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Mode detuning in systems of weakly coupled oscillators

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A system of weakly magnetically coupled oscillating blades is studied experimentally, computationally, and theoretically. It is found that when the uncoupled natural frequencies of the blades are nearly equal, the normal modes produced by the coupling are almost impossible to find experimentally if the random variation level in the system parameters is on the order of (or larger than) the relative differences between mode frequencies. But if the uncoupled natural frequencies are made to vary (detuned) in a smooth way such that the total relative spread in natural frequency exceeds the random variations, normal modes are rather easy to find. And if the detuned uncoupled frequencies of the system are parabolically distributed, the modes are found to be shaped like Hermite functions.
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I. INTRODUCTION

Many phenomena in physics involve the simple harmonic oscillator, and the most interesting cases involve systems of coupled oscillators. One such system [experimentally observed by Weimer *et al.*, at the National Institute of Standards and Technology (Boulder) and numerically modeled by Mason *et al.*, at Brigham Young University] serves as the inspiration for this investigation.[1,2] Weimer's experiment involved a non-neutral electron plasma in a Penning trap. Because of the radial expansion of the cloud, at late times in the experiment the plasma became a very thin pancake-like disk, behaving like a collection of charged rings in a quadratic potential well with weak repulsive coupling between the rings. Distinct frequencies were observed in the experiment when the external potential was "detuned," meaning that the harmonic oscillation of uncoupled electrons at different radii had different frequencies in the external well.

In this paper we model the basic dynamics of this situation by considering many neighboring oscillators with almost the same natural frequency. Their frequencies differ slightly from neighbor to neighbor due to random variations, or they may also vary because they have been purposely detuned. In addition, the oscillators are weakly coupled. Figure 1 shows a photograph of a system having these properties. It consists of 20 hacksaw blades weighted by modeling clay and weakly coupled by ring magnets arranged with parallel dipole moments so that their interaction force along the line of motion of the blades is repulsive. The lumps of clay at the top of the blades all have equal masses, but their shapes (moments of inertia) have been changed so that each oscillator has the same frequency. The ring magnets about half-way down the blades provide the coupling and the lumps of clay pressed onto the ring magnets allow the system to be systematically detuned.

This system is described in Sec. II and computationally studied in Sec. III. Section IV discusses the experimental results, Sec. V discusses an analytic approximation to the system, and Sec. VI concludes the paper.

II. THE COUPLED-OSCILLATOR SYSTEM

Consider a system of harmonic oscillators consisting of nearly identical blades evenly spaced in a row and bending perpendicular to the line connecting them (see Fig. 1). The blades are 26 cm long hacksaw blades and are attached to a wooden plank by screws and washers. To avoid coupling of blades caused by motion of the plank, two lead bricks are placed on the plank to hold it securely to the floor. The blades were not designed for precision oscillation studies and turn out to have elastic constants that vary from blade to blade by about 6%. The lumps of modeling clay at the top of each blade all have the same mass, but their shapes have been carefully adjusted so that each blade has the same natural frequency as all the rest (within about 2%). Once the blades have been tuned in this way they are coupled by attaching ring magnets with neighboring magnets parallel to each other. When the blades are undisturbed, the magnets exert no forces in the direction of motion of the blades. But when neighboring blades are displaced, the magnets repel each other with a force that is linear in the relative displacement (for sufficiently small displacements). Finally, the oscillators can be systematically detuned by pressing lumps of clay onto the ring magnets, as shown in Fig. 1.

A. Dynamics of a single blade

The first thing to consider is the oscillatory behavior of a single blade. We especially wish to determine the dependence of the oscillation frequency on the extra mass M attached to the blades at the ring magnets so that we can understand how changing these masses detunes the system. (The top masses have tuned the blades and are not considered to be adjustable.) A simple model that predicts the behavior of the oscillator is to approximate the blade-mass system (without the extra masses) as a pendulum with a moment of inertia I_0, and a net torsional spring constant κ (elasticity and gravity acting on the masses combined). The displacement angle θ is measured from the base of the blade to the center of the ring magnet, which is a distance L above the base of the blade. The extra mass M then contributes ML^2 to

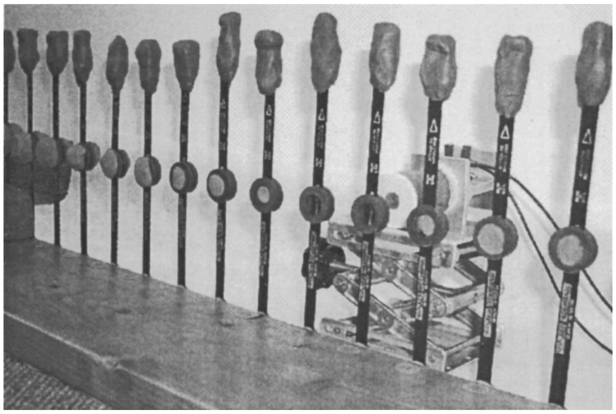

Fig. 1. The coupled oscillator system. Twenty hacksaw blades attached by screws and washers to a wooden plank are weighted at the top by 30-gram lumps of clay whose shapes have been adjusted to make each blade vibrate at the same frequency (when uncoupled from its neighbors.) The ring magnet pairs provide coupling and extra mass, while lumps of clay pressed against the ring magnets systematically detune the system. To prevent the plank from moving, it is pressed against the floor by two lead bricks.

