Energy transport in linear dielectrics

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Abstract: We examine the energy exchanged between an electromagnetic pulse and a linear dielectric medium in which it propagates. While group velocity indicates the presence of field energy (the locus of which can move with arbitrary speed), the velocity of energy transport maintains strict luminality. This indicates that the medium treats the leading and trailing portions of the pulse differently. The principle of causality requires the medium to respond to the instantaneous spectrum, the spectrum of the pulse truncated at each new instant as a given locale in the medium experiences the pulse.

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1 Introduction

It is well verified, both analytically and experimentally [1, 2, 3, 4, 5, 6], that electromagnetic pulses can seemingly propagate through linear dielectric media at speeds greater than c. In these situations, it is important to note that one is tracking the presence of only the electromagnetic field energy when these superluminal observations are made. Of course in a dielectric medium, field energy is only part of the energy picture. Energy is also stored in the medium; the pulse continually exchanges energy with the medium as it propagates. Thus, the overly rapid appearance of electromagnetic energy at one point and its simultaneous disappearance at another point does not require superluminal transport of energy, but merely an exchange between energy forms at individual locations.

In a companion article [7] we discussed how the group delay function tracks the *presence* of field energy in dielectric media (irrespective of whether the field energy is transported from point to point or converted to or from energy stored locally in the medium). In this article, we examine the actual transport of energy and how energy is exchanged between the pulse field and the medium. In section 2 we briefly review Poyntings theorem and the concept of energy transport velocity. Section 3 demonstrates that the global energy transport velocity is strictly bounded by c. We also show that there is no such limit on the velocity at which the centroid of field energy (i.e., the average position of the field energy density) moves, even though the velocity at which field energy is transported from point to point is strictly bounded by c. This effect is a result of the medium exchanging energy asymmetrically with the leading and trailing portions of the pulse.

In section 4 we discuss the exchange of energy between the field of a pulse and a linear causal medium [8, 9]. As was pointed out, it is this exchange which is related to the fact that group velocity is not bounded by c. Exotic behaviors such as superluminal or highly subluminal pulse propagation [10] have often been analyzed using the Lorentz oscillator model (either uninverted [11] or inverted [4]), which is known to be consistent with the principle of causality [3]. In the present work, rather than invoking a specific causal model to intimate the general compliance of the exotic behaviors with causality, we use the principle of causality itself to demonstrate how the exotic behavior is a direct and natural consequence, independent of a specific model. We also demonstrate how the exchange of energy between the field and the medium depends on the instantaneous spectrum [12, 13, 14] of the field.

2 Poynting's theorem and the energy transport velocity

By way of review, Poynting's theorem is a direct consequence of Maxwell's equations and in a linear, non-magnetic, non-conducting medium, and can be written as

$$\nabla \cdot \mathbf{S}(\mathbf{r}, t) + \frac{\partial u(\mathbf{r}, t)}{\partial t} = 0, \tag{1}$$

where the Poynting vector is

$$\mathbf{S}(\mathbf{r}, t) \equiv \mathbf{E}(\mathbf{r}, t) \times \frac{\mathbf{B}(\mathbf{r}, t)}{\mu_0} \tag{2}$$

and the total energy density is given by

$$u(\mathbf{r}, t) = u_{\text{field}}(\mathbf{r}, t) + u_{\text{exchange}}(\mathbf{r}, t) + u(\mathbf{r}, -\infty). \tag{3}$$

This expression for the energy density includes all relevant forms of energy, including a non-zero integration constant $u(\mathbf{r}, -\infty)$, which corresponds to energy stored in the

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medium before the arrival of any pulse. The electromagnetic field energy density is

$$u_{\text{field}}(\mathbf{r},t) \equiv \frac{B^2(\mathbf{r},t)}{2\mu_0} + \frac{\epsilon_0 E^2(\mathbf{r},t)}{2}. \tag{4}$$

The time-dependent accumulation of energy density transferred into the medium from the field is given by

$$u_{\text{exchange}}(\mathbf{r},t) \equiv \int_{-\infty}^{t} \mathbf{E}(\mathbf{r},t') \cdot \frac{\partial \mathbf{P}(\mathbf{r},t')}{\partial t'} \mathrm{d}t'. \tag{5}$$

As u_{exchange} increases, the energy in the medium increases. Conversely, as u_{exchange} decreases, the medium surrenders energy to the electromagnetic field. While it is possible for u_{exchange} to become negative, the combination $u_{\text{exchange}}+u(-\infty)$ (i.e., the net energy in the medium) cannot go negative since a material cannot surrender more energy than it has to begin with. Both u_{field} and u_{exchange} are zero before the arrival of the pulse (i.e. at $t=-\infty$). In addition, the field energy density returns to zero after the pulse has passed (i.e. at $t=+\infty$).

Poynting's theorem has the form of a continuity equation which, when integrated spatially over a small volume V, yields

$$\int_A \mathbf{S} \cdot \mathrm{d}\mathbf{a} = -\frac{\partial}{\partial t} \int_V u \mathrm{d}^3 r, \tag{6}$$

where the left-hand side has been transformed into an area integral representing the power leaving the volume. Let the volume V be small enough to take $\mathbf{S}$ to be uniform throughout. The energy transport velocity (directed along $\mathbf{S}$) is then defined to be the effective speed at which the energy contained in the volume (i.e. the result of the 3-D integral) would need to travel in order to achieve the power transmitted through one side of the volume (e.g., the power transmitted through one end of a tiny cylinder aligned with $\mathbf{S}$). The energy transport velocity as traditionally written [15] is then

$$\mathbf{v}_E \equiv \mathbf{S}/u\,. \tag{7}$$

It is not essential to time-average $\mathbf{S}$ and u over rapid oscillations, although this average is often made [11]. (One may choose to add the curl of an arbitrary vector function to $\mathbf{S}$. However, this possibility should not be injected into (7) since it cannot contribute to the integral in (6).)

