Real-time recoverable and irrecoverable energy in dispersive-dissipative dielectrics

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We discuss the recoverable and irrecoverable energy densities associated with a pulse at a point in the propagation medium and derive easily computed expressions to calculate these quantities. Specific types of fields are required to retrieve the recoverable portion of the energy density from the point in the medium, and we discuss the properties that these fields must have. Several examples are given to illustrate these concepts.

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I. INTRODUCTION

As an optical pulse propagates in a passive dielectric, the pulse field continually exchanges energy with the medium. The process of energy exchange is never perfectly efficient, and some fraction of the exchanged energy is irrecoverably dissipated by the medium. Although it is straightforward to calculate the total energy lost to the medium *after* a point in space has experienced the entire temporal pulse, it is less obvious how to account for energy loss in real time, i.e., *during* the medium's interaction with the pulse.

An example of the relevance of real-time energy accounting arises in the phenomena of "fast" and "slow" light [1–3]. Slow light results when energy from the leading part of the pulse is stored temporarily by the medium and then returned to the pulse's latter portion [4]. The effect is most striking when only a small amount of energy is dissipated and there is a relatively long delay between when energy transfers into the medium and when it eventually returns to the field. Experiments where dramatically slow propagation is achieved implicitly involve engineering the pulse-medium combination so that a large fraction of the energy transferred to the medium remains "recoverable" by the latter portion of the pulse. On the other hand, to observe fast light, experimenters choose pulse-medium combinations where the latter portion of the pulse avoids recovering the energy transferred into the medium by earlier portions of the pulse.

In this article, we identify the fraction of stored energy that could potentially be recovered by a future pulse field and the fraction that is irrecoverably lost to the medium at any given time during the pulse-medium interaction and derive easily computed expressions for calculating these quantities. The resulting expressions are mathematically equivalent to those derived by Polevoi in [5]. The results obtained with this method are independent of the model used to represent the medium, and the relevant quantities can be numerically computed in a straightforward fashion. We illustrate the results for several examples relating to superluminal and subluminal pulse propagation. (This method of classifying energy is not the only possible approach. Various energy accounting methods have been developed for general viscoelastic [6–8] and dielectric media [9–11].)

We restrict our analysis to passive, homogeneous, isotropic linear dielectrics (nonmagnetic) without spatial dispersion. As usual, the polarization $P(t)$ of the medium in response to an electric field $E(t)$ is specified by a susceptibility $\chi(\omega)$ through

$$P(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(\omega)\hat{E}(\omega)e^{-i\omega t}d\omega, \tag{1}$$

where $\hat{E}(\omega)$ is the Fourier transform of $E(t)$. The polarization is temporally nonlocal since $P(t)$ depends on the electric field at times other than t. This dependence is described by an impulse response function $G(t)$

$$P(t) = \int_{-\infty}^{+\infty} G(t-\tau)E(\tau)d\tau = \int_{-\infty}^{t} G(t-\tau)E(\tau)d\tau, \tag{2}$$

where

$$\chi(\omega) = \int_{-\infty}^{+\infty} G(t')e^{i\omega t'}dt' = \int_{0}^{+\infty} G(t')e^{i\omega t'}dt'. \tag{3}$$

Causality is assured by requiring $G(t')$ to be zero for $t' < 0$ (i.e., only past fields can influence the current polarization), as indicated by the second forms of (2) and (3). A straightforward analysis of the second form of (3) shows that $\chi(\omega)$ must be an analytic function of ω in the upper half of the complex plane in order for the system to be causal [12]. In addition, since $G(t')$ is real we have

$$\chi(-\omega) = \chi^*(\omega^*). \tag{4}$$

These properties of $\chi(\omega)$ will be important to our analysis.

In the Lorentz-Heaviside system of units (and disregarding a factor of 4π), Poynting's conservation theorem [12] is given by

$$\mathbf{\nabla} \cdot \mathbf{S}(t) + \frac{\partial u(t)}{\partial t} = 0, \quad u(t) = u_{\text{field}}(t) + u_{\text{int}}(t), \tag{5}$$

$$u_{\text{field}}(t) = \frac{1}{2}E^2(t) + \frac{1}{2}H^2(t), \tag{6}$$

$$u_{\text{int}}(t) := \int_{-\infty}^{t} E(\tau)\dot{P}(\tau)d\tau, \qquad (7)$$

where E, H, and P denote the electric, magnetic, and polarization fields, and $\mathbf{S}(t)$ refers to the usual Poynting vector $\mathbf{E}(t) \times \mathbf{H}(t)$. The total energy density $u(t)$ is composed of two parts: $u_{\text{field}}(t)$ is the field energy density at time t, and $u_{\text{int}}(t)$ is the accumulation of energy density transferred into the medium (at the point of consideration) from the beginning of the pulse-medium interaction until time t.

After a point has experienced the entire pulse (i.e., as $t \to +\infty$), the interaction energy density $u_{\text{int}}(+\infty)$ specifies the amount of energy density that was dissipated into the medium (at that spatial point) over the course of the entire pulse. A straightforward calculation demonstrates that this lost energy depends only on χ and the evolution of the electric field E, not on any specific model giving rise to χ [13]. This suggests the possibility of establishing an equally unambiguous and useful notion of the energy irretrievably lost from a pulse at an arbitrary time t as the medium experiences the pulse.

Barash and Ginzburg studied the feasibility of this sort of dynamic generalization and concluded that any such notion must be tied to a specific model of the medium [14]. Consequently they focused on extending some work by Loudon that described the correct real-time notion of loss for a single oscillator Lorentz medium [15]. Barash and Ginzburg generalized Loudon's analysis to include multiple oscillator Lorentz media. To facilitate comparison, we summarize their work here (with which we disagree).

In the Barash and Ginzburg approach, the susceptibility of the medium is modeled by a sum of Lorentz oscillators,

$$\chi(\omega) = \sum_{n=1}^{N} \chi_n(\omega) = \sum_{n=1}^{N} \frac{f_n \omega_{p_n}^2}{\omega_n^2 - i\gamma_n \omega - \omega^2}. \qquad (8)$$

The phenomenological parameters f_n, ω_{p_n}, ω_n, and γ_n are the oscillator strength, plasma frequency, resonant frequency, and damping rate of the nth Lorentz oscillator. To obtain expressions for loss and energy, u_{int} is written using (8) and then expanded. Terms explicitly depending on phenomenological damping parameters are collected and separated from those that do not. The former are said to represent losses, the latter energy. This procedure yields the following expressions for the different types of energy:

$$u_{\text{int}}(t) = u_e(t) + u_\ell(t), \qquad (9)$$

$$u_e(t) = \sum_{n=1}^{N} \frac{1}{2f_n \omega_{p_n}^2} \dot{P}_n^2(t) + \frac{\omega_n^2}{2f_n \omega_{p_n}^2} P_n^2(t), \qquad (10)$$

$$u_\ell(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2(\tau)d\tau, \qquad (11)$$

where

$$P_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_n(\omega) \hat{E}(\omega)d\omega. \qquad (12)$$

The quantity u_e contains the collected "energy" terms and u_ℓ contains the "loss" terms. Note that the kinetic and potential energies of each individual oscillator (as usually identified) appear in (10), and that the usual viscous or frictional losses of each oscillator are summed in (11). Thus, in this viewpoint, the energy stored in the medium is effectively defined to be the energy that the several oscillators could donate to the field if, despite their distinct resonant frequencies, there were no interference between them as they surrender energy back to the optical pulse.