the moment of inertia and the gravitational torque exerted on the system by M is given by $MgL\sin\theta$. This then leads to the small-angle equation of motion

$$\ddot{\theta}=-\left(\frac{\kappa-MgL}{I_0+ML^2}\right)\theta. \tag{1}$$

Note that $\kappa>MgL$ so that the force on each blade is always restoring.

To determine the parameters of the system we did the following experiments. (i) With $M=0$ and the neighboring coupling magnets removed, the frequency of an isolated blade/magnet oscillator was determined to be $\omega_0=16.4\,\mathrm{s}^{-1}$ (accurate to about 2%). This determines the ratio of κ to I_0 through $\omega^2=\kappa/I_0$. (ii) Mass $M=0.075\,\mathrm{kg}$ was added at the position of the ring magnet and a new oscillation frequency was measured. These two frequency measurements and Eq. (1) give two equations that determine κ and I_0. Solving these two equations and using $g=9.8\,\mathrm{m/s^2}$ and $L=0.11\,\mathrm{m}$ yields $\kappa=1.0\,\mathrm{kg\,m^2/s^2}$ and $I_0=3.7\times10^{-3}\,\mathrm{kg\,m^2}$. (Note that due to differences between the blades, these values of κ and I_0 vary from blade to blade by about 6%, even though their ratio, which determines their frequencies, are consistent to within about 2%). The detuning masses M were placed at $L=0.11\,\mathrm{m}$ and have values ranging from 0.001 kg up to 0.075 kg.

B. Magnetic coupling

The second matter to study is the magnetic coupling. Consider two dipoles oriented side-by-side with parallel dipole moments, constrained to move only in the direction of the dipoles. The force on a magnetic dipole is

$$\vec{F}=\nabla(\vec{m}\cdot\vec{B}). \tag{2}$$

If the first dipole is at the origin and is oriented in the z direction, its field (in spherical coordinates) is

$$\vec{B}_{\mathrm{dip}}=\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r}+\sin\theta\hat{\theta}). \tag{3}$$

Since we are determining $\vec{m}\cdot\vec{B}$ and $\vec{m}=m\hat{z}$, we are only concerned with the $\hat{z}$ components of $\vec{B}$. Thus,

$$\vec{m}\cdot\vec{B}=\frac{\mu_0 m_1 m_2}{4\pi r^3}(3\cos^2\theta-1)$$
$$=\frac{\mu_0 m_1 m_2}{4\pi(x^2+z^2)^{3/2}}\left(3\frac{z^2}{x^2+z^2}-1\right), \tag{4}$$

where x is the equilibrium distance between the magnets. Hence

$$F_z=\frac{\partial}{\partial z}(\vec{m}\cdot\vec{B})$$
$$=\frac{\mu_0 m_1 m_2}{4\pi}\frac{\partial}{\partial z}\left[(x^2+z^2)^{-3/2}\left(3\frac{z^2}{x^2+z^2}-1\right)\right]. \tag{5}$$

Performing the differentiation and doing a Taylor expansion then yields

$$F_z=\frac{\mu_0 m_1 m_2}{4\pi}\frac{3z(3x^2-2z^2)}{(x^2+z^2)^{7/2}}$$
$$=\frac{\mu_0 m_1 m_2}{4\pi}\left(\frac{9}{x^5}z-\frac{75}{2x^7}z^3+\cdots\right), \tag{6}$$

giving a linear (repulsive) coupling constant

$$k=\frac{F_z}{z}=\frac{9\mu_0 m_1 m_2}{4\pi x^5}. \tag{7}$$

If we require that the cubic term be less than 6% of the linear term, then we are restricted to relative displacements $|z_j-z_{j+1}|<x/6\approx0.01\,\mathrm{m}$ in our system (j denotes the number of the oscillator).

Determining the constant k is quite difficult because, as can be seen in the formula above, it depends on the inverse fifth power of the magnet separation. In our system this means that changing this separation by a millimeter shifts k by nearly 10%. To deal with this problem, and also to get another determination of the uncoupled blade frequency ω_0, we did the following.

