15}$$

For mass conservation of a single fluid, Eq. (1) does not involve any dissipation, since there is no diffusion of fluid species relative to one another. For a different plasma model where, say, electrons and ions are treated as separate fluids, then the dissipation associated with the relative diffusion of the components is also a dissipative term and contributes to the generation of entropy. The relative diffusion of species has been treated adequately elsewhere[2] and since these considerations are irrelevant for our single-fluid model, we do not consider them further.

Although in the present single-fluid model it is only Eqs. (2), (3), and (8) that correspond to the generation of entropy; the mass conservation Eq. (1) can still be included in a GEP functional if desired. We assume that $\nabla\cdot\mathbf{v}_0\neq0$ and then

$$\Phi_\rho=(\text{const})\int_V dv\left[\frac{\rho^2-\rho_0^2}{2T_0}(\nabla\cdot\mathbf{v}_0)^2+\frac{(\rho-\rho_0)(\nabla\cdot\mathbf{v}_0)}{T_0}\left[\frac{\partial\rho_0}{\partial t}+\mathbf{v}_0\cdot\nabla\rho_0\right]\right]\ , \tag{16}$$

subjected to the usual form of nonstandard variation results in Eq. (1). The constant in front of the integral in Eq. (16) is chosen to get appropriate dimensions and scaling. No boundary conditions on $\rho_0(\mathbf{x},t)$ are implied by the variation of Eq. (16), and the form of the GEP functional is also chosen so that it makes no contribution to the entropy-production rate $\Phi(\phi_0,\phi_0)$.

We also remark that other forms of $\Phi=\Phi_B+\Phi_v+\Phi_T+\Phi_\rho$ are possible leading to the same balance equations. The functional Φ is not unique. In particular, terms involving only the mean fields ϕ_0 can be added or subtracted without affecting the result of the variation since it is ϕ which is varied and not ϕ_0. Consequently, the entropy-production rate $\Phi(\phi_0,\phi_0)$ is also not unique. There may be other reasons for selecting one form of Φ over another but insofar as the balance equations are concerned, it makes no difference.

For Eqs. (10), (12), (14), and (16) to serve as GEP functionals, it remains to check that indeed $\Phi(\phi,\phi_0)>\Phi(\phi_0,\phi_0)$. We verify this property for Φ_B and leave the verification for the other functionals to the reader. We take $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}$. Consequently,

$$\begin{aligned}\Delta\Phi_B&\equiv\Phi_B(\phi,\phi_0)-\Phi_B(\phi_0,\phi_0)\\&=\int_V dv\left\{\frac{\eta_0}{2T_0}|\delta\mathbf{J}|^2+\frac{\eta_0}{T_0}\mathbf{J}_0\cdot\delta\mathbf{J}+\frac{c}{4\pi}\delta\mathbf{B}\cdot\left[\frac{1}{T_0c}\frac{\partial\mathbf{B}_0}{\partial t}-\frac{1}{T_0c}\nabla\times(\mathbf{v}_0\times\mathbf{B}_0)-\frac{1}{T_0}\nabla\times\mathcal{E}_0-\nabla\left[\frac{1}{T_0}\right]\times\eta_0\mathbf{J}_0\right]\right\}\ .\end{aligned}$$

Using $\delta\mathbf{J}=(c/4\pi)\nabla\times\delta\mathbf{B}$ and integrating the second term in the integral by parts gives

$$\begin{aligned}\Delta\Phi_B=&\int_V dv\frac{\eta_0}{2T_0}|\delta\mathbf{J}|^2+\int_{\partial V}\left[\delta\mathbf{B}\times\frac{c\eta_0}{4\pi T_0}\mathbf{J}_0\right]\cdot\hat{\mathbf{n}}\,da\\&+\int_V dv\frac{c}{4\pi T_0}\delta\mathbf{B}\cdot\left[\frac{1}{c}\frac{\partial\mathbf{B}_0}{\partial t}-\frac{1}{c}\nabla\times(\mathbf{v}_0\times\mathbf{B}_0)\right.\\&\left.-\nabla\times\mathcal{E}_0+\nabla\times\eta_0\mathbf{J}_0\right]\ .\end{aligned}$$