When the total energy density u is used in computing (7), the energy transport velocity is *fictitious* in its nature; it is not the actual velocity of the total energy (since part is stationary), but rather the effective velocity necessary to achieve the same energy transport that the electromagnetic flux alone delivers. There is no behind the scenes flow of mechanical energy. Moreover, if only u_{field} is used in evaluating (7), the Cauchy-Schwartz inequality (i.e., $\alpha^2+\beta^2 \geq 2\alpha\beta$) ensures an energy transport velocity that is strictly bounded by the speed of light in vacuum c. We insist that the total energy density u at a minimum should be at least as great as the field energy density so that this strict luminality is maintained. In this we differ from previous usage of the energy transport velocity in connection with amplifying media [3, 4, 5] where the constant of integration $u(-\infty)$ was left at zero, resulting in the viewpoint of superluminal and negative (opposite to the direction of $\mathbf{S}$) energy transport velocities.

3 Average energy transport velocity

Since the point-wise energy transport velocity defined by (7) is strictly luminal, it follows that the global energy transport velocity (the average speed of *all relevant energy*) is

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also bounded by c. This has been discussed for pulses propagating in vacuum [16]. The analysis given here includes also the effects of a linear medium. To obtain the global properties of energy transport, we begin with a weighted average of the energy transport velocity at each point in space. A suitable weighting parameter is the energy density at each position. The global energy transport velocity is then

$$\langle \mathbf{v}_E \rangle \equiv \frac{\int \mathbf{v}_E u \mathrm{d}^3 r}{\int u \mathrm{d}^3 r} = \frac{\int \mathbf{S} \mathrm{d}^3 r}{\int u \mathrm{d}^3 r} \tag{8}$$

where we have inserted the definition (7), and the integral is taken over all relevant space.

Integration by parts leads to

$$\langle \mathbf{v}_E \rangle = \frac{\int \mathbf{r} \nabla \cdot \mathbf{S} \mathrm{d}^3 r}{\int u \mathrm{d}^3 r} = \frac{\int \mathbf{r} \frac{\partial u}{\partial t} \mathrm{d}^3 r}{\int u \mathrm{d}^3 r} \tag{9}$$

where we have assumed that the volume for the integration encloses all energy in the system and that the field near the edges of this volume is zero. We have also made a substitution from (1). Since the continuity relation (1) is written with no explicit source terms (i.e. zero on the right-hand side), the total energy in the system is conserved and is equal to the denominator of (9). This allows the time derivative in (9) to be brought out in front of the entire expression, giving

$$\langle \mathbf{v}_E \rangle = \frac{\partial \langle \mathbf{r} \rangle}{\partial t}, \tag{10}$$

where

$$\langle \mathbf{r} \rangle \equiv \int \mathbf{r} u \mathrm{d}^3 r \Big/ \int u \mathrm{d}^3 r \; . \tag{11}$$

Equation (11) represents the 'center of mass' or centroid of the total energy in the system [8].

This precise relationship requires the total energy density u. If, for example, only the field energy density u_{field} is used in defining the energy transport velocity, the time derivative cannot be brought out in front of the entire expression as in (10) since the integral in the denominator would retain time dependence. Although (10) guarantees that the centroid of the *total* energy moves strictly luminally (since $\mathbf{v}_E$ is pointwise luminal), there is no such guarantee on the centroid of field energy alone. Mathematically, we have

$$\left\langle \frac{\mathbf{S}}{u_{\mathrm{field}}} \right\rangle \neq \frac{\partial}{\partial t} \frac{\int \mathbf{r} u_{\mathrm{field}} \mathrm{d}^3 r}{\int u_{\mathrm{field}} \mathrm{d}^3 r}. \tag{12}$$

While the left-hand side of (12) is strictly luminal (via the Cauchy-Schwartz inequality), the right hand side can easily exceed c as the medium exchanges energy with the field. Moreover, it is the field energy that is typically "watched" in connection with pulse propagation. In an amplifying medium that exhibits superluminal behavior, for example, the rapid appearance of a pulse downstream is merely an artifact of not recognizing the energy already present in the medium until it converts to the form of field energy[4, 5]. Traditional group velocity is connected to this method of accounting, which is why it can become superluminal.

To see this connection, consider the centroid of field energy appearing in the right-hand side of (12), which defines the pulse's position (according to an "observer" who sees only field energy):

$$\langle \mathbf{r}_{\mathrm{field}} \rangle_t \equiv \int \mathbf{r} u_{\mathrm{field}} (\mathbf{r}, t) \; \mathrm{d}^3 r \Big/ \int u_{\mathrm{field}} (\mathbf{r}, t) \; \mathrm{d}^3 r \; . \tag{13}$$

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As a pulse evolves from an initial time t_0 to time $t_0 + \Delta t$, the difference in the average position of the field energy is given by

$$\Delta \mathbf{r} \equiv \langle \mathbf{r}_{\text{field}} \rangle_{t_0+\Delta t} - \langle \mathbf{r}_{\text{field}} \rangle_{t_0}. \tag{14}$$

In the appendix we sketch how Eq. (14) can be rewritten as the sum of two terms with intuitive interpretations:

$$\Delta \mathbf{r} = \Delta \mathbf{r}_G + \Delta \mathbf{r}_R. \tag{15}$$

(This expression is very similar to Eq. (27) in our companion article [7] in which we considered the time difference between the arrival of pulse energy at two points in space, as opposed to the displacement of the field centroid at two points in time as done here.)