The quantities represented in (10) and (11) are explicitly model-dependent. Barash and Ginzburg point out that the parametrization of a given χ is generally not unique. Furthermore, two distinct parametrizations for the same χ generally result in different energy allocations between u_ℓ and u_e for a given field. Thus, in the Barash and Ginzburg approach one can change the fraction of energy that is allocated as "lost" at a given time by changing the parametrization of χ, even though the choice of parametrization has no effect on physically measurable quantities. We find this unsatisfactory at the physical level. In this paper we develop concepts of recoverable energy and loss that depend only on χ and the electric field, and not on the parametrization of χ.

II. RECOVERABLE AND IRRECOVERABLE ENERGY

We use the following natural definition for energy that is recoverable by the field [16]:

Recoverable energy: *The recoverable energy (density) $u_{\text{rec}}(t)$ at time t, is the supremum of the amount of energy (density) that the dielectric can subsequently return to the field (under the influence of a well-chosen field)*:

$$u_{\text{rec}}[E](t) := \sup_{E_+^t} \{u_{\text{int}}[E](t) - u_{\text{int}}[E_-^t + E_+^t](+\infty)\}$$

$$= u_{\text{int}}[E](t) - \inf_{E_+^t} \{u_{\text{int}}[E_-^t + E_+^t](+\infty)\}. \qquad (13)$$

The complementary dynamical notion of loss is as follows:

Irrecoverable energy: *The irrecoverable energy (density) $u_{\text{irrec}}(t)$ at time t, is the infimum of the amount of energy that must be eventually dissipated by the medium (under the influence of a well-chosen field)*:

$$u_{\text{irrec}}[E](t) := \inf_{E_+^t} u_{\text{int}}[E_-^t + E_+^t](+\infty)$$

$$= u_{\text{int}}[E](t) - u_{\text{rec}}[E](t). \qquad (14)$$

The notation $u_{\text{int}}[E](t)$ in these definitions emphasizes the fact that u_{int} depends on both the field E and the time t. We have explicitly broken E into a fixed "past field" E_-^t and a variable "future field" E_+^t:

$$E_-^t(\tau) := \begin{cases} E(\tau) & \tau \leq t \\ 0 & \tau > t, \end{cases} \qquad (15)$$

$$E_+^t(\tau) := 0 \quad \tau < t, \qquad (16)$$

where the future field E_+^t is indeterminate after the current time. The recoverable and irrecoverable energy densities are explicitly causal (i.e., the field evolution after time t does not affect the energy and loss at that time):

$$u_{\text{irrec}}[E](t) = u_{\text{irrec}}[E_-^t + E_+^t](t) = u_{\text{irrec}}[E_-^t](t),$$

$$u_{\text{rec}}[E](t) = u_{\text{rec}}[E_-^t + E_+^t](t) = u_{\text{rec}}[E_-^t](t).$$

The supremum in (13) and the infimum in (14) are accomplished by treating the past field as fixed and allowing the future field to vary as necessary to accomplish the extrema. Thus $u_{\text{rec}}[E](t)$ gives the largest portion of $u_{\text{int}}(t)$ that could possibly be converted back to field energy after t, given a fixed past field. Conversely, $u_{\text{irrec}}[E](t)$ represents the smallest amount of eventual loss $u_{\text{int}}(+\infty)$ that is required for fields that share a common history prior to t. The second line of (14) makes it clear that both extrema are accomplished by the same future field.

If we look at the medium as a whole (rather than focusing on a single point) we find that the maximum possible increase in total field energy after a given time is given by the total recoverable energy at that time. More precisely,

$$\Delta_t \mathcal{E}(t) \le \int u_{\text{rec}}(t) d\mathbf{x}, \qquad (17)$$

where the total field energy $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) := \int u_{\text{field}}(t) d\mathbf{x} \qquad (18)$$

(see Appendix A for a proof). By comparison, the Barash and Ginzburg notion of energy in (10) overestimate the amount of energy available for return to the field

$$\int u_{\text{rec}}(t) d\mathbf{x} \le \int u_{\text{e}}(t) d\mathbf{x}, \qquad (19)$$

with equality rarely holding for more than one oscillator [$N>1$ in (8)]. In contrast, we find in Sec. IV that it is always possible for the inequality in (19) to be saturated.

The inequality (19) is demonstrated in Appendix B by showing that the following inequalities hold at every point in the medium:

$$u_{\text{rec}}(t) \le u_{\text{e}}(t),$$

$$u_{\text{irrec}}(t) \ge u_\ell(t). \qquad (20)$$

The difference between the irrecoverable energy (14) and the Barash and Ginzburg loss is given by

$$u_{\text{irrec}}[E](t) - u_\ell[E](t) = u_{\ell-\text{fut}}[E](t), \qquad (21)$$

where

$$u_{\ell-\text{fut}}[E](t) := \inf_{E_+^t} \sum_{n=1}^{N} \int_t^{+\infty} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2[E_-^t + E_+^t](\tau) d\tau \quad (22)$$

gives the Barash and Ginzburg loss that is inevitable in the future. At first glance, the result in (21) may seem to indicate

that the loss derived by Barash and Ginzburg describes an "already occurred" loss, and definition (14) simply adds losses that must inevitably occur in the future. As noted before, however, the loss in (11), and so also the "future loss" in (22), are model dependent. Thus, in the Barash and Ginzburg approach the choice of parametrization determines how loss is allocated between past and future.

III. COMPUTING RECOVERABLE AND IRRECOVERABLE ENERGY

Given an explicit representation of χ, one can (in principle) perform the extremizations in (13) and (14) to find an explicit representation for the recoverable and irrecoverable energy densities. For a single Lorentz oscillator the recoverable energy density is given by (10) and the irrecoverable energy density is given by (11) (both with $N=1$). For more complicated models, the extremization is usually fraught with the intractability of finding the roots of a polynomial and the resulting formulas are frequently not particularly enlightening. Thus, it is often not of interest to derive an explicit formula in terms of model parameters. Instead, we focus on obtaining a general algorithm for numerically calculating recoverable and irrecoverable energy densities.

In this section we prove that the irrecoverable energy density defined in the previous section can be calculated using the following expressions:

$$u_{\text{irrec}}[E](t) = R(\infty) \int_{-\infty}^{t} \dot{P}_{\text{eff}}^2[E_-^t](\tau) d\tau, \qquad (23)$$

where

$$P_{\text{eff}}[E](\tau) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{[\mathcal{F}\theta_+\mathcal{F}^{-1} \ln R(\omega)/R(\infty)]} \chi(\omega) \hat{E}(\omega) e^{i\omega\tau} d\omega \tag{24}$$

and

$$R := \text{Re}\left[\frac{1}{-i\omega\chi(\omega)}\right]. \qquad (25)$$

Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, and θ_+ is a Heaviside function supported for positive time. The recoverable energy can be obtained using the relation $u_{\text{rec}} = u_{\text{int}} - u_{\text{irrec}}$. In some respects (23) is similar to (11) (e.g., both have integrals of a positive-definite quantity), but note that (23) depends on $\chi(\omega)$ itself, not on a parametrization of χ. In Sec. IV we demonstrate how to calculate a future field that extracts all of the recoverable energy from the medium, and in Sec. V we discuss several examples illustrating the use of these formulas.