The boundary condition, Eq. (11), insures that the surface integral vanishes and the mean fields satisfy the balance, Eq. (8), making the integral with $\delta\mathbf{B}$ vanish. All that remains is

$$\Delta\Phi_B=\int_V dv\frac{\eta_0}{2T_0}|\delta\mathbf{J}|^2\geq0\ . \tag{17}$$

C. Boundary conditions

The boundary conditions directly affect the form of the GEP functional. The boundary Eqs. (13) and (15) seem adequate for our needs, where, for example, Eq. (15) is satisfied by an insulating boundary or a boundary held at a constant temperature. On the other hand, the boundary conditions implied by Eq. (11) very often do not correspond to the physical situation under study. For example, in cylindrical geometry with $\hat{\mathbf{n}}=\hat{\mathbf{r}}$, Eq. (11) is satisfied by $\delta B_\theta=0$ and $J_\theta=0$ at the boundary. These are the boundary conditions used in computing the examples of Sec. III. However, instead of operating with a constant axial current ($\delta B_\theta=0$) it may be that the axial voltage is constant instead. In such a situation the boundary conditions implied by Eq. (11) are no longer appropriate and consequently the GEP functional in Eq. (10) is no longer suitable. We now demonstrate how the GEP functional may be changed to incorporate the boundary conditions by considering the specific example of a constant axial voltage.

We consider the boundary ∂V to be a flux surface so that $\mathbf{B}_0\cdot\hat{\mathbf{n}}\,|_{\partial V}=0$ and also such that $\mathbf{v}_0\cdot\hat{\mathbf{n}}\,|_{\partial V}=0$. With $\mathcal{E}_0=\langle \mathbf{v}_1\times\mathbf{B}_1\rangle$ (the ensemble average of fluctuations in $\mathbf{v}$ crossed into fluctuations in $\mathbf{B}$) and the usual assumption that the normal component of the fluctuations vanish at the wall, the tangential components of $\mathcal{E}_0$ vanish at the wall. These results are incorporated in the boundary equations

$$[\delta\mathbf{B}\times(\mathbf{v}_0\times\mathbf{B}_0)]\cdot\hat{\mathbf{n}}\,|_{\partial V}=0$$

and (18)

$$(\delta\mathbf{B}\times\mathcal{E}_0)\cdot\hat{\mathbf{n}}\,|_{\partial V}=\delta\mathbf{B}\cdot(\mathcal{E}_0\times\hat{\mathbf{n}})\,|_{\partial V}=0\ .$$

Thus, without changing its value, we can change the integral in Eq. (11) to

$$\int_{\partial V}\frac{c}{4\pi T_0}\left[\delta\mathbf{B}\times\left[-\frac{1}{c}\mathbf{v}_0\times\mathbf{B}_0-\mathcal{E}_0+\eta_0\mathbf{J}_0\right]\right]\cdot\hat{\mathbf{n}}\,da = \int_{\partial V}\frac{c}{4\pi T_0}(\delta\mathbf{B}\times\mathbf{E}_0)\cdot\hat{\mathbf{n}}\,da\ . \quad (19)$$

The last equality follows from Ohm's law, Eq. (7).

If we envision the system of interest to have toroidal geometry and the temperature on the boundary to be uniform, then Eq. (19) may be written in the form

$$\frac{1}{T_B}\int_{\partial V}\frac{c}{4\pi}(\delta B_P E_T-E_P\delta B_T)dl_P dl_T = \frac{1}{T_B}(V_T\delta I_T-V_P\delta I_P)\ , \quad (20)$$