The first term, $\Delta \mathbf{r}_G$ (typically the dominant contributor to the total displacement $\Delta \mathbf{r}$) is a linear superposition of the group velocity given by

$$\Delta \mathbf{r}_G \equiv \Delta t \int \left[\nabla_{\mathbf{k}} \operatorname{Re} \omega \left(\mathbf{k} \right) \right] \rho \left(\mathbf{k}, t \right) \, \mathrm{d}^3 k, \tag{16}$$

where $\rho\left(\mathbf{k}, t\right)$ is a normalized k-space distribution of field energy density (see Eq. (40)) at the final time $t \equiv t_0 + \Delta t$:

$$\rho \left(\mathbf{k}, t \right) \equiv u_{\text{field}} \left(\mathbf{k}, t \right) \bigg/ \int u_{\text{field}} \left(\mathbf{k}, t \right) \, \mathrm{d}^3 k \, . \tag{17}$$

Equation (16) explicitly demonstrates how the group velocity function $\nabla_{\mathbf{k}} \operatorname{Re} \omega \left(\mathbf{k} \right)$ is connected to the presence of field energy. The velocity of the pulse is predicted by an average of the group velocity function weighted by the k-space distribution of field energy in the *final* pulse (i.e. the pulse at $t = t_0 + \Delta t$). To the extent that this k-space distribution of the field energy is modified due to amplification or absorption, the displacement of the centroid changes accordingly. Since, as is well known, the group velocity function can be superluminal or negative, the displacement per time $\Delta \mathbf{r}_G / \Delta t$ can take on virtually any value.

Note that in Eq. (16) we use real wave-vectors associated with complex frequencies. Also, in writing (16) we made the restrictive assumption that $\omega\left(\mathbf{k}\right)$ is single-valued. For details, see the appendix

The term $\Delta \mathbf{r}_R$ in (15) represents a displacement which arises solely from a reshaping of the pulse through absorption or amplification (without considering the dispersion introduced by propagation). This reshaping displacement is the difference between the pulse position at the *initial time* t_0 evaluated without and with the spatial frequency amplitude that is lost during propagation (speaking as though the medium is absorptive). The reshaping displacement is zero if the amplitudes of the spatial frequency components are unaltered during propagation (i.e. if the imaginary part of $\omega\left(\mathbf{k}\right)$ is tiny). The reshaping displacement is also relatively modest (negligible) if the pulse is unchirped before propagation. In addition, it goes to zero in the narrowband limit even if pulses experience strong absorption or amplification. (In the narrowband limit, $\Delta \mathbf{r}_G$ reduces to $\nabla_{\mathbf{k}} \operatorname{Re} \omega \left(\bar{\mathbf{k}} \right)$, where $\bar{\mathbf{k}}$ is the central wave-vector in the pulse. This recovers the standard group velocity obtained using expansion techniques.)

Because $\Delta \mathbf{r}_R$ is ususally small, the presence of field energy is generally tracked by group velocity as shown in Eq. (16). Thus, while the velocity of the centroid of total energy is strictly bounded by c (as demonstrated in Eq. (10)), the centroid of field energy can move with any speed. This is not very mysterious when one recalls that in our discussion of field energy we have made no mention of where this energy comes from. Since a dielectric medium continually exchanges energy with the field of a pulse, the

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rapid movement of the centroid of field energy requires only that the medium exchange energy differently with various portions of the pulse. For example, the centroid of the field can be made to move extra fast if the medium gives energy to the leading portion and takes energy from the trailing portion (very slow propagation requires the converse).

4 Energy exchange and the instantaneous spectrum

In this section, we turn our attention to the exchange of energy between the field and the medium, which is responsible for the seemingly exotic behavior of superluminal and highly subluminal pulses. For this purpose it is enlightening to consider $u_{\rm exhchange}$ given in Eq. (5) within a frequency context. The frequency domain and time domain representation of the electric field $\mathbf{E}$ at a point $\mathbf{r}$ are related by

$$\mathbf{E}(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \mathbf{E}(\mathbf{r},t)\ \mathrm{d}t, \tag{18}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \mathbf{E}(\mathbf{r},\omega)\ \mathrm{d}\omega. \tag{19}$$

(Here we return to the convention of real frequencies ω.) We assume a linear, isotropic medium so that the polarization is connected to the electric field in the frequency domain via

$$\mathbf{P}(\mathbf{r},\omega) = \epsilon_0 \chi(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega)\,. \tag{20}$$

Homogeneity need not be assumed here. We take all fields to be real in the time domain, so that the following symmetries hold in the frequency domain:

$$\mathbf{E}(\mathbf{r},-\omega) = \mathbf{E}^*(\mathbf{r},\omega) \tag{21}$$

$$\mathbf{P}(\mathbf{r},-\omega) = \mathbf{P}^*(\mathbf{r},\omega) \tag{22}$$

$$\chi(\mathbf{r},-\omega) = \chi^*(\mathbf{r},\omega)\,. \tag{23}$$

The energy density (5), can immediately be written as

$$\begin{aligned} u_{\rm exchange}(\mathbf{r},t) = & \int_{-\infty}^{t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},\omega')\, e^{-i\omega' t'}\ \mathrm{d}\omega'\right] \\ & \cdot \left[\frac{-i\epsilon_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega\chi(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega)\, e^{-i\omega t'}\ \mathrm{d}\omega\right]\ \mathrm{d}t'. \end{aligned} \tag{24}$$

With a rearrangement of integration order, the expression becomes

$$u_{\rm exchange}(\mathbf{r},t) = -i\epsilon_0 \int_{-\infty}^{\infty} \mathrm{d}\omega\omega\chi(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega) \cdot \int_{-\infty}^{\infty} \mathrm{d}\omega' \mathbf{E}(\mathbf{r},\omega) \frac{1}{2\pi} \int_{-\infty}^{t} e^{-i(\omega+\omega')t'} \mathrm{d}t'. \tag{25}$$

The final integral in (25) becomes the delta function when t goes to $+\infty$. In this case, the middle integral can also be performed. Therefore, after the point $\mathbf{r}$ has experienced the entire pulse, the total amount of energy density that the medium has exchanged with the field is

$$u_{\rm exchange}(\mathbf{r},+\infty) = -i\epsilon_0 \int_{-\infty}^{\infty} \omega\chi(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega) \cdot \mathbf{E}(\mathbf{r},-\omega)\,\mathrm{d}\omega. \tag{26}$$

Finally, we use the symmetries (21) and (23) to obtain

$$u_{\rm exchange}(\mathbf{r},+\infty) = \epsilon_0 \int_{-\infty}^{\infty} \omega\,\mathrm{Im}\,\chi(\mathbf{r},\omega)\,|\mathbf{E}(\mathbf{r},\omega)|^2\,\mathrm{d}\omega. \tag{27}$$

The above formula is well known and appears in a textbook by Landau and Lifshitz [17]. However, to our knowledge, the argument that follows is presented here for the first time.

