If Coulomb's law were not inverse square: The charge distribution inside a solid conducting sphere

Ross L. Spencer

Citation: American Journal of Physics **58**, 385 (1990); doi: 10.1119/1.16179
View online: http://dx.doi.org/10.1119/1.16179
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/58/4?ver=pdfcov
Published by the American Association of Physics Teachers

If Coulomb's law were not inverse square: The charge distribution inside a solid conducting sphere

Ross L. Spencer
Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602

(Received 21 March 1988; accepted for publication 6 December 1988)

The distribution of charge between concentric conducting shells has been at the heart of the most sensitive tests of the exponent in Coulomb's law since the days of Henry Cavendish. But it appears that no one has ever answered the question of how an excess of charge would distribute itself throughout the interior of a *solid* conductor if Coulomb's law were other than inverse square. Spherically symmetric solutions to this problem have been found under the assumption that the potential of a point charge varies either as e^{-kr}/r or as $1/r^n$.

I. INTRODUCTION

During the course of his research on electricity, Benjamin Franklin wrote from Philadelphia to his friend, Joseph Priestley, the English scientist, telling him of the results of some experiments on the electrifying of conductors. In these experiments Franklin had discovered that the charge always resided on the outer surface of the conductor. Priestley repeated these experiments, and reasoning by analogy with Newton's theory of gravitation, concluded that this experimental result implied that electrical attraction and repulsion also obeyed the inverse-square law. In Priestley's own words:

> May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation and is therefore according to the squares of the distances; since it is easily demonstrated that were the earth in the form of a shell a body on the inside of it would not be attracted to one side of it more than another.[1]

Priestley's observation was first made into a quantitative experiment by Henry Cavendish who studied the electrification of concentric spherical shells.[2] His work was improved upon by James Clerk Maxwell,[3] by Samuel Plimpton and Willard Lawson,[4] and most recently by Edwin Williams, James Faller, and Henry Hill,[5] who established that Coulomb's law holds to about one part in 10^{16}. (These experiments and many others are discussed in the excellent review article of Goldhaber and Nieto.[6] See also Fulcher[7] for a recent downward revision of the possible deviation from Coulomb's law allowed by the experiment described in Ref. 5.)

All of these researchers based their experiments on the mathematical analysis of nested conducting shells under the assumption that the electrostatic force law was other than inverse square. This is, however, not the only geometry of interest. In all elementary textbooks on electromagnetism, for instance, Gauss' law is employed to show that any excess charge on a solid conductor resides on its surface. But the question of how an excess of charge distributes itself in a solid conductor in the event that the electrostatic force law is other than inverse square is rarely discussed. (It is interesting to note that Fulcher's analysis[7] of the four concentric conductors of the Williams, Faller, and Hill experiment is a step in the direction of a solid conductor.) A prominent exception is the paper by Shaw[8] in which the question of unique solutions to the problem of the solid conductor is discussed. The computation of spherically symmetric solutions, whose unavailability Shaw laments, is the subject of this paper.

Consider a solid conducting sphere made up of a fixed positive background, having density ρ_+, and a continuous negative fluid. The electrostatic interaction between the charges is assumed to be mediated by a potential law that varies either as e^{-kr}/r (Yukawa potential) or as $1/r^n$ (inverse-power-law potential). The quantity r is the distance from a point charge and n, the exponent, is a real number rather than an integer. The quantity k is proportional to the mass of the photon. Coulomb's law corresponds either to $n = 1$ or to $k = 0$. Imagine that an excess of charge is placed on the sphere. The negative fluid will be set in motion, seeking to find a distribution of charge that produces zero electric field throughout the interior of the fluid. (At the outer surface of the sphere it is assumed that microscopic forces act to keep excess negative fluid from leaving the sphere.[9–12]) In the case of Coulomb's law the end result of this process is that all of the excess charge is concentrated at the outer surface of the sphere. However, if the electrostatic force law is different from Coulomb's law, the excess charge is distributed in a rather more complicated way. (Note: In general, it is necessary to distinguish between the cases of positive and negative excess charge. In our world, however, where excess charge amounts are nearly vanishing fractions of the total charge available, the distinction between the two cases is usually uninteresting. Hence, in this paper only the simpler case of negative excess charge will be discussed in detail.)