where T_B is the temperature on the boundary and the subscripts T and P refer to the toroidal and poloidal directions, respectively. We see again that the previously mentioned boundary conditions with $V_P=0$ and $\delta I_T=0$ make the terms in Eq. (20) vanish. However, if we demand instead that $V_T=V_0$, a constant toroidal voltage, then we must add another term to the GEP functional to insure that this boundary condition is satisfied. We define $\mathbf{E}_a$ (which is never varied) to be a constant applied electric field such that the line integral of $\mathbf{E}_a$ around a toroidal circuit is V_0. We then reverse the previous computations to find an altered GEP functional appropriate to the constant toroidal voltage boundary condition:

$$\overline{\Phi}_B=\int_V dv\left\{\frac{1}{T_0}\left[\frac{\eta_0}{2}\mathbf{J}-\mathbf{E}_a\right]\cdot\mathbf{J}+\frac{c}{4\pi}\mathbf{B}\cdot\left[\frac{1}{T_0 c}\frac{\partial\mathbf{B}_0}{\partial t}-\frac{1}{T_0 c}\nabla\times(\mathbf{v}_0\times\mathbf{B}_0)-\frac{1}{T_0}\nabla\times\mathcal{E}_0-\nabla\left[\frac{1}{T_0}\right]\times(\eta_0\mathbf{J}_0-\mathbf{E}_a)\right]\right\}\ . \quad (21)$$

Equation (20) suggests other possibilities that can be handled in a similar fashion. It is straightforward to check that $\Delta\overline{\Phi}_B=\Delta\Phi_B$ with $\Delta\Phi_B$ given in Eq. (17).

D. Special cases

To justify calling $\Phi(\phi,\phi_0)$ a generalized entropy-production functional and to build intuition it is helpful to consider some special cases. Consider Eq. (21) for the case of a solid conductor at a constant temperature in steady state. As a result of these assumptions all terms in the square brackets in Eq. (21) vanish and we find only

$$\overline{\Phi}_B=\frac{1}{T_0}\int_V dv\left[\frac{\eta_0}{2}\mathbf{J}-\mathbf{E}_a\right]\cdot\mathbf{J}\ , \quad (22)$$

a result consistent with that reported in Ref. 10. Using the energy balance relation

$$\int_V \eta_0 J_0^2 dv=\int_V \mathbf{J}_0\cdot\mathbf{E}_a dv\ , \quad (23)$$

we find the entropy-production rate to be

$$\overline{\Phi}_B(\phi_0,\phi_0)=-\frac{1}{2T_0}\int_V \eta_0 J_0^2 dv\ , \quad (24)$$

also obtained in Ref. 10 (to within a factor of $1/T_0$).

At this point we can contrast the differences in physical interpretation adopted in this work with that in Refs. 2 and 10. The entropy-production rate in the steady state is the integral

$$\frac{1}{T_0}\int_V dv\left[\frac{\eta_0}{2}\mathbf{J}_0-\mathbf{E}_a\right]\cdot\mathbf{J}_0\ , \quad (25)$$

not the GEP functional of Eq. (22). The GEP functional of Eq. (22) is a function of $\mathbf{J}$, which is not the physically measurable variable $\mathbf{J}_0$ obtained in the minimization of Eq. (22). The mathematical variations contained in $\mathbf{J}$ do not necessarily represent physical fluctuations and should not be considered as contributing to the entropy-production rate. We consider the contribution to the entropy-production rate from the fluctuations to be contained in the transport coefficients which embody the dissipation and any ensemble-averaged pieces such as the dynamo field $\mathcal{E}(\mathbf{B})$. The entropy-production rate depends on the mean field and transport, not the mean field and its variation.

We need not restrict the application of Eq. (21) in a solid conductor to the steady state. With

$$\overline{\Phi}_B=\int_V dv\left[\frac{1}{T_0}\left[\frac{\eta_0}{2}\mathbf{J}-\mathbf{E}_a\right]\cdot\mathbf{J}+\frac{1}{4\pi T_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}_0}{\partial t}\right] \tag{26}$$

as the GEP functional, we get the entropy-production rate by substituting for the variation fields the correct mean fields, i.e., the ones with zero subscripts satisfying the balance equations. A straightforward substitution using Faraday's law, Eq. (4), and an integration by parts shows the entropy-production rate for the time-evolving state to be given by the integral in (25) with $\mathbf{E}_a$ replaced by $\mathbf{E}_0$. Ohm's law, Eq. (7), shows the integral in (24) to be unchanged.

