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# Classical connection between near-field interactions and far-field radiation and the relevance to quantum photoemission

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Interference in the far-field radiation pattern emitted from a classical current distribution implies near-field work between different spatial portions of the distribution. We examine this relationship and the essential role of system geometry for the case of two oscillating dipoles and for a Gaussian current distribution. This analysis offers a compelling argument as to why the radiation from a large single-electron quantum wave packet should not exhibit the same destructive interference as that associated with a comparable classical charge density. Our discussion draws attention to the *ad hoc* heuristics motivating the original derivation of a quantum electron's radiation profile.

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## I. INTRODUCTION

It is well known that the founders of quantum mechanics disagreed (for a time) about how to interpret the single-particle wavefunction  $\psi(\mathbf{r}, t)$ . In an early paper,<sup>1</sup> Schrödinger suggested that the quantity  $e|\psi(\mathbf{r}, t)|^2$  be interpreted as a classical charge density for the electron, coupling naturally to classical electrodynamics via Maxwell's equations. This suggestion was superseded by the probabilistic interpretation introduced by Born in 1926,<sup>2</sup> which held steadfast even through the developments of second quantization and quantum electrodynamics (QED). The Born interpretation is the commonly held viewpoint today,<sup>3,4</sup> although alternative perspectives have been explored (for example, cf. Refs. 5–9). Even so, the unwieldiness of QED often compels researchers to resort to first-quantized calculations. In this context, it is easy to revert to Schrödinger-like intuition, especially in connection with radiative processes. We refer in particular to the radiation generated by a laser-driven single-electron wave packet.

The purpose of this paper is to improve general intuition for quantum radiation scattering by studying the implications of classical electrodynamics. Classical charge and current distributions  $\rho$  and  $\mathbf{J}$  generate (Lorenz-gauge) potentials,

$$\begin{aligned}\Phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t_r)}{\iota} \quad \text{and} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}', t_r)}{\iota},\end{aligned}\quad (1)$$

where  $\iota = |\mathbf{r} - \mathbf{r}'|$  and  $t_r = t - \iota/c$  is the retarded time.<sup>10</sup> The electric and magnetic fields are generated from these potentials by  $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ . If the classical current  $\mathbf{J}$  oscillates across its distribution, then the radiation can be strongly suppressed in certain directions due to destructive interference. Thus, it seems natural to ask: does this same suppression occur if the source is a (large) single-electron quantum wave packet? A cursory appeal to Born's probabilistic interpretation gives no immediate intuition on the matter; hence, it may be tempting to suppose that  $e|\psi(\mathbf{r}, t)|^2$  (or a relativistic analog) radiates like a classical charge density in spite of being a fundamentally different object. Indeed, Refs. 11 and 12 found pronounced radiative interference when Eq. (1) is sourced by a quantum probability current.

We recently demonstrated in the context of QED that light scattered from a single-electron wave packet is independent of the size of the wave packet.<sup>13–15</sup> This is in sharp contrast with classical electrodynamics wherein the radiated light depends sensitively on the spatial extent of the oscillating source. QED predicts that the radiation emitted from different regions of an electron wave packet do not interfere—no radiative suppression occurs. The numerical quantum field simulations by Cheng *et al.* agree with our conclusions.<sup>16</sup> Unfortunately, these approaches generally only enlighten those who are already well-versed in second quantization.

In this paper, we present intuitive arguments to show that radiation from a single-electron wave packet, computed classically from Eq. (1), leads to inconsistencies with quantum mechanics. We appeal only to the principles of classical electrodynamics, showing that far-field interference in the radiated light would imply that different regions of the electron wave packet do work on each other via the near fields. This is at odds with ordinary single-particle quantum mechanics because one does not include a Hamiltonian term for the near-field interaction between different parts of the same electron wavefunction. (Such a term would completely alter the well-understood hydrogenic ground state, for example.) At the same time, the examples given in this article are interesting classical electrodynamic problems in their own right.

In Secs. II and III, we compute the far-field radiation from two dipoles and we demonstrate that near-field work accounts for the interference between the dipoles. This relates to recent work by Berman, who looked at energy exchange between two well-separated radiating dipoles.<sup>17</sup> Section IV generalizes the analysis to a Gaussian current distribution, investigating the radiative interference that a quantum electron wave packet would exhibit if  $e|\psi(\mathbf{r}, t)|^2$  were a classical charge density, as Schrödinger suggested. Finally, Sec. V discusses the Schrödinger interpretation in the context of early calculations of single-electron radiation scattering.

## II. RADIATION FROM A TWO-DIPOLE SYSTEM

In this section, we analyze interference in the radiation generated by a pair of oscillating dipoles. First, consider a single dipole at position  $\mathbf{r}_1$ , oriented along the  $z$ -direction, and oscillating with frequency  $\omega = ck$ ,

$$\mathbf{p}_1 = \hat{\mathbf{z}} p_0 \cos(-\omega t + \varphi_1). \quad (2)$$

The parameters  $p_0$  and  $\varphi_1$  represent the amplitude and phase. The electric field surrounding the radiating dipole is given by<sup>18</sup>

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) = & \frac{p_0}{4\pi\epsilon_0} \left[ 3(\hat{\mathbf{z}} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} - \hat{\mathbf{z}} \right] \left[ \frac{\cos(kz - \omega t + \varphi_1)}{z^3} \right. \\ & \left. + \frac{k \sin(kz - \omega t + \varphi_1)}{z^2} \right] \\ & - \frac{p_0}{4\pi\epsilon_0} [(\hat{\mathbf{z}} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} - \hat{\mathbf{z}}] \frac{k^2 \cos(kz - \omega t + \varphi_1)}{z}, \end{aligned} \quad (3)$$

where  $\vec{r} \equiv \mathbf{r} - \mathbf{r}_1$  and  $\hat{\mathbf{i}} \equiv \vec{r}/z$ . A similar expression can be written for the field  $\mathbf{E}_2(\mathbf{r}, t)$  emanating from a parallel dipole positioned at  $\mathbf{r}_2$  and oscillating with the same frequency. The pair of dipoles is depicted in Fig. 1. Of course, the net field is given by  $\mathbf{E}_1 + \mathbf{E}_2$ .