With no extra masses attached, we measured the frequency of each blade while holding its nearest neighbors fixed. Assuming constant k for each pair of oscillators, the equation of motion for a blade with two neighbors would be

$$\ddot{\theta}=-\omega_0^2\theta+2\,\delta\omega^2\theta, \tag{8}$$

which predicts an oscillation frequency ω given by

$$\omega^2=\omega_0^2-2\,\delta\omega^2, \tag{9}$$

where

$$\delta\omega^2=\frac{kL^2}{I_0}. \tag{10}$$

The value of ω was found by measuring it for each blade with two neighbors, then averaging. This average value is $\omega=15.5\,\mathrm{s}^{-1}$.

We then measured the beat frequency between two neighboring blades with their other neighbors held fixed. This was done by setting one blade in motion and measuring the time T_b between successive instants of zero amplitude in the motion of the other blade. The beat frequency is then defined to be $\omega_b=2\pi/T_b$ and is related to the two natural modes of oscillation of the pair [(i) both blades moving together at ω_t and (ii) both blades moving oppositely at ω_{opp}] by $\omega_b = \omega_t - \omega_{opp}$. (Note that the opposite mode has *lower* frequency than the together mode because the coupling is repulsive.) As might be expected from the sensitivity of k on magnet separation, the beat periods varied by 10–20 % from pair to pair. (The coupling is so weak that this rather large variation does not affect the consistency of the single blade measurements.) On the other hand, because of this sensitivity the separations can be adjusted by moving the blades slightly closer or further apart without appreciably changing the other parameters of the system. So we measured all of the beat periods and then averaged them to get a target beat period. We then started at the middle pair of masses and worked our way out to the ends, moving blades closer or further apart to get the desired common beat period. This average value was then used to find $\omega_b = 0.98\ \mathrm{s}^{-1}$.

Once we have approximately the same coupling for each pair, we may write for any pair of blades (their other neighbors are fixed)

$$\ddot{\theta}_1 = -\omega_0^2 \theta_1 + \delta\omega^2 (2\theta_1 - \theta_2),$$
$$\ddot{\theta}_2 = -\omega_0^2 \theta_2 + \delta\omega^2 (2\theta_2 - \theta_1). \qquad (11)$$

A normal mode analysis of these coupled equations leads to the following expression for the beat frequency:

$$\sqrt{\omega_0^2 - \delta\omega^2} - \sqrt{\omega_0^2 - 3\,\delta\omega^2} = \omega_b. \qquad (12)$$

Solving Eqs. (9) and (12) simultaneously then yields

$$\delta\omega^2 = \omega_b \sqrt{\omega^2 - \omega_b^2/4}, \quad \omega_0^2 = \omega^2 + 2\,\omega_b \sqrt{\omega^2 - \omega_b^2/4},$$

which then leads, in our system, to $\omega_0 = 16.6\ \mathrm{s}^{-1}$ (which is within 2% of the earlier value of 16.4 s^{-1}), $\delta\omega^2 = 15.2\ \mathrm{s}^{-2}$, and $k = 2.7$ N/m. These parameters will now be used to analyze the full system.

III. THE COUPLED SYSTEM

With the N blades magnetically coupled together, the equations of motion for the system are (assuming nearest-neighbor coupling because of the $1/x^5$ decrease in the coupling force between blades)

$$\ddot{\theta}_j = -\left(\frac{\kappa - M_j g L}{I_0 + M_j L^2}\right)\theta_j - \frac{kL^2}{I_0 + M_j L^2}(\theta_{j+1} - 2\theta_j + \theta_{j-1})$$
$$\text{for} \quad 1 < j < N \qquad (13)$$

for interior blades and

$$\ddot{\theta}_1 = -\left(\frac{\kappa - M_1 g L}{I_0 + M_1 L^2}\right)\theta_1 - \frac{kL^2}{I_0 + M_1 L^2}(\theta_2 - \theta_1)$$

and

$$\ddot{\theta}_N = -\left(\frac{\kappa - M_N g L}{I_0 + M_N L^2}\right)\theta_N - \frac{kL^2}{I_0 + M_N L^2}(\theta_{N-1} - \theta_N) \qquad (14)$$

at the ends.

Consider first the case where we have no detuning masses, i.e., $M_j = 0$ for all j. Assuming that we have a normal mode so that $\theta_j(t) = A_j e^{i\omega t}$, we obtain from this set of equations an eigenvalue problem which is easily solved numerically (we used Matlab) using the parameters given in Sec. II. Figure 2 shows the amplitude distribution of the first 3 modes. The frequencies corresponding to these modes are 16.40, 16.39, and 16.37 s^{-1}, beginning with the "flat" mode and ending with the mode with two nodes. There are three important things to notice. First, unlike spring systems with attractive coupling where frequencies go up with increasing nodes, with repulsive coupling the frequencies go down. Second, these frequencies are very close together. Third, the eigenmode extends across the entire system. The combination of the latter two facts means, in practice, that normal modes are very nearly impossible to observe in this system.