We begin our derivation of (23) with the interaction energy density defined in (7). Landau and Lifshitz showed that after the medium has experienced the entire pulse, the energy density remaining in the medium (i.e., the loss) can be expressed as

$$u_{\text{int}}[E](+\infty) = \int_{-\infty}^{+\infty} \omega \, \text{Im}[\chi(\omega)] |\hat{E}(\omega)|^2 d\omega. \quad (26)$$

In previous articles [4,17] we showed that (26) can be generalized to any time during the pulse as

$$u_{\text{int}}[E](t) = \int_{-\infty}^{+\infty} \omega \, \text{Im}[\chi(\omega)] |\hat{E}_-^t(\omega)|^2 d\omega, \quad (27)$$

where $\hat{E}_-^t(\omega)$ is the "instantaneous spectrum," i.e., the Fourier transform of the field $E(\tau)$ whose future has been eliminated:

$$\hat{E}_-^t(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E_-^t(\tau) e^{i\omega\tau} d\tau := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} E(\tau) e^{i\omega\tau} d\tau. \quad (28)$$

In a passive medium, we have

$$0 \leq \omega \, \text{Im}[\chi(\omega)] \quad (29)$$

with equality only at $\omega = 0, \infty$. Thus, (27) demonstrates that the medium never returns more energy to the field than it has received from it. This allows u_{rec} to be well-defined by (13). Since truncating a time evolution broadens the associated spectrum, (27) shows that the effective overlap of the field spectrum and medium resonances (described by $\omega \, \text{Im}[\chi(\omega)]$) changes as the medium experiences a pulse. This explains why u_{int} is rarely monotonic.

We proceed with the development by manipulating (27) as follows:

$$u_{\text{int}}[E](t) = \int_{-\infty}^{+\infty} \frac{\text{Im}[\chi(\omega)]}{\omega |\chi(\omega)|^2} |-i\omega\chi(\omega)\hat{E}_-^t(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{+\infty} R(\omega) |-i\omega\chi(\omega)\hat{E}_-^t(\omega)|^2 d\omega, \quad (30)$$

where

$$R(\omega) := \frac{\text{Im}[\chi(\omega)]}{\omega |\chi(\omega)|^2} = \text{Re}\left[\frac{1}{-i\omega\chi(\omega)}\right]. \quad (31)$$

It is useful to normalize $R(\omega)$,

$$\bar{R}(\omega) := \frac{R(\omega)}{R(\infty)},$$

$$R(\infty) := \lim_{\omega \to \infty} R(\omega), \quad (32)$$

and then factor the normalized $\bar{R}(\omega)$ as

$$\bar{R}(\omega) = |\bar{R}^+(\omega)|^2 = \bar{R}^+(\omega)\bar{R}^{+*}(\omega) = \bar{R}^+(\omega)\bar{R}^+(-\omega)$$

$$= \bar{R}^+(\omega)\bar{R}^-(\omega). \quad (33)$$

We require the factors $\bar{R}^+(\omega)$ and $\bar{R}^-(\omega)$ to be analytic and nonvanishing in the upper and lower half planes, respectively. Thus, it is insufficient, for example, to simply choose $\bar{R}^+(\omega) = \sqrt{\bar{R}(\omega)}$.

The identification of factors analytic in complementary regions of the complex plane as called for in (33) is called a "homogeneous Riemman-Hilbert problem." Since the factors in (33) are commuting scalar-valued functions, there is a formulaic solution. In Appendix C we carry out the factorization of $\bar{R}$ and find

$$\bar{R}^+(\omega) = \exp \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} \ln \bar{R}(\omega), \quad (34)$$

where θ_+ is a Heavyside (step) function supported for positive times, and $\mathcal{F}^{+1}$ and $\mathcal{F}^{-1}$ indicate the Fourier transform and its inverse. Note that the normalized $\bar{R}(\omega)$ ensures that the logarithm goes to zero as $\omega \to \infty$, which makes the following transforms and other operations well behaved. (If we had not normalized, we would be faced with the awkward situation of trying to apply θ_+ to a function with a delta distribution spike directly at the step discontinuity.)

With definitions (32) and (33) we rewrite (30) as

$$u_{\text{int}}[E](t) = R(\infty) \int_{-\infty}^{+\infty} |-i\omega\bar{R}^+(\omega)\chi(\omega)\hat{E}_-^t(\omega)|^2 d\omega, \quad (35)$$

and complete our manipulation by transforming (35) into the time domain using Parseval's theorem,

$$u_{\text{int}}[E](t) = R(\infty) \int_{-\infty}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t](\tau) d\tau. \quad (36)$$

The time derivative comes from the $-i\omega$ factor in (35), and the "effective polarization" P_{eff} is defined by

$$P_{\text{eff}}[E](\tau) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \chi_{\text{eff}}(\omega)\hat{E}(\omega)e^{-i\omega\tau} d\omega. \quad (37)$$

Because of our careful choice of $R^+(\omega)$, the "effective susceptibility"

$$\chi_{\text{eff}}(\omega) := \bar{R}^+(\omega)\chi(\omega) \quad (38)$$

shares many properties with $\chi(\omega)$: it is real symmetric, analytic in the upper-half complex ω plane, and also has the same asymptotics as $\chi(\omega)$ there. As discussed in relation to (2), these properties ensure that the behavior of P_{eff} is causally linked to the field. Specifically, we can replace the past field E_-^t in (36) with the past field plus an arbitrary future field without changing any of the past values of $P_{\text{eff}}[E](t)$. If we had been less careful in factoring $\bar{R}$, the effective polarization would not have this property.

The effective polarization P_{eff} can be thought of as the response that a medium with susceptibility χ_{eff} would have to the electric field. While P_{eff} is only indirectly related to the physical polarization, it is instructive to consider its behavior. As an important example, note that the integral in (36) cannot be truncated after time $\tau = t$. If we separate the integral into two pieces

$$u_{\text{int}}[E](t) = R(\infty) \int_{-\infty}^{t} \dot{P}_{\text{eff}}^2[E_-^t](\tau)d\tau + R(\infty) \int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t](\tau)d\tau,$$
(39)

the second integral represents the "ringing" of P_{eff} that would occur if the field were suddenly turned off after the excitation of the past field E_-^t.

We can use (39) together with definition (14) to find an expression for the irrecoverable energy:

$$u_{\text{irrec}}[E](t) := \inf_{E_+^t} u_{\text{int}}[E_-^t + E_+^t](+\infty)$$

$$= \inf_{E_+^t} \left\{ R(\infty) \int_{-\infty}^{t} \dot{P}_{\text{eff}}^2[E_-^t + E_+^t](\tau)d\tau + R(\infty) \right.$$

$$\left. \times \int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t + E_+^t](\tau)d\tau \right\}$$

$$= R(\infty) \int_{-\infty}^{t} \dot{P}_{\text{eff}}^2[E_-^t](\tau)d\tau + \inf_{E_+^t} R(\infty)$$

$$\times \int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t + E_+^t](\tau)d\tau.$$
(40)

The last term is calculated by considering all possible future fields appended to the fixed past field, and finding the future field that minimizes the "ringing" of the effective polarization after the current time (i.e., we want a future field that relaxes P_{eff} to its equilibrium state with as small an integral over $\dot{P}_{\text{eff}}^2$ as possible). In the next section we show that it is always possible to find a future field for which the ringing is arbitrarily small, so that

$$\inf_{E_+^t} R(\infty) \int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t + E_+^t](\tau)d\tau = 0$$
(41)

(i.e., the effective oscillator can be made to essentially stop ringing after any initial excitation E_-^t via some future excitation E_+^t). If we combine (40) and (41) and also recall that $u_{\text{int}}[E](t) = u_{\text{rec}}[E](t) + u_{\text{irrec}}[E](t)$, we arrive at the final expressions for the recoverable and irrecoverable energies:

$$u_{\text{irrec}}[E](t) = R(\infty) \int_{-\infty}^{t} \dot{P}_{\text{eff}}^2[E_-^t](\tau)d\tau,$$

$$u_{\text{rec}}[E](t) = R(\infty) \int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t](\tau)d\tau.$$
(42)

Theorems (42) have the interpretation that the irrecoverable energy is the "frictional/ohmic" losses incurred by the effective oscillator, whereas the recoverable energy is a measure of that oscillator's tendency to ring after active excitation. Although it is written in a different form, the expression for u_{rec} in (42) is mathematically equivalent to the form derived in [5].