We can also examine the stability of a state minimizing Φ_B. We take the time derivative of $\Delta\Phi_B$ as given by Eq. (17). In the present example we use

$$\frac{1}{c}\frac{\partial \delta\mathbf{B}}{\partial t}=-\nabla\times\eta_0\delta\mathbf{J}\ , \tag{27}$$

and the boundary condition that $\delta\mathbf{J}$ vanishes at the surface of the solid conductor ($\eta_0\mathbf{J}_0=\mathbf{E}_a$). We find

$$\frac{d}{dt}\Delta\Phi_B=-\frac{c^2}{4\pi T_0}\int_V dv\ |\nabla\times\eta_0\delta\mathbf{J}|^2<0\ .$$

Consequently, the variation of Φ_B from the minimum value, i.e., the entropy-production rate, decreases with time. The state obtained by the minimization of the GEP integral for a solid conductor is secularly stable.

A second example considered in Ref. 10, is that of a uniform temperature plasma in steady state. Making use of the boundary conditions in Eq. (18), from Eq. (21) we find for a GEP functional,

$$\overline{\Phi}_B=\int_V dv\frac{1}{T_0}\left[\frac{\eta_0}{2}J^2-\mathbf{J}\cdot\left[\mathbf{E}_a+\frac{1}{c}(\mathbf{v}_0\times\mathbf{B}_0)+\mathcal{E}_0\right]\right] \tag{28}$$

It is also straightforward to show that once again (24) is obtained for the entropy-production rate. As shown in Ref. 10, the assumption that $\mathbf{J}_0=\lambda\mathbf{B}_0$ with λ constant (a Taylor state) gives a minimum for the entropy-production rate (24) when the magnetic energy is a minimum. However, we emphasize that this Taylor state is not obtained by minimizing any form for the entropy production rate and it seems premature to assert that the "relaxed state" achieved in RFP fusion experiments is a state of minimum entropy production.

III. APPLICATION TO A CYLINDRICAL PLASMA

In order to gain experience in applying the GEP formulation we have considered the evolution of a single-component, DMHD plasma in cylindrical geometry. We have in mind a simplified cylindrical approximation to a reversed-field pinch (RFP). We do not attempt here a comprehensive simulation of an RFP plasma, but hope to retain only enough physics in the following example to make the calculation interesting while exploring the GEP method. Within the context of this view we list the further assumptions and reductions defining the model.

A. Physical model

We assume translational symmetry along the cylindrical axis, and rotational symmetry about this same axis, i.e., all field quantities depend only on the radial coordinate. We take $\mathbf{v}=v(r)\hat{\mathbf{r}}$ and $\mathbf{B}=[0,B_\theta(r),B_z(r)]$.

Pressure balance is assumed for all time

$$\nabla p=\frac{1}{c}\mathbf{J}\times\mathbf{B}\ . \tag{29}$$

Thus, viscous damping plays no role in this example and the contribution to the GEP functional given by Eq. (12) becomes irrelevant, and is dropped from further consideration.

By assuming force balance as in Eq. (29), we choose to ignore the true dynamics of the velocity field. However, one may construct a diffusion velocity in the following manner: (1) one first integrates the force balance relation over r and then differentiates with respect to time to obtain an expression for $\partial\rho/\partial t$. (2) One then substitutes this expression into the continuity equation, Eq. (1), and after integrating over r again, one solves this resultant expression for the diffusion velocity. For suitably small flows this approximation is reasonable and for numerical purposes, we assume this model to be valid, provided

$$\frac{\left|\rho\dfrac{\partial \mathbf{v}}{\partial t}+\rho\mathbf{v}\cdot\nabla\mathbf{v}\right|}{|\mathbf{J}\times\mathbf{B}|}\le 1\times10^{-6}\ .$$

In what follows we let $u(x,\tau)$ be the scaled radial velocity, where the radial variable x is scaled by the chamber radius and the time variable τ is scaled by a resistive diffusion time τ_R. The magnetic field $\mathbf{B}$ and the temperature T are scaled by arbitrary factors B_s and T_s, respectively. The mass density ρ is replaced by the particle density n and is scaled by a factor of n_s. (The scaling factors used in the numerical results to follow, are the same as those used in Ref. 9.) A scaling constant $K=4\pi n_s T_s/B_s^2$ serves to scale the magnetic variables to the thermal variables.