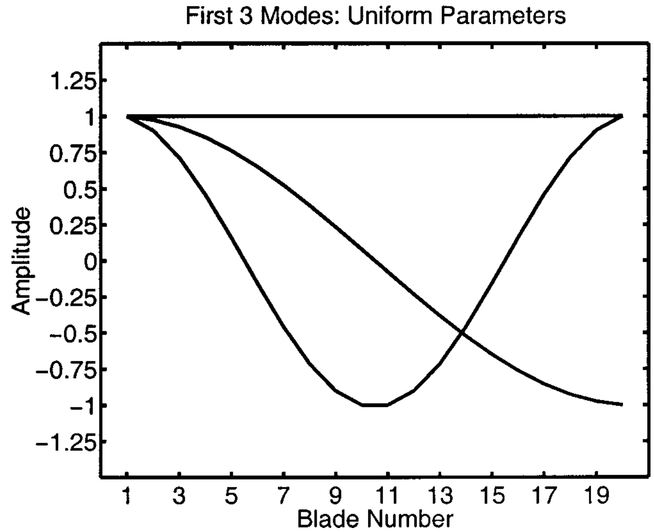

Fig. 2. Computed normal mode eigenfunctions for the first three modes of the system with all blades tuned to the same uncoupled frequency. The shapes are almost, but not quite, sinusoidal because of the boundary conditions at the end of this finite-length system.

To make matters worse, in the system we have built there are variations of order 6% in the parameters because of the varying elasticity constants of the blades. To study this effect, we used Matlab's random number generator to add variations at the 6% level to κ and variations at the 2% level to ω_0 and k. The result for a typical choice of randomly varying parameters is shown in Fig. 3, with corresponding frequencies 16.58, 16.53, 16.50 s^{-1}. Now it's even worse:

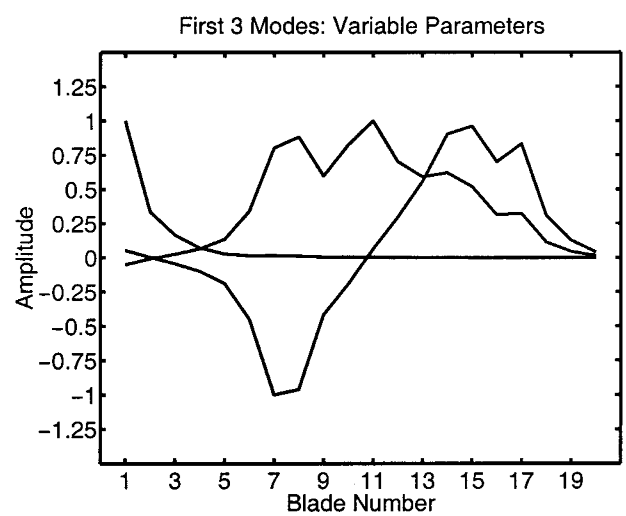

Fig. 3. Computed normal mode eigenfunctions for the tuned system with random variations in the system parameters at the few-percent level, as discussed in the article. The modes bear no resemblance to those of Fig. 2.

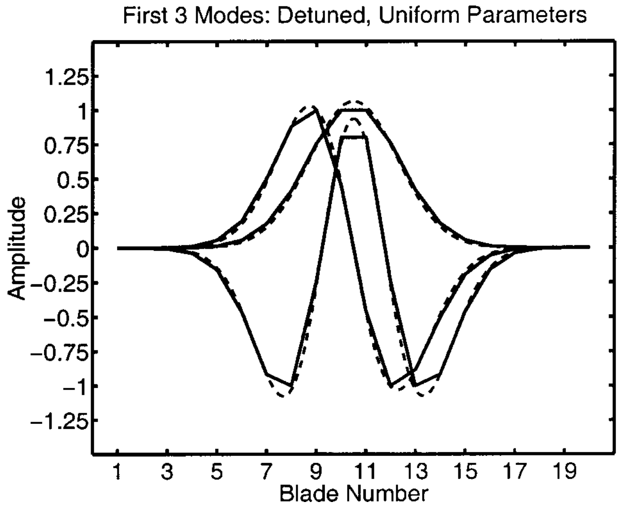

Fig. 4. Computed eigenfunctions for the detuned system with no random variation in system parameters. The dashed curves are the approximate Hermite eigenfunctions discussed in Sec. V.