IV. RECOVERY FIELDS

In order for a pulse to actually retrieve the recoverable energy available in the medium at a given time, an optimal recovery field needs to be appended to the fixed past of a pulse. Before discussing how to calculate this field, we illustrate some of its general properties. Consider the eventual loss (26) that is incurred by a pulse. The recovery field seeks to minimize the integral in this formula. The frequency components of the total pulse spectrum $\hat{E}(\omega)$ that are located at frequencies where $\omega \, \text{Im}[\chi(\omega)]$ is zero make no contribution to the integral, and we expect the recovery field to be composed of these frequency components. There are two obvious frequencies where this happens. $\text{Im}[\chi(\omega)]$ is odd so it has a zero at $\omega = 0$, indicating that the recovery field has a d.c. component. At high frequencies we have $\lim_{\omega \to \pm\infty}(\omega \, \text{Im}[\chi(\omega)]) = 0$. These high-frequency components can combine to form a delta distribution spike at time t in the recovery field.

For the type of media discussed in this paper, $\omega = 0, \pm\infty$ are the only real frequencies for which $\omega \, \text{Im}[\chi(\omega)] = 0$. However, there are complex frequencies (corresponding to exponentially decaying oscillations) that also satisfy the condition, and these components are also present in the recovery field. Thus, a general recovery field will have these three features: a delta spike, a d.c. tail, and exponentially decaying oscillations. The relative importance of the three features depends on the state of the medium when the recovery field is initiated.

To find a recovery field, we need to show for each permissible past field that there is a future field E_+^t for which, given any positive ϵ,

$$\int_{t}^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t + E_+^t](\tau)d\tau < \epsilon,$$
(43)

so that the infimum goes zero. Note that the limiting sense of infimum is important here since this expression implicitly requires that the "motionless" effective oscillator relax to its equilibrium position: $P_{\text{eff}}(+\infty) = 0$. For this to occur, a sequence of evolutions is required with "remainders" (43) tending to zero.

Equation (43) dictates that we find a recovery field E_+^t that satisfies

$$\dot{P}_{\text{eff}}[E_-^t + E_+^t](\tau) = \theta_-^t(\tau)\dot{P}_{\text{eff}}[E_-^t](\tau),$$
(44)

where θ_-^t is a unit step function supported before t. By linearity of $\dot{P}_{\text{eff}}$'s dependence on its field argument, and noting that $1 = \theta_-^t + \theta_+^t$ (θ_+^t is the unit step function supported after t), (44) becomes

$$\dot{P}_{\text{eff}}[E_+^t](\tau) = -\theta_+^t(\tau)\dot{P}_{\text{eff}}[E_-^t](\tau),$$
(45)

which says that the future field E_+^t must produce an oscillation exactly opposite to the ringing that would result from a truncated past field.

To find an explicit solution for the optimal future field we need to invert (45), and this is most easily done in the frequency domain. For notational convenience, we define the quantity

$$\hat{\Lambda}_+ := -i\omega\chi_{\text{eff}}(\omega) = -i\omega\bar{R}^+(\omega)\chi(\omega), \qquad (46)$$

which allows us to write the Fourier transform of $\dot{P}_{\text{eff}}[E](\tau)$ as

$$\mathcal{F}(\dot{P}_{\text{eff}}[E](\tau)) = \hat{\Lambda}_+(\omega)\hat{E}(\omega). \qquad (47)$$

Taking the Fourier transform of (45) gives

$$\hat{\Lambda}_+\hat{E}_+^t(\omega) = -\mathcal{F}\theta_+^t\dot{P}_{\text{eff}}[E_-^t](\tau) = -\mathcal{F}\theta_+^t\mathcal{F}^{-1}\mathcal{F}\dot{P}_{\text{eff}}[E_-^t](\tau)$$

$$= -\mathcal{F}\theta_+^t\mathcal{F}^{-1}\hat{\Lambda}_+\hat{E}_-^t(\omega), \qquad (48)$$

and, finally,

$$\hat{E}_+^t(\omega) = -\hat{\Lambda}_+^{-1}\mathcal{F}\theta_+^t\mathcal{F}^{-1}\hat{\Lambda}_+\hat{E}_-^t(\omega). \qquad (49)$$

Transforming back to the time domain, we have an explicit expression for the recovery field,

$$E_+^t(\tau) = -\mathcal{F}^{-1}\hat{\Lambda}_+^{-1}\mathcal{F}\theta_+^t\mathcal{F}^{-1}\hat{\Lambda}_+\mathcal{F}E_-^t(\tau). \qquad (50)$$

It is important to verify that (50) is self-consistent: the left-hand side claims to be a field evolution supported only for times $\tau > t$, and we need to confirm that the right-hand side produces such. By noting asymptotic and analytic properties of the solution (49), it is straightforward to show that the solution would be well behaved and have these properties except that $\hat{\Lambda}_+(\omega)$ has a (provably simple) zero at $\omega = 0$, and $\hat{E}_+^t(\omega)$ will have a simple pole there. Thus (50), which involves Fourier transformation over the real frequency axis, cannot directly constitute an algorithm for calculating the field E_+^t. This complication is related to the contradictory requirements of having that the effective oscillator instantly stop and yet somehow relax to zero as $t \to \infty$.

The remedy for this problem is straightforward: we allow the oscillator to relax slowly and take the limit as the relaxation time goes to infinity. Mathematically, this corresponds to taking a sequence of positive ϵ's tending to zero and using

$$\hat{\Lambda}_{+\epsilon}(\omega) := -i(\omega + i\epsilon)\chi_{\text{eff}}(\omega) = (-i\omega + \epsilon)\chi_{\text{eff}}(\omega) \quad (51)$$

to rewrite (50) as

$$E_{+\epsilon}^t = -\mathcal{F}^{-1}\hat{\Lambda}_{+\epsilon}^{-1}\mathcal{F}\theta_+^t\mathcal{F}^{-1}\hat{\Lambda}_+\mathcal{F}E_-^t. \qquad (52)$$

This gives a sequence of absolutely integrable fields $E_{+\epsilon}^t$ supported only for times $\tau > t$. The integrability occurs because the pole of $\hat{\Lambda}_+^{-1}$ at $\omega = 0$ has been shifted a finite distance away to $\omega = -i\epsilon$ in the lower half plane. In the limit of small ϵ we have

$$\lim_{\epsilon \to 0^+} \int_t^{+\infty} \dot{P}_{\text{eff}}^2[E_-^t + E_{+\epsilon}^t](\tau)d\tau = 0, \qquad (53)$$

so that (41) holds and the main energy theorems (42) are confirmed.