We calculate the power radiated by the pair of dipoles following standard far-field analysis. In the far-field (i.e.,  $z \rightarrow$  large), the final term in Eq. (3) dominates. Moreover, outside of the cosine argument we may write  $z \approx r$ . Within the cosine argument, we make the approximation  $|\mathbf{r} - \mathbf{r}_1| = \sqrt{r^2 + r_1^2 - 2\mathbf{r}_1 \cdot \mathbf{r}} \approx r - \mathbf{r}_1 \cdot \hat{\mathbf{r}}$  (and similarly  $|\mathbf{r} - \mathbf{r}_2| \approx r - \mathbf{r}_2 \cdot \hat{\mathbf{r}}$ ). The net field from the two dipoles in the far-field then becomes

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t) \approx & -\frac{p_0}{4\pi\epsilon_0} [(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{z}}] \frac{k^2}{r} \\ & \times [\cos(kr - k\mathbf{r}_1 \cdot \hat{\mathbf{r}} - \omega t + \varphi_1) \\ & + \cos(kr - k\mathbf{r}_2 \cdot \hat{\mathbf{r}} - \omega t + \varphi_2)]. \end{aligned} \quad (4)$$

In spherical coordinates, we have  $(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{z}} = \hat{\theta} \sin \theta$ . The magnitude of the Poynting flux (which is directed along  $\hat{\mathbf{r}}$ ) is then

$$\begin{aligned} S(\mathbf{r}, t) = & \epsilon_0 c |\mathbf{E}_1 + \mathbf{E}_2|^2 \\ = & \frac{cp_0^2 \sin^2 \theta k^4}{(4\pi)^2 \epsilon_0 r^2} [\cos^2(kr - k\mathbf{r}_1 \cdot \hat{\mathbf{r}} - \omega t + \varphi_1) \\ & + \cos^2(kr - k\mathbf{r}_2 \cdot \hat{\mathbf{r}} - \omega t + \varphi_2) \\ & + 2 \cos(kr - k\mathbf{r}_1 \cdot \hat{\mathbf{r}} - \omega t + \varphi_1) \\ & \times \cos(kr - k\mathbf{r}_2 \cdot \hat{\mathbf{r}} - \omega t + \varphi_2)]. \end{aligned} \quad (5)$$

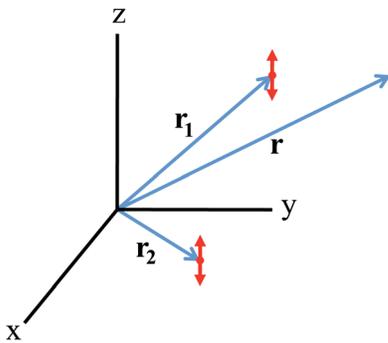


Fig. 1. Two oscillating dipoles at arbitrary locations  $\mathbf{r}_1$  and  $\mathbf{r}_2$  with orientations parallel to the  $z$ -axis.

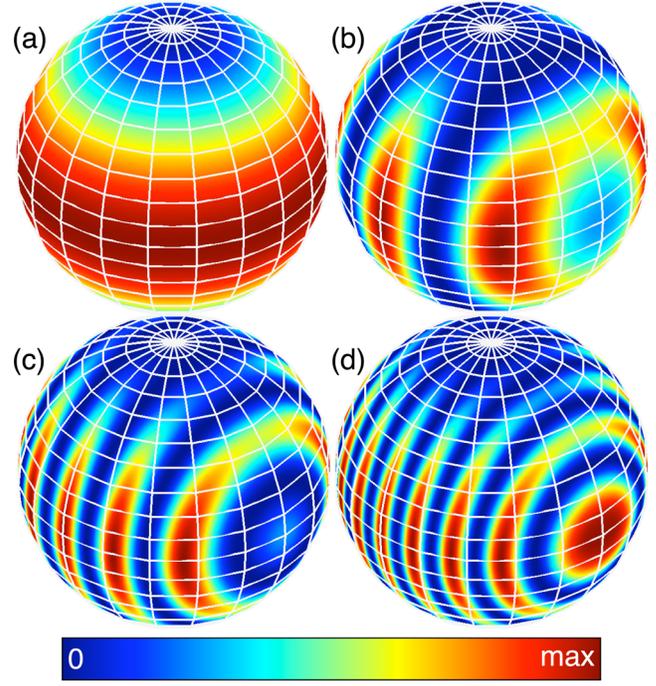


Fig. 2. (Color online) Far-field average Poynting-flux distribution from two parallel dipoles separated by (a)  $k\tilde{a} = 0$ , (b)  $k\tilde{a} = 4\pi/3$ , (c)  $k\tilde{a} = 8\pi/3$ , and (d)  $k\tilde{a} = 4\pi$ . Here, we have taken  $ka_z = 0$  and  $\varphi = 0$ ;  $a_z$  and  $\tilde{a}$  are defined following Eq. (8).

The time average of this expression reduces to

$$\langle S(\mathbf{r}, t) \rangle_t = \frac{cp_0^2 \sin^2 \theta k^4}{(4\pi)^2 \epsilon_0 r^2} [1 + \cos(ka \cdot \hat{\mathbf{r}} - \varphi)], \quad (6)$$

where  $\mathbf{a} \equiv \mathbf{r}_2 - \mathbf{r}_1$  and  $\varphi \equiv \varphi_2 - \varphi_1$ . This distribution is shown in Fig. 2 for various dipole separations. As the dipoles get farther apart, the distribution shows rich interference structure.

To obtain the average power radiated into the far-field, we integrate the average Poynting flux over the surface of a large sphere of radius  $r$ ,

$$\langle P \rangle_t = r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \langle S \rangle_t. \quad (7)$$

The integration is worked out in Appendix A and the resulting expression for radiated power is

$$\begin{aligned} \langle P \rangle_t = & \frac{cp_0^2 k^4}{4\pi\epsilon_0} \left[ \frac{2}{3} + \cos \varphi \left( \frac{\tilde{a}^2 \sin ka}{a^2 ka} + \frac{\tilde{a}^2 - 2a_z^2 \cos ka}{a^2 (ka)^2} \right. \right. \\ & \left. \left. - \frac{\tilde{a}^2 - 2a_z^2 \sin ka}{a^2 (ka)^3} \right) \right], \end{aligned} \quad (8)$$

where  $a_z^2 \equiv (z_2 - z_1)^2$ ,  $\tilde{a}^2 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$ , and  $a^2 = \tilde{a}^2 + a_z^2$ . Figure 3 shows the total radiated power as a function of dipole separation. As is evident, the geometry of the dipole arrangement not only affects the distribution of the far-field Poynting flux but also the overall amount of emitted power.

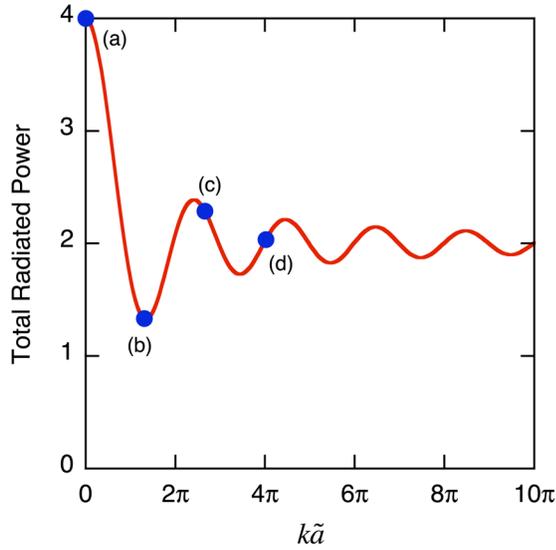


Fig. 3. Total average power emitted from a pair of parallel oscillating dipoles as a function of separation [in units of the power radiated from a single dipole given by Eq. (12)]. As in Fig. 2, we take  $ka_z = 0$  and  $\varphi = 0$ ; separations corresponding to those shown in Fig. 2 are indicated.