The expression (27) describes the net exchange of energy density after all interaction between the pulse and the medium has ceased at the point $\mathbf{r}$. We can modify this formula in a simple and intuitive way so that it describes u_{exchange} for any time during the pulse. This requires no approximations; the slowly-varying envelope approximation need not be made. The principle of causality guides us in considering how the medium perceives the electric field for any time.

Since the medium is unable to anticipate the spectrum of the entire pulse before experiencing it, the material must respond to the pulse according to the history of the field up to each instant. In particular, the material at all times must be prepared for the possibility of an abrupt cessation of the pulse, in which case all exchange of energy with the medium ceases. If the pulse were in fact to abruptly terminate at a given moment, then obviously (27) would immediately apply since the pulse would be over; it would not be necessary to integrate the Fourier transform (18) beyond the termination time t for which all contributions are zero. Causality requires that the medium be indifferent to whether the pulse actually ceases at a given instant before that instant arrives. Therefore, (27) in fact applies at all times where the spectrum (18) is evaluated over that portion of the field previously experienced by the medium.

The following is then an exact representation for the exchange energy density defined in (5):

$$u_{\text{exchange}}(\mathbf{r},t) = \epsilon_0 \int_{-\infty}^{\infty} \omega \,\mathrm{Im}\,\chi(\mathbf{r},\omega) \left|\mathbf{E}_t(\mathbf{r},\omega)\right|^2 \mathrm{d}\omega, \tag{28}$$

where

$$\mathbf{E}_t(\mathbf{r},\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \mathbf{E}(\mathbf{r},t')\, e^{i\omega t'}\ \mathrm{d}t'. \tag{29}$$

The time dependence enters only through $\left|\mathbf{E}_t(\mathbf{r},\omega)\right|^2$, the *instantaneous power spectrum*, which has been used to describe the response of driven electronic circuits [12], the acoustical response of materials to sound waves [13], and the behavior of photon counters [14].

The causality argument presented above comprises a sufficient proof of (28) and (29). It is essentially the same argument as that used to justify that the susceptibility has no poles in upper half of the complex $\chi(\mathbf{r},\omega)$ plane, which leads to the Kramers-Kronig relations [18]. We have given formal proof starting from this more familiar context of causality in Ref. [8], while including the possibility of both material anisotropy and diamagnetism. A streamlined proof is given in Ref. [9] for an isotropic non-magnetic dielectric.

The expression (28) reveals physical insights into the manner in which causal dielectric materials exchange energy with different parts of an electromagnetic pulse. It is clear from (28) that the magnitude of u_{exchange} depends on the overlap that the instantaneous spectrum has with the resonances in the medium (described by $\mathrm{Im}\,\chi(\mathbf{r},\omega)$). Since the function $\mathbf{E}_t(\mathbf{r},\omega)$ is the Fourier transform of the pulse truncated at the current time and set to zero thereafter, it can include frequency components that are not present in the pulse taken in its entirety. As a point in the medium experiences the pulse, the instantaneous spectrum can lap onto or off of resonances in the medium, causing u_{exchange} to change accordingly. As discussed in section 2, as u_{exchange} increases the medium absorbs energy from the pulse and as u_{exchange} decreases the medium surrenders energy to the pulse. Thus a point in the medium may amplify the pulse at one instant while absorbing at another. As noted at the end of section 3, this allows for the possibility

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of dramatic superluminal or highly subluminal effects when observing the field energy alone. In section 5 we discuss specific examples in which this exotic behavior occurs.

Before proceding, we briefly note that the expressions (28) and (29) manifestly contain the Sommerfeld-Brillouin result [11, 19] that a sharply defined signal edge cannot propagate faster than c. If a signal edge begins abruptly at time t_0, the instantaneous spectrum $\mathbf{E}_t(\omega)$ clearly remains identically zero until that time. In other words, no energy may be exchanged with a material until the field energy from the pulse arrives. Since, as was pointed out in connection with Eq. (7), the Cauchy-Schwartz inequality prevents the field energy from traveling faster than c, at no point in the medium can a signal front exceed c.

5 Discussion

In this section we discuss several specific examples which illustrate the concepts discussed above. We begin with a situation in which a pulse propagates superluminally (as reckoned by observing the centroid of field energy) in an amplifying medium. We choose the pulse so that the spectrum of the entire pulse is in the neighborhood of an amplifying resonance, but not on the resonance [3, 4, 5]. The instantaneous spectrum during the leading portion of the pulse is wider than the entire spectrum, and can therefore interact with the nearby gain peak. The medium accordingly amplifies this perceived spectrum, and the front of the pulse grows. During the latter portion of the pulse the instantaneous spectrum narrows and withdraws from the gain peak and energy is absorbed from the trailing portion. The net result is that the centroid of field energy moves forward at a speed greater than c. The effect is not only consistent with the principle of causality (as has been previously demonstrated via the Lorentz model [3, 4, 5]), but it is in fact a direct and general consequence of causality as demonstrated by Eqs. (28) and (29).