In the process described above for obtaining the diffusion velocity we must add an extra equation to our system for consistency and closure. To see how this arises, take the pressure as $p(x,t)=Kn(x,t)T(x,t)$ and use the usual screw-pinch form[17] of Eq. (29); integrating yields

$$\left[KnT+\frac{B^2}{2}\right]\Bigg|_0^x=-\int_0^x\frac{B_\theta^2}{s}ds\ . \quad (30)$$

In evaluating the limits to obtain $n(x,\tau)$, one must determine $[KnT+(B^2/2)]|_0$. To do this, one integrates Eq. (30) over the whole plasma volume ($\int_0^1 x\,dx$), and introduces the scaled global thermal energy of the system

$$E_T(\tau)=\int_0^1 xnT\,dx\ . \quad (31)$$

The global variable E_T is a useful choice for closing our simplified system, and in the following paragraph an equation for E_T is derived that is added to the GEP system of equations.

Expressing the lower limit of Eq. (30) in terms of E_T, the density may be expressed as

$$n(x,\tau)=2\frac{E_T}{T}+\frac{1}{KT}\left[\int_0^1 B_z^2 x\,dx+\int_x^1\frac{B_\theta^2}{s}ds-\frac{B^2}{2}\right]. \quad (32)$$

As described above, one obtains the plasma velocity $u(x,\tau)$ by an integration of the reduced form of Eq. (1),

$$u(x,\tau)=\frac{-1}{nx}\int_0^x \dot{n}x\,dx\ , \quad (33)$$

where an overdot on n denotes $\partial n/\partial\tau$. Assuming reasonable behavior of $n(x,\tau)$, we obtain $u(x,\tau)\to 0$ as $x\to 0$, and at $x=1$ (the wall value),

$$u(1,\tau)=\frac{-1}{n(1,\tau)}\int_0^1 \dot{n}x\,dx=\frac{-\dot{N}}{n(1,\tau)}\ , \quad (34)$$

where $N(\tau)\equiv\int_0^1 nx\,dx$ is proportional to the total number of particles in the plasma. Clearly, $u(1,\tau)=0$ implies $\dot{N}=0$. This particle-conserving boundary condition is imposed for the results reported here, although a global energy conserving formulation has also been considered. Fixing the velocity at the wall to be zero also has the implication that the total kinetic energy in the system is constant.

With $u(1,\tau)=0$, a time derivative of Eq. (32) substituted into Eq. (34) yields a differential equation for E_T. The result is

$$\dot{E}_T=\frac{1}{2\alpha}\int_0^1 \dot{T}^{-1}nT\,x\,dx-\frac{1}{K}\int_0^1 B_z\dot{B}_z x\,dx$$
$$-\frac{1}{\alpha}\int_0^1\frac{x}{KT}dx\int_x^1\frac{B_\theta\dot{B}_\theta}{s}ds$$
$$-\frac{1}{2\alpha}\int_0^1\frac{x\mathbf{B}\cdot\dot{\mathbf{B}}}{KT}dx\ , \quad (35)$$

where $\alpha\equiv\int_0^1(x/T)dx$. From Eq. (33), one sees that $u(x,\tau)$ depends on $\dot{E}_T$, as well as the other fields.

Equation (35) is to be solved simultaneously with the minimization of the GEP functional to provide a closed system. Thus, the variables to be obtained for a solution are $T^{-1}(x,\tau)$, $B_\theta(x,\tau)$, $B_z(x,\tau)$, and $E_T(\tau)$, for $\tau>0$ and $0\le x\le 1$.