In the case of the Yukawa potential, the equilibrium charge distributions are similar to those for Coulomb's law, with the exception that a constant charge density fills the region interior to the edge charge distributions. This constant inner charge density is proportional to $(ka)^2 \Delta Q$, where a is the radius of the sphere and ΔQ is the total excess charge. The inner charge density is, therefore, proportional to the square of the mass of the photon. This case is discussed in detail in Sec. II. In the case of the inverse-power law, the situation is much more complicated, and only the relatively simple case of excess negative charge with $n \geqslant 1$ will be fully discussed. In this case, the equilibrium charge density is singular, having the form $\rho(r) \propto (a^2 - r^2)^{(n-3)/2}$. As in the Yukawa case, there is excess charge distributed throughout the interior of the sphere; in this case, the inner charge density is proportional to $(n-1)\Delta Q$. This calculation is the subject of Sec. III. The other inverse-power-law cases will be qualitatively discussed in Sec. IV, and the article concludes with a summary in Sec. V. The special inverse-power-law case $n=2$ is discussed in Appendix A and equilibria containing regions from which all free charge is removed are discussed in Appendix B.

II. THE YUKAWA POTENTIAL

Suppose that the potential of a point charge q is given by the formula

$$\phi(r)=qe^{-kr}/4\pi\epsilon_0 r, \tag{1}$$

where k is related to the mass of the photon m_p by $k=m_p c/\hbar$ and where SI-appearing units have been chosen to make the formulas look more familiar. The potential ϕ_s at a distance r from the center of an infinitely thin shell of radius r' and charge q can be obtained from Eq. (1) by integrating over the shell:

$$\phi_s(r)=(q/8\pi\epsilon_0 krr')(e^{-k|r-r'|}-e^{-k|r+r'|}). \tag{2}$$

If the distribution of charge within a sphere of radius a is described by a spherically symmetric charge density $\rho(r)$, then the potential at any radius r can be obtained by integrating over concentric shells of charge. It is convenient to define the new variable $f(r)=r\rho(r)$ and to extend r from the interval $0\leqslant r\leqslant a$ to the interval $-a\leqslant r\leqslant a$ by taking the odd extension of $f(r)$. With these changes, the potential at any radius r due to the charge density $\rho(r)=f(r)/r$, can be written

$$\phi(r)=\frac{1}{2\epsilon_0 kr}\int_{-a}^{a}e^{-k|r-r'|}f(r')dr'. \tag{3}$$

If we now require that $\rho(r)$ be in electrostatic equilibrium, then the potential must be constant wherever there is movable negative fluid.

It is possible to make a wide variety of equilibria, even in the case of Coulomb's law, by allowing regions from which all negative fluid has been removed, for then the potential need not be constant everywhere but only in the regions containing negative fluid. This is only of mathematical interest in our world, however, since completely removing all of the electrons from any significant region would produce enormous electric fields. In addition, such equilibria are almost always unstable; they are briefly discussed in Appendix B. Here, we shall assume that the potential is uniform throughout the sphere. The equation determining the function $f(r)$ that produces such a potential is obtained by taking ϕ_0 to be constant in Eq. (3):

$$\int_{-1}^{1}e^{-k|r-r'|}f(r')dr'=2\epsilon_0\phi_0 kr. \tag{4}$$

This equation is a Fredholm integral equation of the first kind,[13] and its kernel, $e^{-k|r-r'|}$, is the so-called Lalesco kernel.[14] Differentiating Eq. (4) twice with respect to r gives rise to a delta function that effects a partial solution; the solution is completed by judicious guesswork. In any case it is easy to verify that Eq. (4) is solved by

$$\rho(r)=f(r)/r=k^2\epsilon_0\phi_0+[(ka+1)/a]\epsilon_0\phi_0\,\delta(r-a), \tag{5}$$

i.e., the density consists of a constant distribution throughout the sphere and a thin shell concentrated at its outer surface. (Note: If the excess charge is positive, then the infinitely thin outer layer is replaced by a thin, but finite, outer positive region. As the ratio of the excess positive charge to the total positive charge in the sphere approaches zero, the difference between the cases of positive and negative excess charge vanishes.)

It is also possible to obtain this result by solving the Proca equation, the differential equation for the Yukawa potential that corresponds to Poisson's equation for the Coulomb potential. The Proca equation is

$$\nabla^2\phi-k^2\phi=-\rho/\epsilon_0$$

for the electrostatic units being used here. If spherical symmetry is assumed, then inside the sphere we have $\phi(r)=\phi_0$ while outside we have $\phi(r)=\phi_0 e^{-kr}/r$. The Proca equation may now be solved for ρ to obtain Eq. (5). As pointed out by Shaw,[8] the existence of a differential relation between ϕ and the charge density is a powerful aid. Here, it means that the solution given in Eq. (5) is unique.