V. EXAMPLES

To illustrate the use of the formulas developed in Secs. III and IV, we consider several examples. While this formalism does not depend on the model used to describe the medium, in these examples we will restrict our attention to a two resonance Lorentz medium [specified by (8) with $N=2$] to facilitate comparison with the Barash and Ginzburg approach (which is only valid for Lorentz media). We use a Gaussian envelope on a cosine oscillation to describe the pulse:

$$E(t) = E_0 e^{-t^2/T^2}\cos(\bar{\omega}t). \qquad (54)$$

For our first example we use the following set of parameters:

$$\gamma = \gamma_1 = \gamma_2,$$

$$\omega_1 = 140\gamma,$$

$$\omega_2 = 160\gamma,$$

$$f_1\omega_{p_1}^2 = f_2\omega_{p_2}^2 = 100\gamma^2,$$

$$\bar{\omega} = 150\gamma,$$

$$T = 0.3/\gamma. \qquad (55)$$

The two resonance features are well separated [Fig. 1(a)] and the pulse spectrum sits directly between them [Fig. 1(b)]. Figure 1(c) plots the temporal profile of the pulse and Fig. 1(d) shows the various energy densities as the medium experiences the pulse. The pulse in this example propagates with mildly subluminal delays, with energy transferred into the medium during the early portion of the pulse and returned to the field during the latter portion. A fraction of the energy in the medium remains "recoverable" even after the medium stops returning energy to the field, but this energy slowly dissipates after a time. For comparison purposes, we have also plotted the Barash and Ginzburg loss u_ℓ. Both the u_ℓ and u_{irrec} increase monotonically, but small beats are observed in u_{irrec} (at the beat frequency $\omega_2 - \omega_1$) as the two resonances go in and out of phase. Note that u_{irrec} is larger than u_ℓ as specified by (20).

To retrieve the recoverable energy from the medium, an appropriate recovery field needs to be appended to the pulse (see Sec. IV). Figure 1(e) illustrates the recovery field initiated at $t=0$ for the pulse shown in Fig. 1(c) (the recovery field is appended to the past field). The three generic features of a recovery field are clearly evident: a delta spike, decaying oscillations, and a d.c. component (that is allowed to relax very slowly). Figure 1(f) shows the various energy densities as the medium experiences this recovery pulse. The energy densities before $t=0$ are identical to the unmodified pulse. At $t=0$ the irrecoverable energy is immediately "flat-lined" by the recovery field. The delta spike at $t=0$ transfers some additional energy into the medium, but this extra energy is recoverable by the future field. The medium surrenders all of its recoverable energy to the field through the oscillatory portion of the recovery field.
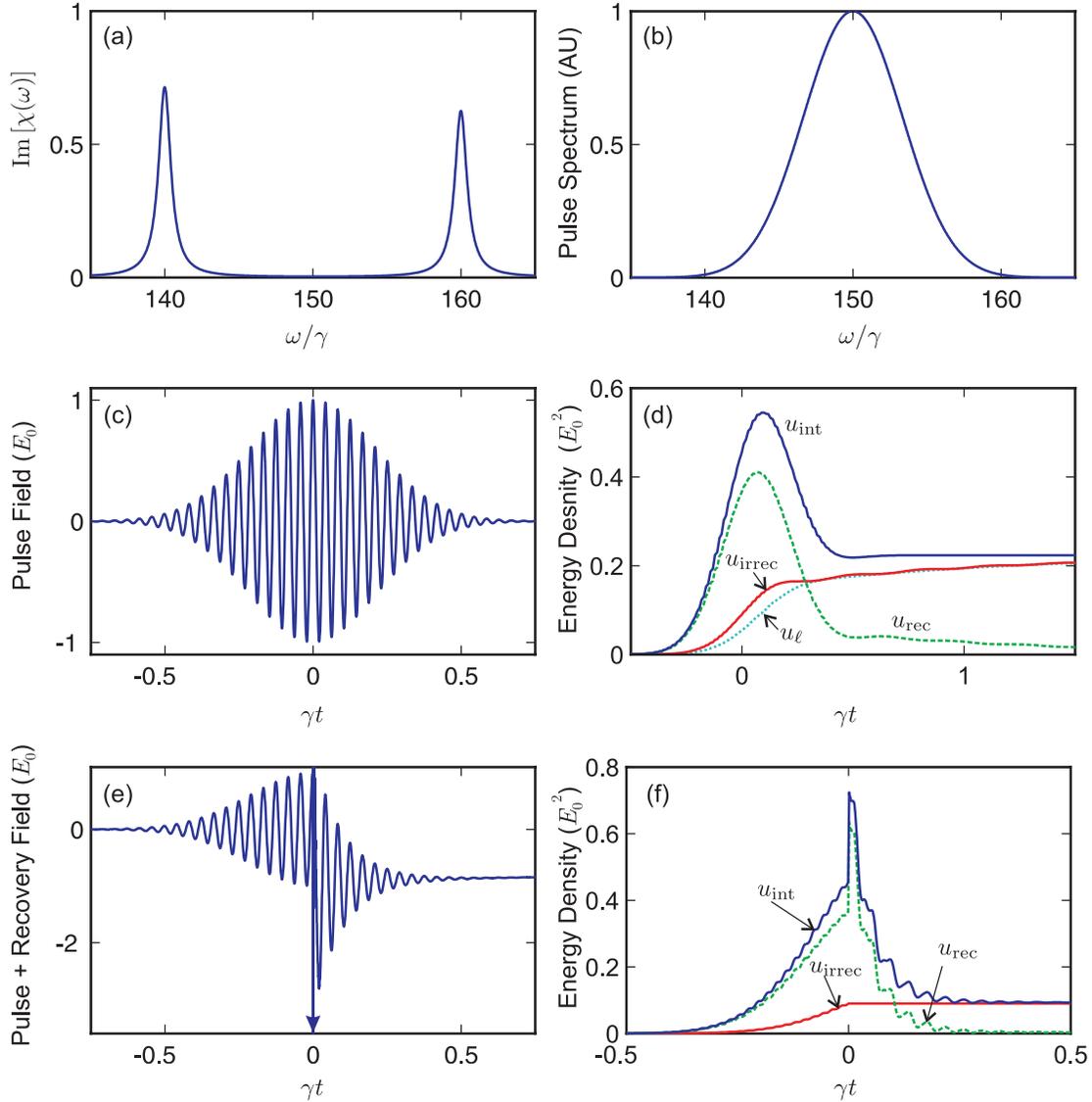
FIG. 1. (Color online) Quantities related to the parameters in (55). (a) Im$[\chi(\omega)]$. (b) The spectrum $|\hat{E}(\omega)|^2$ of pulse field (arbitrary units). (c) Time evolution of electric field. (d) Energy densities associated with the pulse-medium interaction for the field in (c). (e) The field with a recovery field appended after $t=0$. (f) Energy densities associated with the field in (e).

For a second example, consider the following set of parameters:

$$\gamma = \gamma_1 = \gamma_2, \quad \omega_1 = 149\gamma, \quad \omega_2 = 151\gamma,$$

$$f_1\omega_{p_1}^2 = f_2\omega_{p_2}^2 = 200\gamma, \quad \bar{\omega} = 150\gamma, \quad T = 1/\gamma. \quad (56)$$

Figure 2 shows a set of plots for these parameters that are analogous to Fig. 1 for the previous example. The absorption resonances in this example have significant overlap, and the pulse spectrum encompasses the compound resonance structure. Because there is more overlap between the pulse spectrum and the resonance structure, more energy is absorbed than in the previous example. However, the absorption is not uniform over the course of the pulse. In Fig. 2(d) it is clear that the medium absorbs a larger fraction of energy from the trailing edge than from the leading edge of the pulse. As the

pulse propagates this asymmetric absorption shifts the "center-of-mass" of the pulse to earlier times (compared to an unattenuated pulse traveling at c), which results in "superluminal" pulse propagation.