### III. NEAR-FIELD WORK IN A TWO-DIPOLE SYSTEM

We next evaluate the work required of the agent causing the dipole oscillations in the presence of the near fields. As we will see, the missing power in the far-field is accounted for in the near-field work. The power required to drive the oscillations of dipole 2 in the presence of field  $\mathbf{E}_1$  is

$$P_{21} = -\mathbf{E}_1(\mathbf{r}_2, t) \cdot \frac{d\mathbf{p}_2}{dt}, \quad (9)$$

where  $\mathbf{p}_2 = \hat{\mathbf{z}}p_0 \cos(-\omega t + \varphi_2)$ . After explicit substitution from Eq. (3), this expression becomes

$$P_{21} = \frac{ck^4 p_0^2 \sin(\omega t - \varphi_2)}{4\pi\epsilon_0} \left\{ \left[ -(ka)^2 \left( \frac{a_z^2}{a^2} - 1 \right) + \left( 3 \frac{a_z^2}{a^2} - 1 \right) \frac{\cos(ka - \omega t + \varphi_1)}{(ka)^3} + \left( 3 \frac{a_z^2}{a^2} - 1 \right) \frac{\sin(ka - \omega t + \varphi_1)}{(ka)^2} \right] \right\}, \quad (10)$$

and with time averaging it reduces to

$$\langle P_{21} \rangle_t = \frac{ck^4 p_0^2}{8\pi\epsilon_0} \left\{ \left[ (k\tilde{a})^2 + \left( 3 \frac{a_z^2}{a^2} - 1 \right) \right] \frac{\sin(ka + \varphi)}{(ka)^3} - \left( 3 \frac{a_z^2}{a^2} - 1 \right) \frac{\cos(ka + \varphi)}{(ka)^2} \right\}. \quad (11)$$

Starting from the above expression, it is possible to determine the power required to drive dipole 2 in the presence of its own field. We simply make the substitution  $\varphi_1 \rightarrow \varphi_2$  (or set  $\varphi = 0$ ) and take the limits  $\tilde{a} \rightarrow 0$  and  $a_z \rightarrow 0$ , in either order. This produces the well-known Larmor radiation formula,

$$\langle P_{22} \rangle_t = \frac{ck^4 p_0^2}{12\pi\epsilon_0}, \quad (12)$$

in this case derived directly from the interplay between the dipole and the near-field rather than indirectly by invoking energy conservation in the far-field.

We may also determine the power required to drive dipole 1 in the presence of field  $\mathbf{E}_2$  as

$$P_{12} = -\mathbf{E}_2(\mathbf{r}_1, t) \cdot \frac{d\mathbf{p}_1}{dt}. \quad (13)$$

Similar to Eq. (11), the time average is

$$\langle P_{12} \rangle_t = \frac{ck^4 p_0^2}{8\pi\epsilon_0} \left\{ \left[ (k\tilde{a})^2 + \left( 3 \frac{a_z^2}{a^2} - 1 \right) \right] \frac{\sin(ka - \varphi)}{(ka)^3} - \left( 3 \frac{a_z^2}{a^2} - 1 \right) \frac{\cos(ka - \varphi)}{(ka)^2} \right\}. \quad (14)$$

Equations (11) and (14) differ only by the sign in front of  $\varphi$ . The expression for  $\langle P_{11} \rangle_t$  is of course identical to Eq. (12).

The total power required of the agent causing the dipole oscillations is  $\langle P \rangle_t = \langle P_{12} \rangle_t + \langle P_{21} \rangle_t + \langle P_{11} \rangle_t + \langle P_{22} \rangle_t$ , which works out to be identical to Eq. (8), as expected. It is satisfying to see that the injected power matches precisely the power radiated away. It is interesting that only the final term in Eq. (3) contributes to the radiation calculation, whereas the computation of the injected power requires one to take into account *all* of the field terms in Eq. (3). This calculation illustrates the crucial role played by the near-field work between the dipoles, as it accounts for the diminished radiated power resulting from destructive interference.

### IV. RADIATION FROM A GAUSSIAN CURRENT DISTRIBUTION

The electric field arising from an arbitrary current and charge distribution (denoted by  $\mathbf{J}$  and  $\rho$ , respectively) is given by Jefimenko's equation,<sup>10</sup>

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[ \frac{\rho(\mathbf{r}', t_r)}{r^2} \hat{\mathbf{z}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c r} \hat{\mathbf{z}} - \frac{\mathbf{J}(\mathbf{r}', t_r)}{c^2 r} \right] d^3 r', \quad (15)$$

where  $\vec{r} \equiv \mathbf{r} - \mathbf{r}'$  and  $t_r \equiv t - r/c$ . We will find it helpful to eliminate the dynamic part of  $\rho$  in favor of  $\mathbf{J}$  by invoking the continuity equation,

$$\dot{\rho} = -\nabla' \cdot \mathbf{J} + \dot{\mathbf{J}} \cdot \hat{\mathbf{z}}/c$$

and

$$\rho = \rho_{\text{static}} + \int^t \dot{\rho} dt' = \rho_{\text{static}} - \int^t \nabla' \cdot \mathbf{J} dt' + \mathbf{J} \cdot \hat{\mathbf{z}}/c. \quad (16)$$

In the above expressions, we have accounted for the fact that  $\mathbf{J}$  depends on  $\mathbf{r}'$  both explicitly and implicitly through  $t_r$ , and it is understood that  $\nabla'$  operates on explicit *and* implicit occurrences of  $\mathbf{r}'$ . The time-independent quantity  $\rho_{\text{static}}$  restores the part of  $\rho$  that does not survive the operation  $\int^t \dot{\rho} dt$  (i.e., an integration constant).