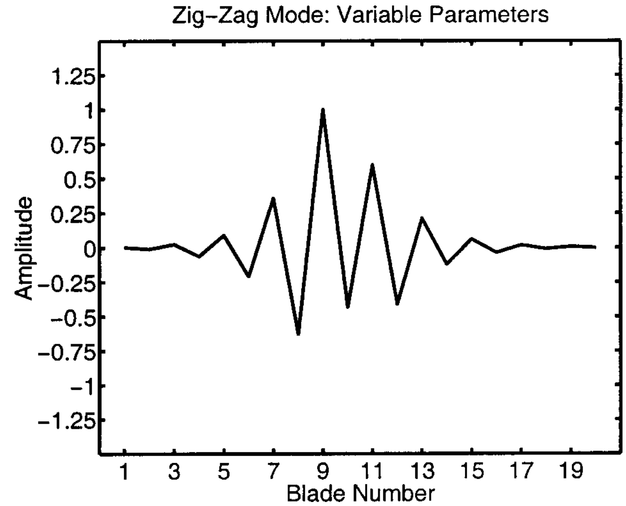

Fig. 6. The computed highest-order mode for the tuned system with random variations.

the frequencies are still close together but the mode shapes are not at all smooth. This trouble is easily seen in the machine itself. Attempting to excite the fundamental mode (where all blades move together in the case of uniform parameters), for instance, leads to rather chaotic motion, as predicted by Fig. 3, in which it can be seen that the ''fundamental'' does not have all oscillators moving together.

Numerical experiments show that this difficulty with the modes occurs as the level of random variation in the system parameters approaches the relative mode spacing. In our experiment, for example, the relative mode spacing is on the order of $0.15/16.4 = 0.1\%$, so errors at the 6% level should, and do, lead to almost randomly shaped eigenfunctions. In Weimer's experiment,[1] the mode spacing was about 0.25%, so errors even below the 1% level could have made the modes hard to find.

Detuning the system, however, has dramatic consequences. Figure 4 shows the numerical result of choosing a mass distribution M_j that gives a quadratic dependence of uncoupled blade frequency. The two center blades have no extra mass attached, the first and last blades have rather large, and equal, masses attached, and blades in between are weighted in pairs, measured from the center, to give a quadratic variation in ω_{0j}, symmetric about the center of the array of blades. The end oscillators have uncoupled frequencies 14% lower than the oscillators in the middle. The modes are now more localized in space, and their frequencies (using uniform parameters) are more spread out: 16.33, 16.19, 16.04 s^{-1}. Even if the random variations are put in, the modes retain their basic shapes, as seen in Fig. 5, and the frequencies don't shift too much.

Numerical experiments show that this restoration of smooth mode shapes occurs when the detuning variation (14% here) exceeds the level of random variation (6%). In Weimer's experiment, as analyzed later by Mason,[2] the detuning variation was about 2%. This leaves a window of nearly a factor of 10 between the mode spacing at 0.25% and

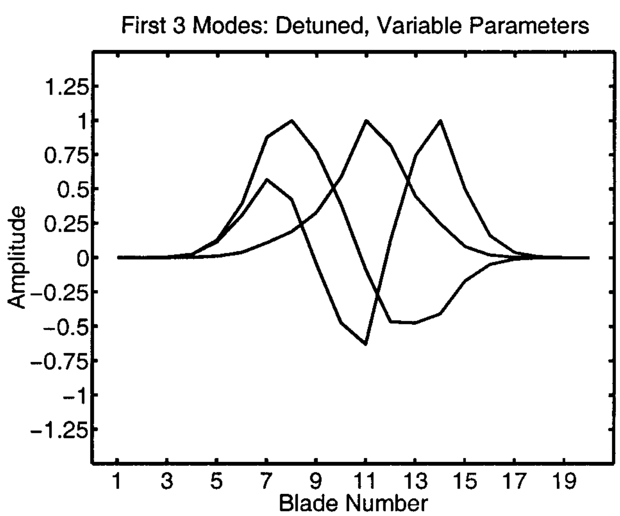

Fig. 5. Computed eigenfunctions for the detuned system with random variations. Note that the mode shapes still resemble those of Fig. 4. (The third mode in this figure has the opposite sign of the corresponding mode in Fig. 4, but sign is arbitrary.)

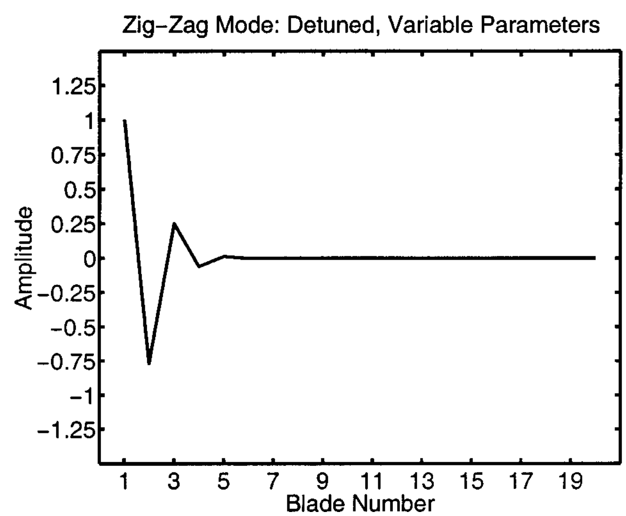

Fig. 7. The computed highest-order mode for the detuned system, random variations included. There is a similar high-order mode localized on the right-hand side.