The boundary conditions on these fields are natural and satisfy Eqs. (11) and (15),

$$T(1,\tau)=T_w=\text{const},\quad B_\theta(1,\tau)=\text{const},\quad J_\theta(1,\tau)=0\ . \quad (36)$$

These boundary conditions correspond to the reasonable physical situation where the temperature is fixed at the wall, the total axial current is fixed, and the poloidal electric field vanishes at the wall. For a fixed toroidal voltage, one uses the boundary condition $\eta J_z(1,\tau)=E_a$, instead of $B_\theta(1,\tau)=\text{const}$ and the GEP function $\overline{\Phi}_B$, rather than Φ_B. From symmetry

$$T'(0,\tau)=0,\quad B_\theta(0,\tau)=0,\quad B_z'(0,\tau)=0\ , \quad (37)$$

where the prime denotes differentiation with respect to x.

B. Numerical scheme

We arbitrarily divide the x interval into a number of finite elements (we have used from 6 to 12). The fields are then expanded for the interval [0,1] in a set of cubic **B** splines,

$$\phi_\alpha(x,\tau)=\sum_i C_\alpha^i(\tau)B_i(x)\ , \quad (38)$$

where the sum is over the spline functions B_i having support over four finite elements. The field ϕ_α denotes the field B_θ, B_z, or T^{-1}, according to the value of the subscript α. At the end points of the interval, linear combinations of the basis splines B_i may be taken to form new functions, giving expansion sets for the fields that automatically satisfy the boundary conditions, Eqs. (36) and (37).

The spline functions contain all the spatial dependence for the appropriately expanded fields. Thus, the spatial integrations for the GEP functional can be carried out and $\Phi(\tau)$ becomes a function of the coefficients $C_\alpha^i(\tau)$ in

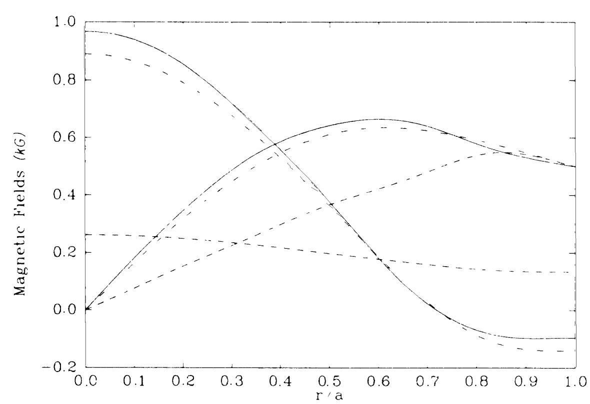

FIG. 1. Toroidal and poloidal magnetic field components are plotted for both the case in which no dynamo has been included in the GEP equations, and for the case in which an "α-effect" dynamo has been included. The poloidal fields are those that are zero at the origin and fixed at the wall. The solid lines represent the initial profiles. The short-dashed lines represent the "no-dynamo" fields at $6\tau_r$, and the long-dashed lines are the components of the "α-effect dynamo" fields at $20\tau_r$. The "no-dynamo" fields have nearly evolved in $6\tau_r$ to the decayed steady state with $B_z=\text{const}$, and $B_\theta=B_\theta(a)r$. The "α-effect dynamo" fields have changed only slightly in $20\tau_r$.

the expansion of the variation fields. Minimizing Φ with respect to these coefficients, followed by the application of the subsidiary condition ($C^i_\alpha \to C^i_{\alpha 0}$), in which the variation fields are equated to the mean fields, is equivalent to requiring the system to be consistent with the induction equation and the local power balance, or temperature equation, of the DMHD model. Evaluating the generalized entropy-production rate at this minimum supplies a value of $\Phi(\phi_0,\phi_0)$. More specifically, setting $\partial\Phi/\partial C^i_\alpha$ equal to zero and applying the subsidiary condition yields a first-order system of nonlinear, time-dependent, ordinary differential equations for $C^i_{\alpha 0}(\tau)$. Together with Eq. (35) for E_T, these equations guide the evolution of the system.