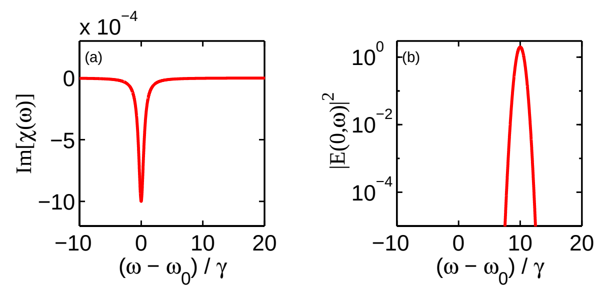

Fig. 1. (a) The imaginary part of $\chi(\omega)$ (b) Spectrum of the initial pulse in units of $(E_0/\gamma)^2$

To illustrate this effect, we employ the Lorentz model with a single resonance at ω_0 and a damping frequency γ. (Note that the results derived above are independent of any specific model.) In this model, the linear susceptibility is $\chi(\omega) = f{\omega_p}^2 / \left({\omega_0}^2 - \omega^2 - i\gamma\omega\right)$, where ω_p is the plasma frequency and f is the oscillator strength, which is negative for an inverted medium. We have chosen the medium parameter values as follows: $\omega_0 = 1\times10^5\gamma$, $f{\omega_p}^2 = -100\gamma$, and consider propagation through a thickness of $1.9\,(c/\gamma)$. Figure 1(a) shows the imaginary parts of $\chi(\omega)$ obtained using these parameters. The electric field of the initial pulse is chosen to be Gaussian, $\mathbf{E}(0,t) = \mathbf{E}_0\exp\left(-t^2/\tau^2\right)\cos(\bar{\omega}t)$, with the following parameters: $\tau = 2/\gamma$ and $\bar{\omega}-\omega_0 = 10\gamma$. Thus, the resonance structure is centered a modest distance above the pulse carrier frequency, and there is only minor spectral overlap between the pulse and the resonance structure. Figure 1(b) shows the total spectrum of the initial pulse.

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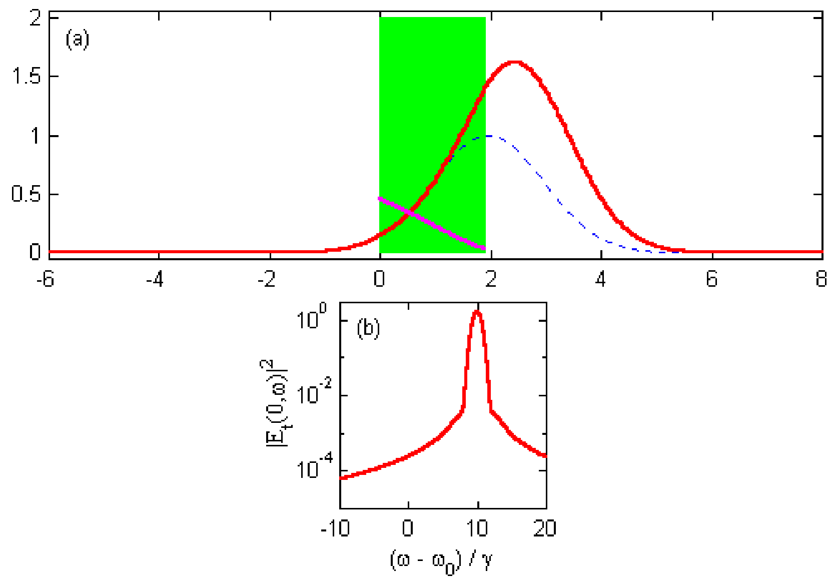

Fig. 2. (a) Animation of energy densities for the Gaussian pulse traversing the medium (distances are in units of c/γ and energy densities are in units of E_0^2/ϵ_0) (b) Instantaneous spectrum of the pulse at the point where it enters the medium (1.5 MB)

Figure 2(a) shows an animation of the energy densities associated with the pulse as it propagates through the medium. The solid rectangle in the middle represents the medium with vacuum on either side. The solid red line indicates the field energy density. The purple line in the medium represents the combination $u_{\text{exchange}} + u(-\infty)$ (energy density in the medium). We have assigned $u(-\infty)$ to be the same value at each point in the medium, chosen such that the energy density in the medium never becomes negative at any point. For reference, the dashed line represents the field energy density of a pulse that propagates exactly at c (i.e. as if the medium were not there). The actual pulse exiting the medium is ahead of this pulse, indicating that the centroid of field energy moved superluminally through the medium. Figure 2(b) shows the instantaneous spectrum for the first point in the medium. Notice that as this point experiences the leading portion of the pulse, the amount of overlap of the instantaneous spectrum with the resonance (at ω_0) increases and the medium surrenders energy to the leading portion of the pulse. As this point experiences the entire pulse, the instantaneous spectrum withdraws from the resonance, and energy is returned to the medium from the trailing portion of the pulse (notice that the energy in the medium rebounds slightly at the end of the pulse).

In Fig. 2 we have examined the instantaneous spectrum of the first point in the medium, so that at large t the instantaneous spectrum withdrew from the resonance. However, at points farther in the medium (after the pulse has experienced modification), the spectrum of the pulse taken in its entirety acquires significant on-resonance spectral components. Therefore, as the pulse propagates farther into the medium the instantaneous spectrum does not withdraw entirely from the resonance during the trailing portion of the pulse. Because the instantaneous spectrum has a greater overlap with the resonance in the trailing portion than the leading portion, the trailing portion of the pulse tends to be amplified to a greater extent than the leading portion. This explains why superluminal propagation in an amplifying medium does not occur over indefinite lengths. (For the pulse shown in Fig. 2, the transition from superluminal to subluminal

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transit times occurs when the medium thickness is increased from $1.9\,(c/\gamma)$ to $2\,(c/\gamma)$.)

The recent Wang experiment [6] in which superluminal propagation is observed in an amplifying medium is similar to the example just discussed. In this experiment the pulse spectrum is centered between two amplifying peaks, so that the broad instantaneous spectrum in the early portion of the pulse accesses the resonances on both sides and then withdraws in the later portions. In their report of this experiment the authors specifically deny that the superluminal effect was associated with amplification of the front edge of the pulse since the pulse taken in its entirety contained essentially no spectral components resonant with the nearby gain lines. However, the instantaneous spectrum reveals how the leading portion of the pulse may be amplified even in this circumstance.