We note that the observed superluminal propagation of the pulse in Figs. 2(c) and 2(d) is not predicted by the traditional narrowband context of group delay. In this traditional context, pulse propagation delay is predicted by evaluating the group delay function at the carrier frequency (i.e., $\partial k/\partial \omega|_{\bar{\omega}}$), which for this pulse-medium combination is highly *sub*luminal. This illustrates the necessity of using a broadband analysis to calculate the delay. In the broadband context the total delay is predicted using an average over group delays for all frequencies present in the pulse, weighted by the spectral content of the pulse [18,19]. Since the group delay for frequencies near ω_1 and ω_2 (i.e., on either side of $\bar{\omega}$) is superluminal and the pulse has significant spec-

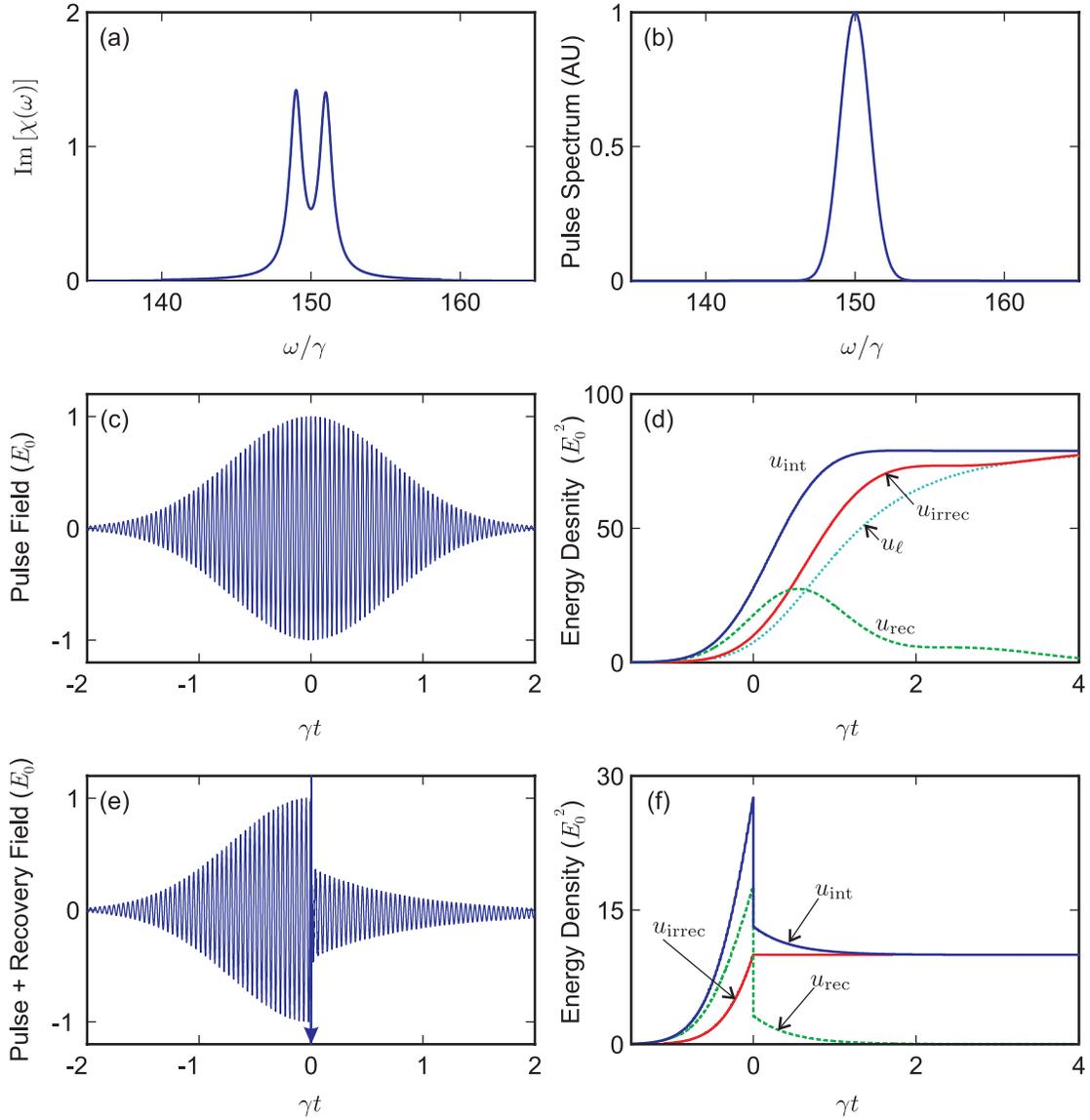FIG. 2. (Color online) Quantities related to the parameters in (57). (a) Im[$\chi(\omega)$]. (b) The spectrum $|\hat{E}(\omega)|^2$ of pulse field. (c) Time evolution of electric field. (d) Energy densities associated with the pulse-medium interaction for the field in (c). (e) The field with a recovery field appended after $t=0$. (f) Energy densities associated with the field in (e).

tral content at these frequencies, the weighted average predicts the observed superluminal delay for the pulse as a whole [20]. If we want to observe subluminal propagation in this medium, we could use a pulse with a narrower bandwidth so that all of its spectral components are associated with the subluminal group delays near $\bar{\omega}$.

Figure 2(e) shows a recovery field initiated at $t=0$ appended to the initial portion of the pulse Fig. 2(c). As before, the recovery field extracts all of the recoverable energy in the medium. The superluminal behavior observed in Fig. 2(d) disappears for the recovery pulse in Fig. 2(e). Note that this recovery field has essentially no d.c. component. This occurs because the effective polarization $P_{\text{eff}}[E](t)$ is near zero at $t=0$, so the delta spike can simply stop the effective polarization at equilibrium and there is no need for the slow relaxation observed in Fig. 1(e).

The superluminal effect in Fig. 2 is accompanied with severe attenuation. It is more fashionable (and often more practical) to observe superluminal or subluminal propagation in situations where absorption is minimized. For our final example, we use a pulse whose spectrum is centered on a narrow low-loss portion of the medium. This can be modeled by making the oscillator strength of one of the oscillators negative (corresponding to an amplifying resonance) as follows:

$$\gamma_2 = 0.1\gamma_1, \quad \omega_1 = 10\gamma_1, \quad \omega_2 = 10\gamma_1,$$