As we substitute Eq. (16) into Eq. (15), we further note (via integration by parts) that

$$\int d^3r' (\nabla' \cdot \mathbf{J}) \frac{\hat{\mathbf{z}}}{\iota^n} = \int d^3r' \frac{\mathbf{J}}{\iota^{n+1}} - (n+1) \int d^3r' (\mathbf{J} \cdot \hat{\mathbf{z}}) \frac{\hat{\mathbf{z}}}{\iota^{n+1}}. \quad (17)$$

With the above substitutions, Jefimenko's equation takes the form,

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = & \frac{1}{4\pi\epsilon_0} \left\{ \frac{\rho_{\text{static}}(\mathbf{r}')}{\iota^2} \hat{\mathbf{z}} \right. \\ & + \int^{t_r} \frac{3[\mathbf{J}(\mathbf{r}', t') \cdot \hat{\mathbf{z}}] \hat{\mathbf{z}} - \mathbf{J}(\mathbf{r}', t')}{\iota^3} dt' \\ & + \frac{3[\mathbf{J}(\mathbf{r}', t_r) \cdot \hat{\mathbf{z}}] \hat{\mathbf{z}} - \mathbf{J}(\mathbf{r}', t_r)}{c\iota^2} \\ & \left. + \frac{[\dot{\mathbf{J}}(\mathbf{r}', t_r) \cdot \hat{\mathbf{z}}] \hat{\mathbf{z}} - \dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\iota} \right\} d^3r'. \quad (18) \end{aligned}$$

It is interesting to note that if the current is chosen to be  $\mathbf{J}(\mathbf{r}', t_r) = \hat{\mathbf{z}} p_0 \sin(-\omega t_r + \varphi_1) \delta^3(\mathbf{r}' - \mathbf{r}_1) / \omega$  with  $\rho_{\text{static}} = 0$ , Eq. (18) reduces to the field surrounding a single oscillating dipole located at  $\mathbf{r}_1$  [identical to Eq. (3)].

We now consider a classical current distribution designed to mimic a free quantum wave packet for a charged particle stimulated by a laser field. If the wave packet is sufficiently diffuse to make quantum spreading slow and if an applied laser field avoids the relativistic regime such that we may neglect geometrical distortions to the shape, we may contemplate a current distribution of the form,

$$\mathbf{J}(\mathbf{r}', t) = \hat{\mathbf{z}} J_0 e^{-r'^2/r_0^2} \sin(kx' - \omega t). \quad (19)$$

This scenario arises when a Gaussian charge distribution is stimulated by an external electric field polarized in the  $z$ -direction and traveling in the  $x$ -direction with frequency  $\omega = ck$ . This current distribution is depicted in Fig. 4.

When Eq. (19) is inserted into Eq. (18), we obtain

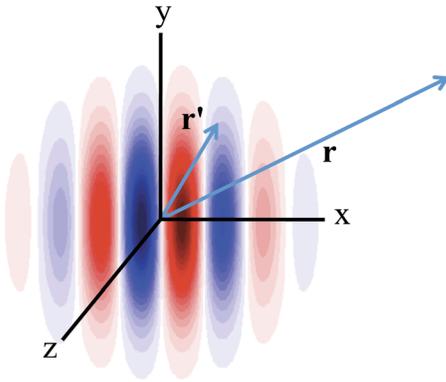


Fig. 4. (Color online) Current distribution defined by Eq. (19) at an instant in time. The direction of the current alternates with each stripe and is polarized in the  $z$ -direction. The internal waves propagate in the  $x$ -direction at speed  $c$  while the overall Gaussian distribution remains centered on the origin.

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = & \frac{J_0}{4\pi\epsilon_0\omega} \int e^{-r'^2/r_0^2} \left\{ [3(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \hat{\mathbf{z}} - \hat{\mathbf{z}}] \right. \\ & \times \left[ \frac{\cos(kx' - \omega t + k\iota)}{\iota^3} + \frac{k \sin(kx' - \omega t + k\iota)}{\iota^2} \right] \\ & - [(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \hat{\mathbf{z}} - \hat{\mathbf{z}}] \frac{k^2 \cos(kx' - \omega t + k\iota)}{\iota} \left. \right\} d^3r' \\ & + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho_{\text{static}}(\mathbf{r}')}{\iota^2} \hat{\mathbf{z}}. \quad (20) \end{aligned}$$

Not surprisingly, this field is identical to that arising from a distribution of oscillating dipoles. Aside from the (inconsequential) electrostatic component, Eq. (20) is nothing more than a Gaussian superposition of dipole fields described by Eq. (3), with  $\mathbf{r}_1 = \mathbf{r}'$  and  $\varphi_1(\mathbf{r}') = kx'$ . In this context,  $\mathbf{J}_0/\omega$  may be thought of as the peak medium polarization (in units of dipoles per volume).

Only the term involving  $1/\iota$  in Eqs. (18) or (20) survives in the far-field limit. As before, in this limit, we make the approximation  $\iota \approx r$  except in the cosine argument where we write  $\iota \approx r - \mathbf{r}' \cdot \hat{\mathbf{r}}$ . Equation (20) then reduces to

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) \approx & -\frac{kJ_0}{4\pi\epsilon_0 cr} [(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \hat{\mathbf{z}}] \int e^{-r'^2/r_0^2} \\ & \times \cos\left(kx' - \omega t + kr - k\frac{x'x + y'y + z'z}{r}\right) d^3r' \\ = & -\hat{\theta} \frac{kJ_0 \sin\theta}{4\pi\epsilon_0 c} \Re \frac{e^{i(kr - \omega t)}}{r} \int_{-\infty}^{\infty} dx' e^{-x'^2/r_0^2} e^{-ik(x/r-1)x'} \\ & \times \int_{-\infty}^{\infty} dy' e^{-y'^2/r_0^2} e^{-iky'y'/r} \int_{-\infty}^{\infty} dz' e^{-z'^2/r_0^2} e^{-ikzz'/r}. \quad (21) \end{aligned}$$

After performing the Gaussian integrals, the electric field simplifies to

$$\mathbf{E}(\mathbf{r}, t) \approx -\hat{\theta} J_0 \frac{k\sqrt{\pi}r_0^3}{4\epsilon_0 c} \sin\theta \frac{\cos(kr - \omega t)}{r} e^{-k^2 r_0^2 (1 - \sin\theta \cos\phi)/2}, \quad (22)$$

with  $x/r = \sin\theta \cos\phi$ .

The average Poynting flux (directed along  $\hat{\mathbf{r}}$ ) is

$$\langle S \rangle_t = \epsilon_0 c \langle |\mathbf{E}|^2 \rangle_t = \frac{k^2 \pi J_0^2 r_0^6}{32\epsilon_0 c} \frac{\sin^2\theta e^{-k^2 r_0^2 (1 - \sin\theta \cos\phi)}}{r^2}, \quad (23)$$

which is shown in Fig. 5. As the size of the source current distribution grows, interferences cause the emitted radiation to be suppressed in every direction except along the  $x$ -axis, the direction of the traveling wave responsible for stimulating the current. This constructive interference in a preferred direction is commonly referred to as phase matching.