These ordinary (in time) differential equations for the coefficients of expansion $C^i_{\alpha 0}$ and E_T, take the form

$$A\dot{\mathbf{X}}+B\mathbf{X}+D=0\ , \tag{39}$$

where $\mathbf{X}=(C^i_{\alpha 0},E_T)$ is a vector containing the spline expansion coefficients plus E_T. The elements of the matrices A, B, and D, involve spatial integrations over the interval [0,1] of spline combinations and fields resulting from the GEP functional that has been differentiated as explained above. Because these matrices are themselves functions of $\mathbf{X}(\tau)$, the system is highly nonlinear. Requiring Eq. (39) to govern the system guarantees that one obtains $\Phi(\phi_0,\phi_0)$ at each time step.

Since we generate the fields and their derivatives as the computation proceeds, we have periodically substituted these solutions in the differential equations as a check on the GEP method.

Boundary conditions may be changed by constructing new combinations of B splines that explicitly satisfy the new set of conditions. A simple switch is included in our code to change from a set for constant axial current, to a set used for constant toroidal loop voltage.

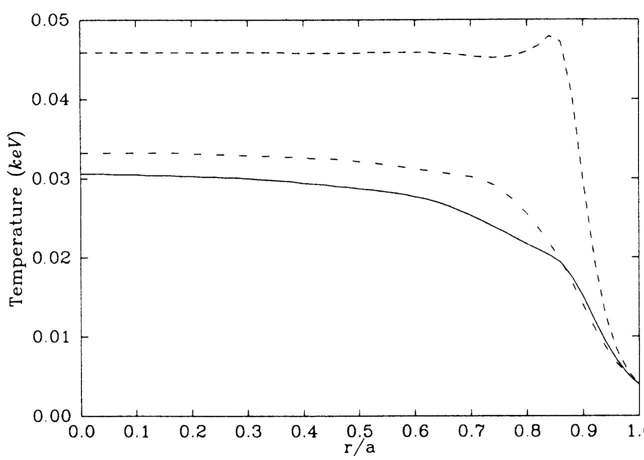

FIG. 2. Temperature profiles are shown for both cases. The solid line is the initial temperature profile. Again, the short-dashed line represents the temperature for the "no-dynamo" case at $6\tau_r$ and the long-dashed line represents the "α-effect dynamo" case at $20\tau_r$. In the former case resistive decay has significantly heated the plasma and flattened the temperature profile throughout the bulk of the plasma, while in the latter case only a small smoothing and flattening of the temperature is experienced.

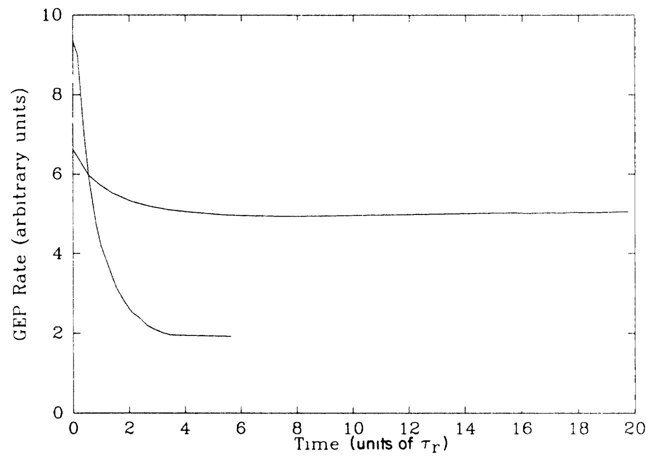

FIG. 3. Generalized entropy-production rate $\Phi(\phi_0,\phi_0)$ is plotted in arbitrary units as a function of time for each of the two cases. In both instances, the system rapidly evolves into a state in which the entropy-production rate is nearly constant. The time is given in units of τ_r.

We use the commercial integration package, DGEAR, from International Mathematical and Statistical Libraries, Inc., to integrate the system of Eq. (39) forward in time.