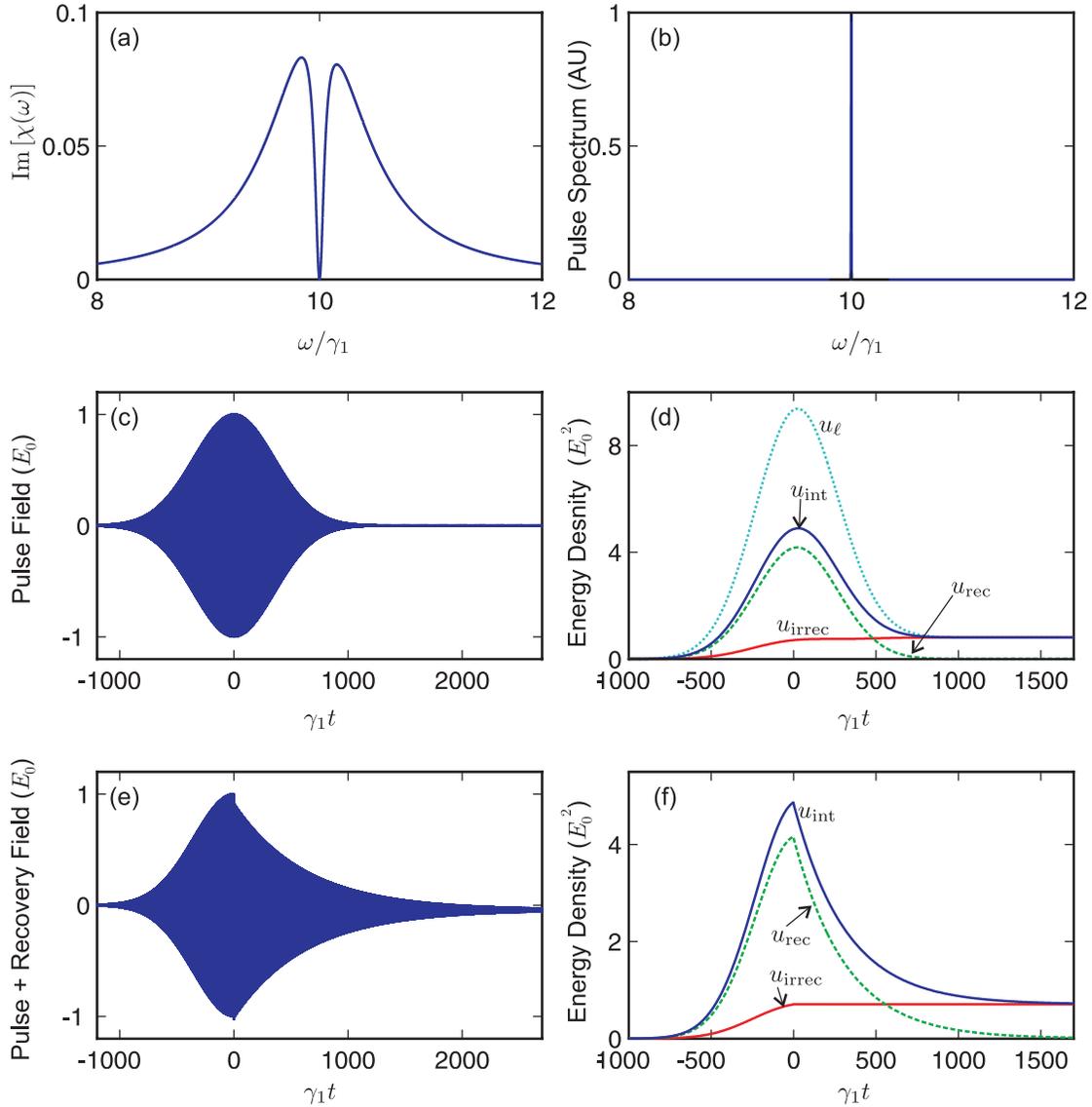$$f_1\omega_{p_1}^2 = 1\gamma_1^2, \quad f_2\omega_{p_2}^2 = -0.999\gamma_1^2,$$

FIG. 3. (Color online) Quantities related to the parameters in (57). (a) Im$[\chi(\omega)]$. (b) The spectrum $|\hat{E}(\omega)|^2$ of pulse field. (c) Time evolution of electric field. (d) Energy densities associated with the pulse-medium interaction for the field in (c). (e) The field with a recovery field appended after $t=0$. (f) Energy densities associated with the field in (e).

$$\bar{\omega} = 10\gamma_1, \quad T = 1/\gamma_1. \tag{57}$$

Although we have included a phenomenological amplifier, the overall medium is still passive since Im$[\chi(\omega)]\geq 0$. Figure 3 is a set of plots for the parameters in (57) that are analogous to Figs. 1 and 2. The pulse bandwidth [Fig. 3(b)] is quite narrow and sits right in the low-absorption window of the medium [Fig. 3(a)]. In Fig. 3(d) we again see the transfer of energy from the leading edge of the pulse to the trailing edge which is characteristic subluminal propagation. Note that u_ℓ gives nonsensical results in this case: the "lost" energy does not monotonically increase and even becomes larger than u_{int} (i.e., more energy has been "lost" than has been transferred into the medium). This reflects the fact that the Barash and Ginzburg approach is not appropriate with a negative oscillator strength. Also note that (20) does not hold for this case since it was derived under the assumption that

$f_n > 0$ (as Barash and Ginzburg also assumed).

The energy dissipated after $t=0$ in Fig. 3(d) is quite minimal (i.e., u_{irrec} does not change much). This indicates that the field after $t=0$ in Fig. 3(c) is a close approximation to a recovery field. As in the previous examples, Fig. 3(e) shows an actual recovery field initiated at $t=0$, and Fig. 3(f) shows the various energy densities as a point in the medium experiences the recovery field.

The imaginary part of χ in Fig. 3(a) is very small at $\omega = 10\gamma_1$, but still greater than zero. As discussed at the beginning of Sec. IV, the decaying oscillation portion of a recovery field is associated with complex frequencies for which Im$[\chi(\omega)]=0$. If we were to modify the parameters in (57) so that Im$[\chi(\omega)]$ approaches zero at $\omega=10\gamma_1$, the oscillations of the recovery field would decay more slowly and approach an unattenuated plane wave. This is because the complex frequencies associated with zero in Im$[\chi]$ move closer to the

real axis and the decay rate is determined by the imaginary part of the complex frequency. Thus, the formalism developed here requires $\text{Im}[\chi(\omega)] > 0$ for all frequencies except $\omega = 0, \pm\infty$ in order to have an integrable recovery field.

VI. SUMMARY

We have discussed natural concepts of the recoverable and irrecoverable energies associated with a pulse at a point in the propagation medium. These concepts lead to quantities that can be calculated with straightforward techniques. As we have shown in the examples, the quantities can give insights into how energy is exchanged in superluminal and subluminal pulse propagation. To get a more complete picture of the energy exchanged in pulse propagation, it is necessary to repeat the analysis done in our examples at many points in the medium, since the temporal form of the pulse experienced at one point is different from the temporal form experienced at another.

ACKNOWLEDGMENT

APPENDIX A: GLOBAL RECOVERABLE ENERGY

In this appendix we demonstrate that the net change in total field energy is less than or equal to the total recoverable energy at any time. We note that by the usual arguments Poynting's theorem (5) is a statement regarding the conservation of the spatial integral of u,

$$\frac{\partial}{\partial t} \int u(t) d\mathbf{x} = 0 \tag{A1}$$

or

$$\int u(t) d\mathbf{x} =: \mathcal{W} = \text{constant}. \tag{A2}$$

The total field energy $\mathcal{E}$ is not independent of time, but given by

$$\mathcal{E}(t) := \int u_{\text{field}}(t) d\mathbf{x} = \int u(t) d\mathbf{x} - \int u_{\text{int}}(t) d\mathbf{x}$$

$$= \mathcal{W} - \int u_{\text{int}}(t) d\mathbf{x}. \tag{A3}$$

Thus the net change in the total field energy after the distinguished time t is

$$\Delta_t \mathcal{E} := \mathcal{E}(+\infty) - \mathcal{E}(t) = \int (u_{\text{int}}(t) - u_{\text{int}}(+\infty)) d\mathbf{x} = \int u_{\text{rec}}(t) d\mathbf{x}$$

$$- \int (u_{\text{int}}(+\infty) - u_{\text{irrec}}(t)) d\mathbf{x} \leqslant \int u_{\text{rec}}(t) d\mathbf{x}, \tag{A4}$$

which demonstrates the natural notion of "recoverable" energy.