To compute the average power radiated into the far-field, as before, we insert Eq. (23) into Eq. (7) to get

$$\langle P \rangle_t = \frac{k^2 \pi J_0^2 r_0^6}{32\epsilon_0 c} e^{-k^2 r_0^2} \int_0^\pi d\theta \sin^3\theta \int_0^{2\pi} d\phi e^{k^2 r_0^2 \sin\theta \cos\phi}. \quad (24)$$

The integration yields<sup>19,20</sup>

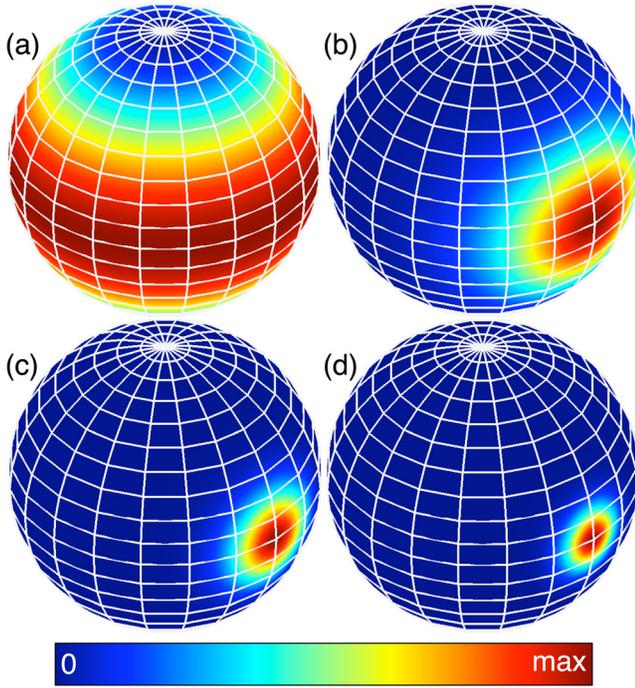


Fig. 5. (Color online) Far-field average Poynting flux from a Gaussian current distribution according to Eq. (23) with distribution sizes (a)  $kr_0 = 0$ , (b)  $kr_0 = 2\pi/3$ , (c)  $kr_0 = 4\pi/3$ , and (d)  $kr_0 = 2\pi$ .

$$\langle P \rangle_t = \frac{\pi^2 J_0^2 r_0^4}{16\epsilon_0 c} \left[ 1 - e^{-2(kr_0)^2} - \frac{1 + e^{-2(kr_0)^2}}{(kr_0)^2} + \frac{1 - e^{-2(kr_0)^2}}{(kr_0)^4} \right]. \quad (25)$$

Note that in the limit  $kr_0 \ll 1$ , this expression reduces to the power radiated from a single dipole [Eq. (12)], as it should, with effective dipole moment  $p_0 = \pi^{3/2} J_0 r_0^3 / kc$ . A graph showing the total power is shown in Fig. 6. As the size of the (normalized) current distribution increases, the relative

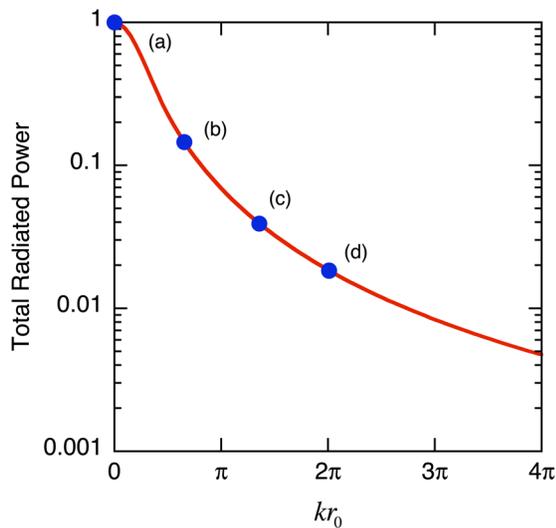


Fig. 6. Total average power emitted from a Gaussian current distribution as a function of distribution size according to Eq. (25). Distribution sizes corresponding to those shown in Fig. 5 are indicated. Power is expressed in units of  $k^2 \pi^2 J_0^2 r_0^6 / (12\epsilon_0 c)$ , which is the effective radiated dipole power for small  $kr_0$ .

overall power drops off as the emission into directions other than the  $x$ -axis is suppressed.

To summarize, Eq. (25) gives the average power radiated from a Gaussian current distribution comprised of linearly polarized oscillations that travel at  $c$  in the  $x$ -direction. This expression may be understood as the time average of the left-hand side of Poynting's theorem,  $\oint \mathbf{S} \cdot \hat{\mathbf{n}} d^2r + \partial/\partial t \int (\epsilon_0 E^2/2 + B^2/2\mu_0) d^3r = -\int \mathbf{E} \cdot \mathbf{J} d^3r$ , since the average field energy for our temporally periodic system is time independent. Thus, Poynting's theorem guarantees that on average the radiated power precisely balances the power necessary to maintain the current, as described by

$$P = -\int \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{J}(\mathbf{r}, t) d^3r. \quad (26)$$

When Eqs. (19) and (20) are explicitly inserted in Eq. (26), we arrive at the following six-dimensional integral:

$$\langle P \rangle_t = \frac{J_0^2 k^2}{8\pi\epsilon_0 c} \int d^3r e^{-r^2/r_0^2} \int d^3r' e^{-r'^2/r_0^2} \times \left\{ \left[ \frac{3(\hat{\mathbf{z}} \cdot \hat{\mathbf{\lambda}})^2 - 1}{(k\lambda)^3} - \frac{(\hat{\mathbf{z}} \cdot \hat{\mathbf{\lambda}})^2 - 1}{k\lambda} \right] \sin(k\lambda - k\lambda_x) - \frac{3(\hat{\mathbf{z}} \cdot \hat{\mathbf{\lambda}})^2 - 1}{(k\lambda)^2} \cos(k\lambda - k\lambda_x) \right\}, \quad (27)$$

where we have used  $\langle \sin(kx - \omega t) \cos(kx' - \omega t + k\lambda) \rangle_t = -\sin(k\lambda - k\lambda_x)/2$  and  $\langle \sin(kx - \omega t) \sin(kx' - \omega t + k\lambda) \rangle_t = \cos(k\lambda - k\lambda_x)/2$ , with  $\lambda_x \equiv x - x'$ . The integration is carried out in Appendix B and as expected it agrees precisely with Eq. (25). Again, it is interesting that the near-field terms play an important role in computing Eq. (27), whereas only the far-field portion is needed to compute Eq. (25). As in the two-dipole case, the near-field work accounts for the diminished radiated power.

## V. HISTORIC BAIT AND SWITCH

Our discussion here is immediately relevant to the original derivation of a quantum electron's scattering cross section. Before the advents of second quantization and QED, Walter Gordon derived the intensity profile for radiation scattered by a single (spinless) electron,<sup>21</sup> which turned out to be correct to lowest order in scalar QED. We provide here a general outline of his semiclassical argument, writing formulas with modernized notation and units.