In the Wang experiment, the time required for a 4μs pulse to traverse a $\Delta r = 6$cm amplifying medium was $\Delta t \approx -63$ns, meaning the pulse moved forward in time by about 1% of its width. The strength of the wings in the instantaneous spectrum can be approximated as $E_t(\omega) \sim E(t)/(\omega - \bar{\omega})$, where $\bar{\omega}$ represents a carrier frequency and $E(t)$ is the strength of the field at the moment the pulse is truncated. The imaginary part of the susceptibility in a low-density vapor is approximately $\mathrm{Im}\,\chi(\omega) \approx cg/\omega$, where g is the frequency dependent gain coefficient (in the Wang experiment, $g \approx 0.1\mathrm{cm}^{-1}$ at a spectral shift of $\delta\omega \equiv \omega - \bar{\omega} \approx 2\pi\,(2\mathrm{MHz})$). A crude approximation to the integral (28) renders $u_{\mathrm{exchange}} = \epsilon_0 E^2(t)\, cg/\delta\omega$. This suggests that in the case of the Wang experiment the front of the pulse extracts about $250 \times \epsilon_0 E^2(t)$ in energy density from the medium (i.e. 250 times the energy density in the electromagnetic field of the pulse). This energy density (extracted from the 6cm vapor cell) is distributed over about a kilometer, corresponding to the duration of the front half of the pulse. Thus, the the electromagnetic field energy on the forward part of the pulse is enhanced by several percent and similarly the field energy diminishes on the trailing edge. This is consistent with the data presented in the paper. (The traditional group velocity analysis used by Wang is perhaps a more convenient way to predict the transit time of the pulse. The utility of (28) lies primarily in its interpretation of how the pulse and the medium interact. Neither analysis substitutes for the full solution to Maxwell's equations, but rather indicates some features of the solution.)

Superluminal behavior can also result when the spectrum of the overall pulse is centered on an absorption resonance (i.e., Garret and McCumber effect [1]). The instantaneous spectrum during early portions of the pulse is broader and laps off of the absorption peak so that there is less attenuation. During the trailing portion of the pulse, its instantaneous spectrum narrows onto the resonance peak and the back of the pulse experiences greater attenuation. Subluminal behavior is the converse of the superluminal examples given above. It occurs when the spectrum of the pulse (taken in entirety) is just off of a near-by absorption resonances or if it centered on an amplifying resonance. We have illustrated all of the scenarios discussed here in another work [9]. Our explanation of the asymmetric absorption using the instantaneous spectrum is consistent with the explanation given by Crisp [20], who considered the time dependence of the polarization for the Lorentz model. He described a kind of delayed response by the oscillator to the stimulus of the pulse field.

As a final example, we consider a dielectric medium constructed with a narrow absorbing resonance superimposed on a wide amplifying resonance (both centered at the same frequency). This type of resonance structure is generally chosen because of group velocity considerations, but it is also interesting to consider in the present context. A relatively narrowband pulse whose spectrum is centered on-resonance can be sent through this medium with relatively little spectral modification. During the early portions of the pulse, the wide wings of the instantaneous spectrum spread away from resonance

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to access the broad amplifying resonance. During the latter portion of the pulse, the spectrum narrows and the trailing edge is attenuated by the absorbing resonance.

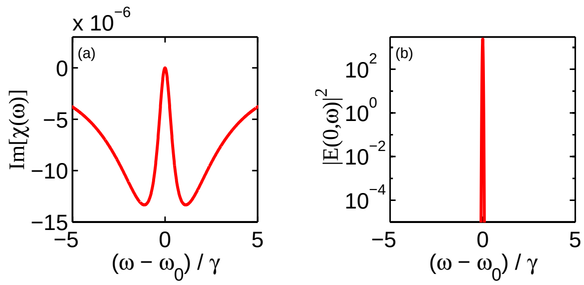

Fig. 3. (a) Animation of energy densities for the Gaussian pulse traversing the medium (distances are in units of c/γ and energy densities are in units of E_0^2/ϵ_0) (b) Instantaneous spectrum of the pulse at the point where it enters the medium (1.5 MB)

We employ a double resonance Lorentz oscillator so that the linear susceptibility is $\chi(\omega) = f_1{\omega_p}^2 / \left({\omega_0}^2 - \omega^2 - i\gamma_1\omega\right) + f_2{\omega_p}^2 / \left({\omega_0}^2 - \omega^2 - i\gamma_2\omega\right)$. For this example we choose the medium parameter values as follows: $\omega_0 = 1 \times 10^5\gamma_1$, $\omega_p = 10\gamma_1$, $f_1 = 0.02$, $f_2 = -0.1$, and $\gamma_2 = 5\gamma_1$. Figure 3(a) illustrates the imaginary part of $\chi(\omega)$ for these parameters. The pulse is Gaussian as before, with parameters as follows: $\tau = 70/\gamma_1$ and $\bar{\omega} = \omega_0$. Figure 3(b) shows the spectrum of the initial pulse.

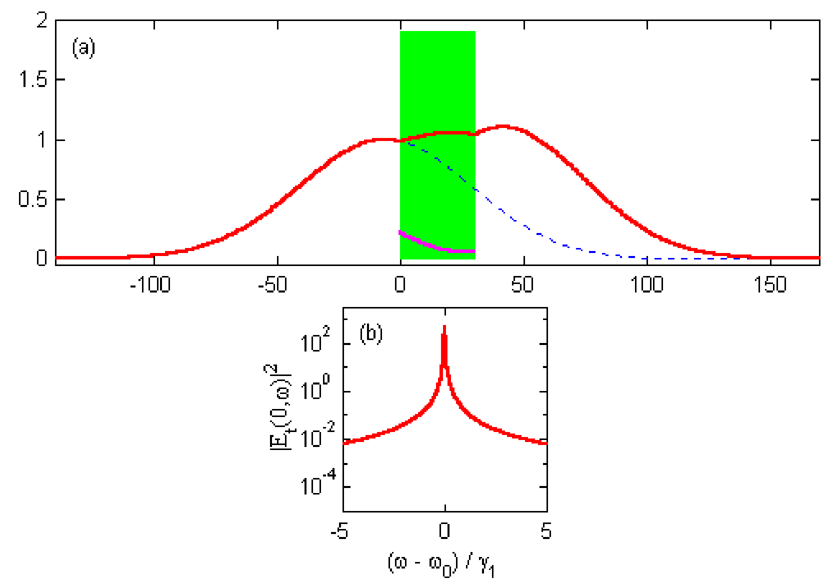

Fig. 4. (a) Animation of a Gaussian pulse traversing an amplifying medium. (b) Instantaneous spectrum of the pulse as it enters the medium (1.5 MB)

Figure 4(a) is an animation of the energy densities associated with this pulse as it traverses a medium of thickness $30\,(c/\gamma_1)$. Again, the purple line in the medium represents the combination $u_{\text{exchange}} + u(-\infty)$. Figure 4(b) shows the instantaneous spectrum for the first point in the medium as it experiences the pulse. In this case, the enhancement of the leading portion and the absorption of the trailing portion causes the exiting pulse to emerge from the medium before the incoming pulse enters.