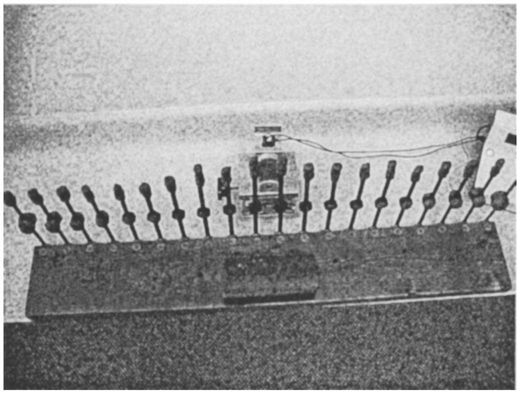

Fig. 8. The lowest-order mode in the detuned experimental system (compare to Fig. 4). The signal generator and driving electromagnet are also shown.

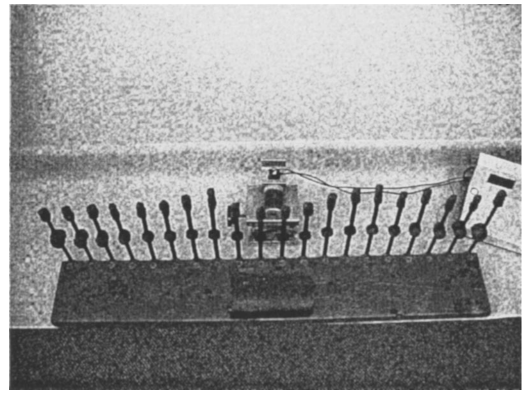

Fig. 10. The second harmonic in the detuned experimental system (compare with Fig. 4).

the detuning at 2% in which to place the unknown error level, if the physics discussed here describes what was happening in that experiment.

The higher-order "zig–zag" modes are also affected by the detuning. Figure 6 shows the highest order mode for the case $M_j = 0$ and Fig. 7 shows the same mode for the detuning discussed above (using parameters with random variations in both cases.) As in the case of long-wavelength modes, the tuned system has a mode shape that involves all of the blades while the detuned system has a mode localized near the ends of the device. (Note that this highest-order mode is asymmetric because of the lack of symmetry in the random variations. There is a similar high-order mode localized on the right, but its frequency is a bit higher than the one shown in Fig. 7.)

IV. EXPERIMENTS

Once the blade-mass-magnet system was built according to the description given in Sec. II and tuned as described in Sec. III, a low-frequency magnetic pulser was borrowed from the demonstration stockroom of the Department of Physics and Astronomy at Brigham Young University. This device sends a pulse of current through an electromagnet at time intervals which can be controlled by turning a dial. A mode of the system is found by using the Matlab calculations of Sec. III to find the approximate frequency of the desired mode, and then, by searching in that neighborhood, the experimental mode frequency. This takes some time because the system has a natural decay time of about 30 seconds and the frequencies are low, but with patience the modes can be isolated.

As predicted by the theory, the normal modes were relatively well separated in frequency and quite easy to find because only a few masses were involved in each mode. Figures 8–10 show the first three modes (corresponding to the calculated modes shown in Fig. 4), which have measured frequencies 16.15, 15.95, and 15.77 s^{-1}. For comparison, the theory (with no random variation) predicts mode frequencies of 16.33, 16.19, and 16.04 s^{-1}. The "zig–zag" mode was also found experimentally and is shown in Fig. 11. Its measured frequency was 13.2 s^{-1}, compared to the calculated value (using uniform parameters) of 13.6 s^{-1}. The discrepancies between the experimental and calculated values are within the expected range given the variations in the system parameters.

V. APPROXIMATE ANALYTIC THEORY

The apparently Gaussian shape of the fundamental mode in the detuned system (see Fig. 4) made us curious. So when we fit a Gaussian to it, and got an almost perfect fit, we went looking for an analytic theory to explain such a simple result. The simple theory involves Hermite functions which obey the differential equation

$$\theta'' + (\beta^2 - \alpha^2 \xi^2)\theta = 0, \quad \theta(\pm\infty) = 0, \tag{15}$$

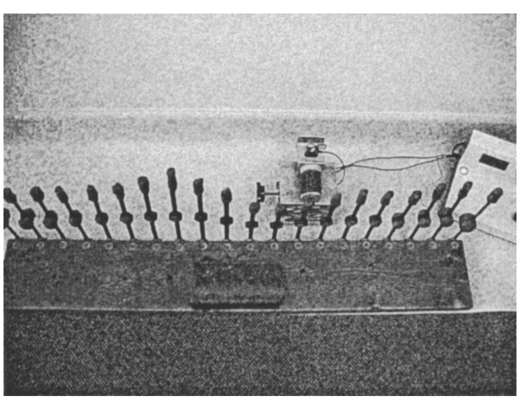

Fig. 9. The first harmonic in the detuned experimental system (compare with Fig. 4).