Gordon's analysis begins with a calculation of approximate momentum states  $\psi_p(\mathbf{r}, t)$  that are dressed by an incident monochromatic plane-wave field (represented by vector potential  $\mathbf{A}_{in}$ ). He superposes these momentum states to form a wave packet,

$$\Psi(\mathbf{r}, t) = \int d^3p \alpha(\mathbf{p}) \psi_p(\mathbf{r}, t), \quad (28)$$

where  $\alpha(\mathbf{p})$  are the momentum amplitudes of the packet. He next uses the quantum wavefunction  $\Psi(\mathbf{r}, t)$  to construct a current density,

$$\mathbf{J}(\mathbf{r}, t) = \frac{e}{m} \Re \{ \Psi^* (-i\hbar \nabla - e\mathbf{A}_{in}) \Psi \}. \quad (29)$$

This current density [with the packet in Eq. (28)] could, for example, look similar to the oscillating Gaussian distribution given in Eq. (19).

Gordon begins the evaluation of the radiation field by using the current density Eq. (29) as a source in Eq. (1). In other words, he appears to treat the quantum current density as a classical current density, consistent with the Schrödinger interpretation.

An interesting development occurs near the end of Gordon's calculation. After substituting Eq. (28) into Eq. (29) and then using Eq. (29) in Eq. (1), one ends up with integrations over  $d^3p$ ,  $d^3p'$ , and  $d^3r'$ , which Gordon expresses in the general far-field form,

$$\mathbf{A}(\mathbf{r}, t) = \Re \int d^3p' \int d^3p \alpha^*(\mathbf{p}') \alpha(\mathbf{p}) \mathbf{A}_{p'p}(\mathbf{r}, t) \delta^3(\tilde{\mathbf{p}}), \quad (30)$$

where

$$\mathbf{A}_{p'p}(\mathbf{r}, t) \delta^3(\tilde{\mathbf{p}}) = \frac{\mu_0 e}{4\pi m r} \int d^3r' \psi_{p'}^*(-i\hbar\nabla - e\mathbf{A}_{\text{in}}) \psi_p. \quad (31)$$

The factor  $\delta^3(\tilde{\mathbf{p}})$  is a momentum-conserving delta function that naturally arises from the integration in Eq. (31), where  $\tilde{\mathbf{p}}$  includes both  $\mathbf{p}$  and  $\mathbf{p}'$  as terms.<sup>22</sup> In a move prescient of QED, Gordon recognizes the latter expression as a kind of transition matrix element with indices  $\mathbf{p}$  and  $\mathbf{p}'$ . Knowing instinctively that the electron ought to transition between momentum states when light is emitted, Gordon concludes by calculating the outgoing light intensity using the *transition potential* defined by Eq. (31) instead of the full potential Eq. (30).

To compute scattered light intensity, classically one expects to square (the derivative of) Eq. (30), which would generate cross terms between the various momentum components via factors of  $\alpha^*(\mathbf{p}')\alpha(\mathbf{p})$ . This immediately would lead to the kinds of interferences described throughout this paper, thereby disagreeing with QED. Rather than squaring the entire integral [Eq. (30)], Gordon squares just a part of the integrand! This departure from classical physics leaves behind Gordon's motivating picture of sourcing Eq. (1) with a wave-packet current. Nevertheless, his final result is applicable to wave packets by summing the *intensities* computed for the different momentum constituents (in contrast with summing fields).<sup>13–15</sup>

It is interesting that Gordon's *ad hoc* step is precisely what is needed to be consistent with the subsequently developed lowest-order scalar QED.<sup>23,24</sup> Gordon's analysis relied on the Klein-Gordon equation. In a subsequent well-known paper, Klein and Nishina<sup>25</sup> applied Gordon's semiclassical framework to the Dirac equation, which correctly incorporates the effects of electron spin. Their result agrees with QED to lowest order.<sup>26</sup> These calculations used notions of wave packets and charge densities to motivate their heuristics, but in the end they produced correct results by making a marked departure from the Schrödinger interpretation.

## VI. CONCLUDING REMARKS

In summary, we have analyzed two examples of radiation in the framework of classical electrodynamics. We considered a pair of oscillating dipoles with arbitrary separation and a Gaussian-shaped current distribution with propagating internal oscillations. In both examples the overall radiated power, as well as the radiation distribution, depends strongly on the source geometry. We demonstrated explicitly the connection between interference in the far-field and the near-

field work between different source components. We note that this connection naturally appears in QED calculations for the photoemission from a *two*-electron wave packet, where far-field interference is possible. Radiative interference for this case is associated with Feynman diagrams that allow the two electrons to perform work on each other through an exchange of virtual photons (demonstrated, for example, following the general procedure in Ref. 15).

The essential role of near-field work, as pertains to far-field radiation, illustrates that a single-electron quantum wave packet with initial wavefunction  $\psi(\mathbf{r}, t)$  and charge  $e$  must radiate quite differently than a classical charge density given by  $\rho(\mathbf{r}, t) = e|\psi(\mathbf{r}, t)|^2$ . At the most basic level, a classical charge density may perform work on itself via Coulomb self repulsion. In contrast, accurate quantum mechanical results for the hydrogen atom are derived by excluding electron self repulsion from the Hamiltonian. The removal of interferences in the scattered radiation from a single quantum electron is as natural (and critical) as the omission of the classical Coulomb self repulsion.

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## APPENDIX A: INTEGRATION OF FAR-FIELD POWER FOR TWO RADIATING DIPOLES

In this Appendix, we work out the integration in Eq. (7), which, when written explicitly, is

$$\begin{aligned} \langle P \rangle_t &= \frac{k^4 c p_0^2}{(4\pi)^2 \epsilon_0} \left[ \frac{8\pi}{3} + \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta \right. \\ &\quad \times \cos(ka_x \sin \theta \cos \phi + ka_y \sin \theta \sin \phi \\ &\quad \left. + ka_z \cos \theta - \varphi) \right]. \quad (A1) \end{aligned}$$

We make the following expansion:

$$\begin{aligned} &\cos(ka_x \sin \theta \cos \phi + ka_y \sin \theta \sin \phi + ka_z \cos \theta - \varphi) \\ &= \cos(ka_x \sin \theta \cos \phi + ka_y \sin \theta \sin \phi - \varphi) \\ &\quad \times \cos(ka_z \cos \theta) - \sin(ka_x \sin \theta \cos \phi \\ &\quad + ka_y \sin \theta \sin \phi - \varphi) \sin(ka_z \cos \theta). \quad (A2) \end{aligned}$$

The final line gives zero when integrated over  $\theta$  because the integrand in the interval  $(0, \pi/2)$  is equal and opposite to the integrand in the interval  $(\pi/2, \pi)$ . Further, we make the expansion,

$$\begin{aligned} &\cos(ka_x \sin \theta \cos \phi + ka_y \sin \theta \sin \phi - \varphi) \\ &= \cos \varphi \cos(ka_x \sin \theta \cos \phi) \cos(ka_y \sin \theta \sin \phi) \\ &\quad - \cos \varphi \sin(ka_x \sin \theta \cos \phi) \sin(ka_y \sin \theta \sin \phi) \\ &\quad + \sin \varphi \sin(ka_x \sin \theta \cos \phi) \cos(ka_y \sin \theta \sin \phi) \\ &\quad + \sin \varphi \cos(ka_x \sin \theta \cos \phi) \sin(ka_y \sin \theta \sin \phi), \quad (A3) \end{aligned}$$

where only the first term survives integration over  $\phi$ , owing to symmetries in the interval  $(0, 2\pi)$ .