There has been some discussion about whether the pulse exiting the medium in superluminal situations arises solely from the leading portion of the incoming pulse.

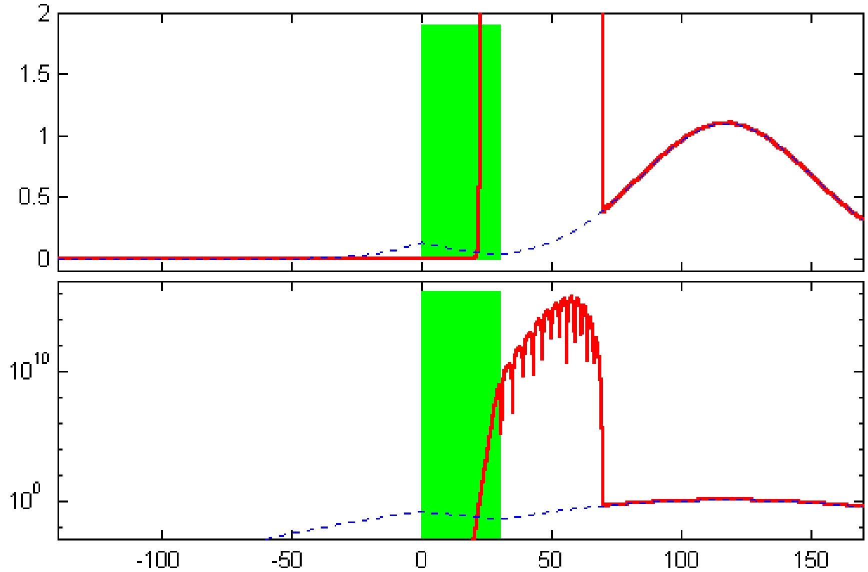

Fig. 5. Animation of a truncated Gaussian pulse traversing an amplifying medium, linear scale in the upper frame and logarithmic in the lower frame (2.0 MB)

This issue becomes clear in light of expression (28). Since the energy exchanged between the pulse and the medium can only depend on the portion of the pulse that a given point has already experienced, later portions of the pulse can have no influence on forward portion. In Fig. 5 we animate the spatial profile of a pulse with a leading edge identical to the pulse of Fig. 4 and the trailing edge set to zero after the peak of the Gaussian profile. The medium is the same as in Fig. 4. The top graph in Fig. 5 shows the pulse on a linear scale while the bottom graph shows it on a logarithmic scale. The field energy density of this truncated pulse is plotted with a solid line. The dashed line shows the untruncated pulse of Fig. 4. Because the truncated pulse taken in its entirety contains a large amount overlap with the amplification resonance it experiences a great deal of amplification in the trailing portion. However, it is clear from the plots that the leading portions of both pulses are identical as causality demands. The entering peak and the exiting peak are not causally connected [21]. The Gaussian appearance of the exiting peak has no connection with the shape of the latter portion of the incoming pulse.

6 Summary

We have discussed energy transport in dielectric media. We examined the centroid of total energy and found that its velocity of transport is strictly luminal. We also pointed out that the velocity at which field energy *transported* from one point to another is strictly bound by c. The centroid of only field energy density can move at any speed, as predicted by group velocity. The overly rapid motion of the centroid of field energy can occur when the medium exchanges energy asymmetrically with the leading and trailing portions of the pulse. The principle of causality requires this asymmetric energy exchange as governed by the instantaneous power spectrum used in Eq. (28).

A Appendix

In this appendix we sketch the derivation of expression (15) which connects group velocity with the presence of field energy. To accomplish this, we solve Maxwell's equations by selecting an instant in time and considering the spatial distribution of the pulse at that instant. This is in contrast to the more common method in which one chooses a point in space and considers the time behavior of the fields at that point. Since the spatial method of obtaining solutions is less common (owing to the fact that the material polarization $\mathbf{P}$ enters into Maxwell's equations through a time derivative as opposed to a spatial derivative) we take a moment to review how the solutions are obtained.

The k-space and spatial distributions of the electric field at an instant t are related by

$$\mathbf{E}(\mathbf{k},t) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{E}(\mathbf{r},t)\ \mathrm{d}^3 r, \tag{30}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{E}(\mathbf{k},t)\ \mathrm{d}^3 k. \tag{31}$$

Analogous expressions for $\mathbf{B}$ and $\mathbf{P}$ give the k-space representation for the magnetic and polarization fields. We take $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and $\mathbf{P}(\mathbf{r},t)$ to be real functions, so that the following symmetry holds for their k-space representations:

$$\mathbf{E}(-\mathbf{k},t) = \mathbf{E}^*(\mathbf{k},t) \tag{32}$$

with analogous expressions for $\mathbf{B}(\mathbf{k},t)$, and $\mathbf{P}(\mathbf{k},t)$. In a homogeneous, isotropic medium, Maxwell's equations have as a solution

$$\mathbf{E}(\mathbf{k},t_0+\Delta t) = \sum_m \mathbf{E}_m(\mathbf{k},t_0)\, e^{-i\omega_m(k)\Delta t}. \tag{33}$$

The initial pulse form $\mathbf{E}(\mathbf{k},t_0)$ is chosen at the instant t_0 for each frequency associated with the wave number $\mathbf{k}$. The solution renders the pulse form $\mathbf{E}(\mathbf{k},t_0+\Delta t)$ (in terms of the initial pulse form) after an arbitrary time interval Δt. The magnetic field is connected to the electric field via

$$\mathbf{B}(\mathbf{k},t) = \sum_m \mathbf{k}\times\mathbf{E}_m(\mathbf{k},t)/\omega_m(k)\ , \tag{34}$$

and the spatial profile of the pulse at the later time is obtained using (31).