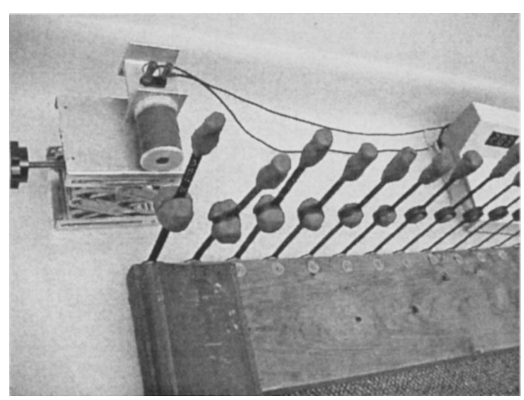

Fig. 11. The highest-order mode in the detuned experimental system (compare with Fig. 7).

where $\theta'' = d^2\theta/d\xi^2$. This equation is the one that appears in the quantum theory for the energy levels of a particle in a quadratic potential well. Its connection to our problem can be found by defining a continuous variable ξ to replace blade number j:

$$\xi = \frac{2j-N-1}{N-1}, \quad \Delta = \frac{2}{N-1}, \tag{16}$$

where Δ is the spacing in ξ between blades. Note that $\xi(1) = -1$, $\xi(N) = 1$, and $\xi = 0$ is in the center of the system. The masses M_j were chosen to make the uncoupled blade frequencies vary quadratically, which is not quite the same as having the masses vary quadratically. But the mass distribution that detuned the masses in our system can be fit by the following quadratic function of ξ to within a maximum error of 7%:

$$M(\xi) = \mu\xi^2, \quad \mu = 0.07\,\text{kg}. \tag{17}$$

Using this approximation, and making the further approximation

$$\theta_{j+1} - 2\theta_j + \theta_{j-1} \approx \Delta^2 \frac{d^2\theta}{d\xi^2} \tag{18}$$

(assuming that we have a long wavelength mode so neighboring blades move similarly), turns Eq. (13) into Eq. (15) with the parameters given by

$$\alpha^2 = \frac{\mu(\omega^2 + g/L)}{k\Delta^2}, \quad \beta^2 = \frac{I_0(\omega_0^2 - \omega^2)}{kL^2\Delta^2}. \tag{19}$$

The end conditions can also be turned into continuous equations in the same way. The symmetry of our system about the center means that the modes also have symmetry, so it is only necessary to look at the right end where $\xi = 1$. Solving Eq. (14) for θ_{N-1} and using the result to build a centered-difference approximation to θ'/θ at the end yields

$$\frac{\theta'(1)}{\theta(1)} = \frac{2(\theta_N - \theta_{N-1})}{\Delta(\theta_N + \theta_{N-1})} = \frac{2(\omega_N^2 - \omega^2)}{\Delta(2\delta\omega_N^2 + \omega^2 - \omega_N^2)}, \tag{20}$$

where

$$\omega_N^2 = \frac{\kappa - \mu g L}{I_0 + \mu L^2}, \quad \delta\omega_N^2 = \frac{kL^2}{I_0 + \mu L^2}. \tag{21}$$

It is possible to solve the differential equation in terms of Whittaker M and W functions, but dealing with these functions is not simpler than doing numerical work. And, in fact, it is not even necessary to solve the problem numerically in our case of weak coupling. For when the coupling is weak so that $\delta\omega_N^2$ is small, then Eq. (20) says that θ'/θ is a large negative number, implying that θ is small. But this is basically the same boundary condition that occurs in the quantum simple harmonic oscillator problem where the wave function vanishes far from the center of the well, so the same condition on α and β that leads to the energy levels of the simple harmonic oscillator[3] determines the mode frequencies in our system:

$$\beta^2 = (2n+1)\alpha. \tag{22}$$

Solving Eqs. (19) and (22) for ω^2 leads to a messy solution of a quadratic equation, but since we are only interested in weak coupling, the solution may be expanded in small k to obtain

$$\omega^2 \approx \omega_0^2 \left(1 - (2n+1)\sqrt{1 + \frac{g}{L\omega_0^2}} \frac{\sqrt{k\mu\Delta^2}L^2}{\omega_0 I_0} \right). \tag{23}$$