Next, we let  $\tilde{a} \equiv \sqrt{a_x^2 + a_y^2}$  and write  $a_x = \tilde{a} \cos \tilde{\phi}$  and  $a_y = \tilde{a} \sin \tilde{\phi}$ , allowing us to write

$$\begin{aligned} & \cos(k a_x \sin \theta \cos \phi) \cos(k a_y \sin \theta \sin \phi) \\ &= \frac{1}{2} \left[ \cos(k \tilde{a} \cos(\phi - \tilde{\phi}) \sin \theta) \right. \\ & \quad \left. + \cos(k \tilde{a} \cos(\phi + \tilde{\phi}) \sin \theta) \right]. \end{aligned} \quad (\text{A4})$$

Because the integration over  $\phi$  is periodic over the interval  $(0, 2\pi)$ , the offset  $\pm \tilde{\phi}$  has no effect on the result and can be dropped. In view of Eqs. (A2)–(A4), Eq. (A1) then reduces to

$$\begin{aligned} \langle P \rangle_t &= \frac{k^4 c p_0^2}{(4\pi)^2 \epsilon_0} \left[ \frac{8\pi}{3} + \cos \varphi \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta \right. \\ & \quad \left. \times \cos(k \tilde{a} \sin \theta \cos \phi) \cos(k a_z \cos \theta) \right]. \end{aligned} \quad (\text{A5})$$

The integration over  $\phi$  is accomplished using the well-known formula  $\int_0^{2\pi} d\phi \cos(\alpha \cos \phi) = 2\pi J_0(\alpha)$ , with  $\alpha \equiv k \tilde{a} \sin \theta$ . The remaining integration over  $\theta$  is accomplished using<sup>27</sup>

$$\begin{aligned} & \int_0^\pi d\theta \sin^3 \theta \cos(\alpha \cos \theta) J_0(\beta \sin \theta) \\ &= \frac{2}{[\alpha^2 + \beta^2]^{3/2}} \left[ \left( \beta^2 + \frac{2\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \right) \sin \sqrt{\alpha^2 + \beta^2} \right. \\ & \quad \left. + \frac{\alpha^2 - 2\beta^2}{\sqrt{\alpha^2 + \beta^2}} \cos \sqrt{\alpha^2 + \beta^2} \right], \end{aligned} \quad (\text{A6})$$

where  $\alpha \equiv k a_z$  and  $\beta \equiv k \tilde{a}$ ; this leads to the result in Eq. (8).

## APPENDIX B: TIME-AVERAGED POWER REQUIRED TO DRIVE A GAUSSIAN CURRENT DISTRIBUTION

In this Appendix, we evaluate Eq. (27). After making substitutions  $x' = x - \iota_x$ ,  $y' = y - \iota_y$ , and  $z' = z - \iota_z$  and treating  $\iota_x$ ,  $\iota_y$ , and  $\iota_z$  as the new integration variables, Eq. (27) can be rewritten as

$$\begin{aligned} \langle P \rangle_t &= \frac{J_0^2}{8\pi \epsilon_0 c} \int d^3 \iota \frac{e^{-\iota^2/r_0^2}}{\iota^2} \times \left\{ \left[ \left( 3 \frac{\iota_z^2}{\iota^2} - 1 \right) \right. \right. \\ & \quad \left. \left. - (k\iota)^2 \left( \frac{\iota_z^2}{\iota^2} - 1 \right) \right] \frac{\sin(k\iota - k\iota_x)}{k\iota} \right. \\ & \quad \left. - \left( 3 \frac{\iota_z^2}{\iota^2} - 1 \right) \cos(k\iota - k\iota_x) \right\} \times \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\ & \quad \times \int_{-\infty}^{\infty} dz e^{-2(x^2+y^2+z^2)/r_0^2} e^{2(\iota_x x + \iota_y y + \iota_z z)/r_0^2}. \end{aligned} \quad (\text{B1})$$

The Gaussian integration of the final line is readily performed and yields  $r_0^3 (\pi/2)^{3/2} e^{\iota^2/2r_0^2}$ . Equation (B1) then becomes

$$\begin{aligned} \langle P \rangle_t &= \frac{J_0^2 r_0^3}{16\epsilon_0 c} \sqrt{\frac{\pi}{2}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\infty d\iota e^{-\iota^2/2r_0^2} \\ & \quad \times \left\{ [(3 \cos^2 \theta - 1) - (k\iota)^2 (\cos^2 \theta - 1)] \right. \\ & \quad \times \frac{\sin(k\iota - k\iota \sin \theta \cos \phi)}{k\iota} \\ & \quad \left. - (3 \cos^2 \theta - 1) \cos(k\iota - k\iota \sin \theta \cos \phi) \right\}, \end{aligned} \quad (\text{B2})$$

where we have made use of spherical coordinates.

We next make the expansions

$$\begin{aligned} \sin(k\iota - k\iota \sin \theta \cos \phi) &= \sin(k\iota) \cos(k\iota \sin \theta \cos \phi) \\ & \quad - \cos(k\iota) \sin(k\iota \sin \theta \cos \phi) \\ \cos(k\iota - k\iota \sin \theta \cos \phi) &= \cos(k\iota) \cos(k\iota \sin \theta \cos \phi) \\ & \quad + \sin(k\iota) \sin(k\iota \sin \theta \cos \phi) \end{aligned} \quad (\text{B3})$$

and note that  $\int_0^{2\pi} d\phi \sin(\alpha \cos \phi) = 0$  and  $\int_0^{2\pi} d\phi \cos(\alpha \cos \phi) = 2\pi J_0(\alpha)$ . The expression for the average power then becomes

$$\begin{aligned} \langle P \rangle_t &= \frac{\pi J_0^2 r_0^3}{8\epsilon_0 c} \sqrt{\frac{\pi}{2}} \int_0^\infty d\iota e^{-\iota^2/2r_0^2} \\ & \quad \times \int_0^\pi d\theta \left\{ 2 \left[ \frac{\sin(k\iota)}{(k\iota)} - \cos(k\iota) \right] \sin \theta \right. \\ & \quad \left. + \left[ ((k\iota)^2 - 3) \frac{\sin(k\iota)}{(k\iota)} + 3 \cos(k\iota) \right] \sin^3 \theta \right\} \\ & \quad \times J_0(k\iota \sin \theta). \end{aligned} \quad (\text{B4})$$