C. Specific results

Classical functional dependence for the transport coefficients has been taken with the resistivity given by $\eta=\eta_0 T^{-3/2}$, and the thermal conductivity by $\kappa=\kappa_0 n^2 B^{-2} T^{-1/2}$, where η_0 and κ_0 are numerical constants.[9] The value of κ_0 is five times that given in Ref. 9. The adiabatic constant γ is $\frac{5}{3}$, and the source (sink) term S_0 in Eq. (14) is set to zero. The arbitrary profiles for the magnetic fields and the temperature are displayed in the Figs. 1 and 2.

Two cases are presented. In the first case, the dynamo term $\mathcal{E}$ in Eq. (7) is set to zero. We use this case to define a computational resistive diffusion time τ_r as the time in which the initial reversed toroidal field is lost through classical resistive decay. The evolution of this system is followed for $6\tau_r$. The computation was halted when it became sufficiently clear that the system would shortly reach the totally decayed steady state, with $B_z=$const, and $B_\theta=B_\theta(a)r$. Initially the fields undergo dramatic resistive decay accompanied by a large increase and flattening of the temperature profile. The generalized entropy-production rate decreases rapidly during this phase and becomes virtually constant after the fields have nearly completed their decay. In the second case an "α-effect" dynamo[15] is taken with $\mathcal{E}=\alpha(x)\mathbf{B}$. Here α is chosen to be a time-independent smooth function of x that is nearly constant in the central region of the plasma and falls off smoothly and rapidly to zero near the wall. The system was allowed to evolve for $20\tau_r$. At this point the diffusion velocity $u(x,\tau)$ of Eq. (33) began to grow, and computation was terminated. In the first $3\tau_r$ of the evolution the fields rapidly adjust themselves to a near-steady state with a concomitant smoothing and flattening of the temperature profile. The system continues to be driven so the fields continue to evolve, but now on a much slower time scale. The generalized entropy-production rate reflects this behavior in rapidly coming to a nearly constant value in the first $3\tau_r$ and then changing only slowly in time, as can be seen by a slight rise in the GEP curve in Fig. 3. In this same time the toroidal field remains reversed at the wall. See Figs. 1 and 2. The generalized entropy-production rate is plotted as a function of time in Fig. 3 for each of the two cases.

[1]I. Prigogine, in *Nonequilibrium Thermodynamics, Variational Techniques, and Stability,* edited by R. J. Donnelly, R. Herman, and I. Prigogine (University of Chicago Press, Chicago, 1966), p. 3.

[2]P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability, and Fluctuations* (Wiley, New York, 1971), Chap. X.

[3]D. F. Hays, in Ref. 1, p. 17.

[4]P. Glansdorff, Physica **32**, 1745 (1966).

[5]R. S. Schechter and D. M. Himmelblau, Phys. Fluids **8**, 1431 (1965).

[6]J. C. Legros and J. K. Platten, J. Non-Equilib. Thermodyn. **2**, 211 (1977).

[7]S. M. Rohde, Physica **61**, 53 (1972);.

[8]S. M. Rohde, in *Advances in Chemical Physics,* edited by I. Prigogine and S. A. Rice (Wiley, New York, 1975), Vol. 32, p. 37.

[9]S. N. Rasband and Y-H. Xu, Phys. Fluids **29**, 2639 (1986).

[10]E. Hameiri and A. Bhattacharjee, Phys. Rev. A **35**, 768 (1987).

[11]N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).

[12]A. Bhattacharjee and E. Harmeiri, Phys. Rev. Lett. **57**, 206 (1986); Phys. Fluids **30**, 1743 (1987).

[13]M. J. Schaffer, Phys. Fluids **27**, 2129 (1984).

[14]A. H. Boozer, J. Plasma Phys. **35**, 133 (1986).

[15]H. K. Moffat, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).

[16]S. R. deGroot and P. Mazur, *Non-equilibrium Thermodynamics* (North-Holland, New York, 1962).

[17]J. P. Freidberg, Rev. Mod. Phys. **54**, 801 (1982).