The corresponding eigenfunction is given by

$$\theta_n(\xi) = H_n(\sqrt{\alpha}\xi) e^{-\alpha\xi^2/2}, \tag{24}$$

where $H_n(x)$ is the nth Hermite polynomial. Figure 4 corresponds to $n = 0,1,2$, and the dashed curves in this figure show $\theta_n(\xi)$ for each mode. The parameter α was obtained from Eq. (19) and the normalization of $\theta_n(\xi)$ was obtained by least-squares fitting to the Matlab eigenfunctions. In our case $\alpha \approx 29$, so the Gaussian factor $e^{-\alpha\xi^2/2}$ has dropped below 1% by about $\xi = 0.6$, which means that the motion at the ends of the system is irrelevant; we do not need to worry about the fact that the system is of finite length for these low-order modes.

Using this theory to find the mode frequencies gives for the first three modes shown in Fig. 4, $\omega = 16.31$, 16.41, and 15.96 s^{-1}, which differs from the discrete theory with uniform parameters by about 1%.

The zig–zag modes can also be approximated by using the same continuous equation. To do so we observe that while θ_j is highly discontinuous for such modes, $\phi_j = (-1)^j\theta_j$ is smooth, allowing us to use the same continuous approximation used above. Making this substitution leads to

$$\phi'' - (\beta^2 - 4 - \alpha^2\xi^2)\phi = 0. \tag{25}$$

Solving Eq. (14) for ϕ_{N-1} and building an end condition for ϕ'/ϕ then leads to

$$\frac{\phi'(1)}{\phi(1)} = \frac{2(\phi_N - \phi_{N-1})}{\Delta(\phi_N + \phi_{N-1})} = \frac{2(2\delta\omega_N^2 + \omega^2 - \omega_N^2)}{\Delta(\omega_N^2 - \omega^2)}. \tag{26}$$

Solving this differential equation numerically and shooting on the boundary condition gives for the lowest-order zig–zag mode $\omega = 13.9\,\text{s}^{-1}$, quite close to the calculated discrete-blade result of 13.6 s^{-1}. The continuous eigenfunction is also very much like the experimental one, with only blades between $\xi = 0.8$ and $\xi = 1$ being involved.

It is of interest to note that the case of attractive coupling involves just a slight modification of these calculations because the principal effect of switching from θ to ϕ is just to change the sign of the second derivative in the differential equation, which is just what changing from repulsion to attraction does. So with attractive coupling, the long wavelength modes are concentrated near the outside edge, and the zig–zag modes have Hermite-function envelopes near the center. Furthermore, note that changing the mass distribution from increasing to decreasing also switches the repulsive and attractive cases with each other because this, too, changes the sign of the second derivative.

Finally, a quick way to see qualitatively what is going on is to note that the physics of this situation is similar to the penetration of evanescent waves into a forbidden region (like FM radio waves reflecting from the ionosphere, or quantum tunneling). There is a region of oscillation which exponentially decays into a region of nonpropagation in all of these cases.

VI. CONCLUSIONS

When dealing with systems of weakly coupled oscillators in which the relative random variations in the system exceed the relative mode spacing, the normal modes can be more cleanly separated both spatially and in frequency by systematically detuning the oscillators. All that is required is to have the relative level of detuning be larger than the random variation level. In our experiments, this effect was used to overcome the difficulties caused by random variations in the elasticity constants of hacksaw blades, but the same effect may also have been at work in the experiments of Weimer *et al.*, in which the vibrations of a thin cloud of plasma in a Penning trap were studied. It was found experimentally that the modes were easier to find if the trap was detuned in such a way that the uncoupled vibration frequency of the electrons decreased with radius.[1,2] In their experiment the amount of detuning was only about 2% across the radius of the plasma, but the coupling was so weak (the relative mode spacing was 0.25%) that even this much detuning could have been significant. But in addition to making the modes easier to find, detuning also shifts the mode frequencies by amounts greater than would be expected from the coupling alone. For instance, in our experiment without detuning, the first three modes would have frequencies 16.40, 16.39, and 16.37 s^{-1}, while with detuning, the frequencies were 16.33, 16.19, and 16.04 s^{-1}. In Weimer's experiments these frequency separations were used to deduce the shape of the plasma, assuming no effect from detuning. This may account for part of the 20% discrepancy they observed in the plasma shapes when they used the first four mode frequencies to get three different values for the plasma aspect ratio (thickness to radius ratio).

Finally, the observation in our experiments and calculations that detuning makes it easier to find modes when there are random errors in the system might also have implications for experiments like Weimer's. These traps have construction errors that introduce electrostatic perturbations, and these errors might make it hard to find modes when the trap is tuned as perfectly as possible. But since detuning can restore the modes, this might explain why they found the modes more easily when the trap was detuned.

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