The  $\theta$  integration is accomplished with the aid of  $\int_0^\pi d\theta \sin \theta \times J_0(u \sin \theta) = \sqrt{2\pi/u} J_{1/2}(u)$  and  $\int_0^\pi d\theta \sin^3 \theta J_0(u \sin \theta) = \sqrt{2\pi/u} [J_{1/2}(u) - J_{3/2}(u)/u]$ , where  $u \equiv k\iota$ .<sup>20</sup> Additionally, we may write  $\sin u/u - \cos u = \sqrt{\pi u/2} J_{3/2}(u)$  and  $\sin u = \sqrt{\pi u/2} J_{1/2}(u)$ , whereupon the average power may be expressed as

$$\begin{aligned} \langle P \rangle_t &= \frac{\pi^2 J_0^2 r_0^3}{8\epsilon_0 c} \sqrt{\frac{\pi}{2}} \int_0^\infty du e^{-u^2/2k^2 r_0^2} [u J_{1/2}^2(u) \\ & \quad - 2J_{1/2}(u) J_{3/2}(u) + 3J_{3/2}^2(u)/u]. \end{aligned} \quad (\text{B5})$$

This final integral<sup>27</sup> (excluding all pre factors) is equal to  $\alpha \sqrt{\pi/2} [1 - e^{-2\alpha^2} - (1 + e^{-2\alpha^2})/\alpha^2 + (1 - e^{-2\alpha^2})/\alpha^4]$ , where  $\alpha \equiv k r_0$ , which finally leads to Eq. (25).

<sup>1</sup>E. Schrödinger, "An undulatory theory of the mechanics of atoms and molecules," *Phys. Rev.* **28**, 1049–1070 (1926).

<sup>2</sup>M. Born, "Zur Quantenmechanik der Stoßvorgänge," *Z. Naturforsch. B* **37**, 863–867 (1926).

<sup>3</sup>D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson, Upper Saddle River NJ, 2005), p. 2.

<sup>4</sup>J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton U.P., Princeton, 1955), p. 198.

<sup>5</sup>A. Barut and J. Kraus, "Nonperturbative quantum electrodynamics: The Lamb shift," *Found. Phys.* **13**, 189–194 (1983).

<sup>6</sup>C. W. Kilmister, *Schrödinger: Centenary celebration of a polymath*, (Cambridge U.P., Cambridge, 1987).

<sup>7</sup>A. Barut, "The revival of Schrödinger's interpretation of quantum mechanics," *Found. Phys. Lett.* **1**, 47–56 (1988).

<sup>8</sup>E. T. Jaynes, in *The Electron*, edited by D. Hestenes and A. Weingartshofer (Kluwer, Dordrecht, 1991), pp. 1–20.

- <sup>9</sup>L. Johansson, *Interpreting Quantum Mechanics* (Ashgate, Burlington VT, 2007).
- <sup>10</sup>D. J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, Boston, 2013), pp. 444, 449.
- <sup>11</sup>P. Krekora, R. E. Wagner, Q. Su, and R. Grobe, "Retardation and kinematic relativistic effects in scattered light," *Laser Phys.* **12**, 455–462 (2002).
- <sup>12</sup>E. A. Chowdhury, I. Ghebregziabihier, and B. C. Walker, "Larmor radiation from the ultra-intense field ionization of atoms," *J. Phys. B* **38**, 517–524 (2005).
- <sup>13</sup>J. Peatross, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, "Photoemission of a single-electron wave packet in a strong laser field," *Phys. Rev. Lett.* **100**, 153601-1–4 (2008).
- <sup>14</sup>J. P. Corson, J. Peatross, C. Müller, and K. Z. Hatsagortsyan, "Scattering of intense laser radiation by a single-electron wave packet," *Phys. Rev. A* **84**, 053831-1–9 (2011).
- <sup>15</sup>J. P. Corson and J. Peatross, "Quantum-electrodynamic treatment of photoemission by a single-electron wave packet," *Phys. Rev. A* **84**, 053832-1–11 (2011).
- <sup>16</sup>T. Cheng, C. C. Gerry, Q. Su, and R. Grobe, "Transfer of source correlations to counter-propagating bosons," *Eur. Phys. Lett.* **88**, 54001-1–6 (2009).
- <sup>17</sup>P. R. Berman, "Energy conservation in collective coherent emission by dipole oscillators," *Am. J. Phys.* **78**, 1323–1330 (2010).
- <sup>18</sup>J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998), Eq. (9.18).
- <sup>19</sup>I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., edited by A. Jeffrey and D. Zwillinger (Academic Press, Burlington, MA, 2007), Formulas 3.534.2, 8.406.3.
- <sup>20</sup>I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., edited by A. Jeffrey and D. Zwillinger (Academic Press, Burlington, MA, 2007), Formulas 6.682.1, 6.683.6, 8.339.2, 8.464.1, 8.464.3.
- <sup>21</sup>W. Gordon, "Der Comptoneffekt nach der Schrödingerschen Theorie," *Zeit. Phys.* **40**, 117–133 (1926).
- <sup>22</sup>It is implied that quantities on the right-hand-side of Eq. (31) are evaluated at the retarded time. Interestingly,  $\vec{p} \equiv \mathbf{p} + \hbar\mathbf{k} - \mathbf{p}' - \hbar\mathbf{k}'\hat{r}$  depends on an incident photon momentum  $\hbar\mathbf{k}$  as well as on a scattered photon momentum  $\hbar\mathbf{k}' \equiv E_p/c + \hbar\mathbf{k} - E_{p'}/c$ , where  $E_p \equiv \sqrt{p^2c^2 + m^2c^4}$ , even though all fields are intrinsically classical. Gordon changes to a different set of momentum variables in order to simplify the integrals, which introduces explicit Doppler factors.<sup>24</sup>
- <sup>23</sup>J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 121.
- <sup>24</sup>J. Lacki, H. Ruegg, and V. L. Telegdi, "The road to Stueckelberg's covariant perturbation theory as illustrated by successive treatments of Compton scattering," *Stud. Hist. Phil. Mod. Phys.* **30**, 457–518 (1999).
- <sup>25</sup>O. Klein and Y. Nishina, "Über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen Quantendynamik von Dirac," *Z. Naturforsch. B* **52**, 853–868 (1929).
- <sup>26</sup>I. Tamm, "Über die Wechselwirkung der freien Elektronen mit der Strahlung nach der Diracschen Theorie des Elektrons und nach der Quantenelektrodynamik," *Z. Naturforsch. B* **62**, 545–568 (1930).
- <sup>27</sup>We were unable to find this integral in a table but have tested it numerically.



### Lead Bells

Yes – you read this right. If you soak these small lead bells in liquid nitrogen, they will ring brightly, until they warm up again. One is cast in a small mold, while the other one seems to be formed from sheet lead. They are in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)