APPENDIX B: LOCAL RECOVERABLE ENERGY

In this Appendix we show that the Barash and Ginzburg approach overestimates the amount of energy that can be recovered from the medium. From (14), (9), and (11) one finds that

$$u_{\text{irrec}}[E](t) := \inf_{E_+^t} u_{\text{int}}[E_-^t + E_+^t](+\infty)$$

$$= \inf_{E_+^t} [u_e[E_-^t + E_+^t](+\infty) + u_\ell[E_-^t + E_+^t](+\infty)]$$

$$\geqslant \inf_{E_+^t} [0 + u_\ell[E_-^t + E_+^t](+\infty)]$$

$$= \inf_{E_+^t} \left[\sum_{n=1}^{N} \int_{-\infty}^{+\infty} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2[E_-^t + E_+^t](\tau) d\tau \right]$$

$$= \sum_{n=1}^{N} \int_{-\infty}^{t} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2[E_-^t](\tau) d\tau$$

$$+ \inf_{E_+^t} \left[\int_{t}^{+\infty} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2[E_-^t + E_+^t](\tau) d\tau \right]$$

$$\geqslant \sum_{n=1}^{N} \int_{-\infty}^{t} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2[E_-^t](\tau) d\tau$$

$$= u_\ell[E_-^t](t) = u_\ell[E](t). \tag{B1}$$

The key ingredients in the development of (B1) are causality, whereby $P_n[E_-^t + E_+^t](\tau) = P_n[E_-^t](\tau)$ for $\tau \leqslant t$, and noting that the Lorentz energy u_e and "future losses"

$$u_{\ell\text{-fut}}[E](t) := \inf_{E_+^t} \sum_{n=1}^{N} \int_{t}^{+\infty} \frac{\gamma_n}{f_n \omega_{p_n}^2} \dot{P}_n^2(\tau) d\tau, \tag{B2}$$

are never negative. In fact one can show that the first inequality in (B1) is always saturated (i.e., at equality) so that

$$u_{\text{irrec}}[E](t) - u_\ell[E](t) = u_{\ell\text{-fut}}[E](t). \tag{B3}$$

Thus the difference between the irrecoverable energy defined here and the Lorentz loss derived by Barash and Ginzburg is precisely the Lorentz loss that is inevitable in the future.

APPENDIX C: HOMOGENEOUS PROBLEM

In this appendix we carry out the factorization (33). We assume that the susceptibility χ is has the following generic asymptotics

$$\chi(\omega) \underset{\omega \to \infty}{\sim} -\left(1 - i\frac{\gamma_\infty}{\omega} + \cdots\right)\frac{\omega_p^2}{\omega^2},$$

$$\chi(\omega) \underset{\omega \to 0}{\sim} \left(1 + i\frac{\gamma_0}{\omega_0^2}\omega + \cdots\right)\frac{\omega_p^2}{\omega_0^2}, \tag{C1}$$

where γ_0 and γ_∞ are nonzero constants. Nongeneric assumptions about these "boundary conditions" (e.g., $\gamma_\infty = 0$, which corresponds to a lossless medium) fundamentally alter the following analysis.

With the constraints of real symmetry and analyticity we obtain a unique factorization (33) as follows. First note that since $\bar{R}(\omega) > 0$ for every real frequency ω, $\ln \bar{R}(\omega)$ may be taken to be real there. Moreover since [by (C1) and (32)]

$$\bar{R}(\omega) \sim_{\omega \to \infty} 1 + \frac{\mu}{\omega^2} + \cdots, \tag{C2}$$

for some (real) μ, one has then that

$$\ln \bar{R}(\omega) \sim_{\omega \to \infty} 0 + \frac{\mu}{\omega^2} + \cdots. \tag{C3}$$

Thus,

$$\ln \bar{R}(\omega) = \ln \bar{R}^+(\omega) + \ln \bar{R}^{+*}(\omega) = \ln \bar{R}^+(\omega) + \ln \bar{R}^+(-\omega)$$
$$= \ln \bar{R}^+(\omega) + \ln \bar{R}^-(\omega) \tag{C4}$$

is the Fourier transform of a real-valued function that is at least continuous, and

$$\ln \bar{R}^+(\omega) \sim_{\omega \to \infty} 0 + \frac{\sqrt{\mu}}{-i\omega} + \cdots \tag{C5}$$

is the Fourier transform of a real-valued function that is no worse than jump discontinuous. Moreover, if it is possible to find a solution to (C4) in which $\ln \bar{R}^+(\omega)$ is analytic in the upper-half ω plane, then we will have $\bar{R}^+(\omega) = \exp \ln \bar{R}^+(\omega)$, $\ln \bar{R}^+(\omega)$ will be the Fourier transform of a real-valued function supported only on positive times, and $\ln \bar{R}^+(-\omega)$ $= \ln \bar{R}^-(\omega)$ will be the transform of the time-reversed function.

Taking the inverse Fourier transform of the *inhomogeneous Riemann-Hilbert* problem (C4) one obtains

$$\mathcal{F}^{-1} \ln \bar{R} = \mathcal{F}^{-1} \ln \bar{R}^+ + \mathcal{F}^{-1} \ln \bar{R}^-$$
$$= \theta_+ \mathcal{F}^{-1} \ln \bar{R}^+ + \theta_- \mathcal{F}^{-1} \ln \bar{R}^-. \tag{C6}$$

Because, as we assume, $\mathcal{F}^{-1} \ln \bar{R}^+$ and $\mathcal{F}^{-1} \ln \bar{R}^-$ are no worse than jump discontinuous and supported on positive and negative times, respectively, multiplication by θ_+ and θ_- can introduce errors for at most one (unmeasurable) point.

So, then, separately multiplying (C6) by θ_+ and θ_- projects the known left-hand side onto the respective supports of the terms on the right. One then obtains (for all points except perhaps one)

$$\mathcal{F}^{-1} \ln \bar{R}^+ = \theta_+ \mathcal{F}^{-1} \ln \bar{R},$$

$$\mathcal{F}^{-1} \ln \bar{R}^- = \theta_- \mathcal{F}^{-1} \ln \bar{R}, \tag{C7}$$

that is,

$$\ln \bar{R}^+ = \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} \ln \bar{R},$$

$$\ln \bar{R}^- = \mathcal{F}^{+1} \theta_- \mathcal{F}^{-1} \ln \bar{R}, \tag{C8}$$

and, finally,

$$\bar{R}^+ = \exp \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} \ln \bar{R},$$

$$\bar{R}^- = \exp \mathcal{F}^{+1} \theta_- \mathcal{F}^{-1} \ln \bar{R}. \tag{C9}$$

Since $\bar{R}$ is an even, real-valued function of real ω, its inverse transform is an even real-valued function. Thus, $\bar{R}^-(\omega)$ is the transform of the time reversal of the inverse transform of $\bar{R}^+(\omega)$, and both functions are real-valued. Hence, the prescription (C9) yields that for real ω, $\bar{R}^-(\omega)$ $= \bar{R}^+(-\omega) = \bar{R}^{+*}(\omega)$, as required. In addition, note that

$$\bar{R}^{+-1} := 1/\bar{R}^+ = \exp - \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} \ln \bar{R} = \exp \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} - \ln \bar{R}$$
$$= \exp \mathcal{F}^{+1} \theta_+ \mathcal{F}^{-1} \ln \bar{R}^{-1}, \tag{C10}$$

and similarly for $\bar{R}^-$'s multiplicative inverse. Thus all four functions, $\bar{R}^\pm$ and their inverses, are analytic in the half planes indicated by their superscripts. Since the susceptibility χ of a passive medium has no zeroes in the (finite) upper-half plane (see [12]), this dictates that the effective susceptibility χ_{eff} also has no zeroes there. Moreover, since from (C9) and (32), $\bar{R}^+$ tends to unity as ω tends to infinity, χ_{eff} has the same asymptotics as χ. These facts are important to the engineering of an optimal field time evolution, E, that actually extracts the maximum energy from the field.

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