The connection between the frequency ω_m and wave number k is:

$$\frac{\omega_m^2}{c^2}\left[1+\chi(\omega_m)\right] = k^2. \tag{35}$$

We choose real values for k and solve Eq. (35) for ω_m. The subscript m and the summations in (33) and (34) reflect the fact that the solution to (35) is in general multi-valued. We take this degeneracy to be countable and therefore use a summation rather than an integral. (For example, a single Lorentz oscillator is four-fold degenerate with two distinct frequencies for a given $\mathbf{k}$ which can each propagate forwards or backwards.) This degeneracy reflects the physical reality that in the presence of a complex linear susceptibility $\chi(\omega)$, different frequencies can correspond to the same wavelength. As mentioned in the text, we make the simplifying assumption that only a single frequency ω is associated with each $\mathbf{k}$, so that we can write the solution to Maxwell's equations as:

$$\mathbf{E}(\mathbf{k},t_0+\Delta t) = \mathbf{E}_0(\mathbf{k},t_0)\, e^{-i\omega(k)\Delta t}. \tag{36}$$

If this assumption is not made one can still derive expressions with the same interpretation as those obtained here. However, the sums involved make the expressions more complicated.

The viewpoint of real $\mathbf{k}$ leads to the use of complex frequencies ω. The meaning of complex frequencies is clear. The susceptibility of a complex frequency is determined by the medium's response to an oscillatory field whose amplitude either decays or builds exponentially in time. If the susceptibility $\chi(\omega)$ is known (measured) only for real values of ω, its behavior for complex frequencies can be inferred through a Fourier transform followed by an inverse Fourier transform with complex frequency arguments. Given the real $\mathbf{k}$ vectors, the complex frequencies correspond to uniform plane waves that decay or build everywhere in space as a function of time. (This is in contrast with the time picture where the pulse is comprised of waves that are steady in time but which decayed or build as a function of position.)

We now consider the average position of field energy at an instant and consider the displacement at a later time. As mentioned in the text, we use the centroid of field energy to define the pulse's position (see (13)):

$$\langle \mathbf{r}_{\text{field}} \rangle_t \equiv \int \mathbf{r} u_{\text{field}}(\mathbf{r}, t) \ \mathrm{d}^3 r \Big/ \int u_{\text{field}}(\mathbf{r}, t) \ \mathrm{d}^3 r \ . \tag{37}$$

Motivated by a desire to make a connection with group velocity, we rewrite (37) in terms of the k-space representation of the fields:

$$\langle \mathbf{r}_{\text{field}} \rangle_t = \mathbf{R}\left[\mathbf{E}(\mathbf{k}, t)\right], \tag{38}$$

where

$$\mathbf{R}\left[\mathbf{E}(\mathbf{k}, t)\right] \equiv i \frac{\int \mathrm{d}^3 k \sum_{j=x,y,z} \left[\frac{\epsilon_0}{2} E_j^*(\mathbf{k}, t) \cdot \nabla_k E_j(\mathbf{k}, t) + \frac{1}{2\mu_0} B_j^*(\mathbf{k}, t) \cdot \nabla_k B_j(\mathbf{k}, t)\right]}{\int u_{\text{field}}(\mathbf{r}, t) \ \mathrm{d}^3 r}. \tag{39}$$

The k-space representation of the energy density is

$$u_{\text{field}}(\mathbf{k}, t) = \frac{\epsilon_0 \mathbf{E}(\mathbf{k}, t) \cdot \mathbf{E}^*(\mathbf{k}, t)}{2} + \frac{\mathbf{B}(\mathbf{k}, t) \cdot \mathbf{B}^*(\mathbf{k}, t)}{2\mu_0}. \tag{40}$$

We have included only the electric field in the argument of the displacement $\mathbf{R}$ since the magnetic field can be obtained through (34).

The expression (39) is not very useful in itself. Its usefulness comes when applied to the difference in the pulse's average position at two different instants in time. Consider a pulse as it evolves from an initial time t_0 to time $t_0 + \Delta t$. The difference in the average position at these two times is

$$\Delta \mathbf{r} \equiv \langle \mathbf{r}_{\text{field}} \rangle_{t_0 + \Delta t} - \langle \mathbf{r}_{\text{field}} \rangle_{t_0} = \mathbf{R}\left[\mathbf{E}(\mathbf{k}, t_0 + \Delta t)\right] - \mathbf{R}\left[\mathbf{E}(\mathbf{k}, t_0)\right] \tag{41}$$

Using the solution (36), the displacement can be written as the sum of two intuitive terms (see (15)):

$$\Delta \mathbf{r} = \Delta \mathbf{r}_G(t) + \Delta \mathbf{r}_R(t_0). \tag{42}$$

The first term in (42), the net group displacement, is given in Eq. (16) and discussed in the text. The second term in (15), the reshaping displacement, is given by

$$\Delta \mathbf{r}_R(t_0) \equiv \mathbf{R}\left[e^{\operatorname{Im} \omega(\mathbf{k}) \Delta t} \mathbf{E}(\mathbf{k}, t_0)\right] - \mathbf{R}\left[\mathbf{E}(\mathbf{k}, t_0)\right]. \tag{43}$$

The reshaping displacement is the difference between the pulse position *at the initial time* t_0 evaluated without and with the spatial frequency amplitude that is lost during propagation. Dispersion effects due to propagation are not included since $\mathbf{E}(\mathbf{k}, t_0)$ is used in both terms of Eq. (43).