Integrating Eq. (5) over the unit sphere shows that the inner potential ϕ_0 and the excess charge ΔQ are related by

$$\phi_0=3\,\Delta Q/4\pi\epsilon_0(k^2a^2+3ka+3)a. \tag{6}$$

It is now convenient to define $Q_c/\Delta Q$, the fraction of the excess charge that resides in the interior of the sphere, i.e., the portion of the excess charge that is represented by the first term on the right-hand side of Eq. (5). Computing the contribution of this term to the excess charge yields

$$Q_c/\Delta Q=k^2a^2/(k^2a^2+3ka+3). \tag{7}$$

At large values of k, most of the charge is inside, but as Coulomb's law is approached by letting k approach zero, the excess charge rapidly becomes concentrated at the outer edge of the sphere. The small amount of excess charge in the interior produces an inner density whose magnitude is approximately given by

$$\rho_{\rm inner}\approx\Delta Q\,k^2/4\pi a. \tag{8}$$

Note that this quadratic dependence of the interior charge on the photon mass is the same dependence that occurs for two concentric shells,[3] and is the same dependence required by the theorem of Goldhaber and Nieto,[6] namely, that the photon mass effects are of order m_p^2. Hence, solid spheres do not offer any obvious improvement in the method of determining the photon mass in Gauss' law experiments.

III. INVERSE-POWER LAW: NEGATIVE EXCESS CHARGE WITH $n>1$

With two possible signs of excess charge and two separate classes of powers, $n>1$ and $n<1$, there are actually four separate cases to consider for the inverse-power-law potential. Of the four, only the case with negative excess charge and $n>1$ is simple enough to present here. The other cases have been solved, however, by the method described in Ref. 15, and will be discussed qualitatively in Sec. IV.

Suppose that the potential of a point charge q is given by an inverse-power law of the form

$$\phi(r)=q/4\pi\epsilon_0 r^n, \tag{9}$$

where D is a constant. (Note that although the formula has the usual SI appearance, ϵ_0 has unusual units.) By integrating Eq. (9) over a shell it can be shown that the potential ϕ_s, at a distance r from an infinitely thin shell of radius r'

and charge q, is given by

$$\phi_s(r) = \begin{cases} [q/8\pi\epsilon_0(2-n)rr'] \\ \quad \times[(r+r')^{2-n} - |r-r'|^{2-n}], & n\neq 2, \\ (q/16\pi\epsilon_0 rr')\ln[(r+r')/(r-r')]^2, & n=2. \end{cases} \tag{10}$$

If the charge distribution is described by a spherically symmetric charge density $\rho(r)$, contained within a sphere of radius a, then the potential at radius r is obtained by integrating Eq. (10) against the density $\rho(r)$. As in Sec. II, it is convenient here to define the variable $f(r) = r\rho(r)$ and to extend the integration region to negative values by using the odd extension of $f(r)$. The resulting equation for the potential is

$$\phi(r) = \frac{1}{r}\int_{-a}^{a} K(r,r')f(r')dr', \tag{11}$$

where

$$K(r,r') = \begin{cases} [1/2\epsilon_0(n-2)]|r-r'|^{2-n}, & n\neq 2, \\ (1/4\epsilon_0)\ln[1/(r-r')]^2, & n=2. \end{cases}$$

Suppose, now, that the excess negative charge is distributed throughout the interior of a conducting sphere of radius a, as in Sec. II. In this case the equilibrium requirement is that the potential throughout the sphere be constant, and it takes the form

$$\frac{1}{2\epsilon_0(n-2)}\int_{-a}^{a}|r-r'|^{2-n}f(r')dr' = \phi_0 r, \tag{12}$$

where ϕ_0 is the value of the constant potential, and where $0\leqslant r\leqslant a$. [Note that except for differences in notation, this equation is Shaw's Eq. (20).[8] Note also that the special case $n=2$ will be discussed briefly in Appendix A, but will otherwise be ignored.]

As in Sec. II, this equation is a Fredholm integral equation of the first kind. The kernel of this integral equation, $|r-r'|^{2-n}$, is the so-called Carleman kernel.[14] This equation is solved by noting that it is a special case of a general identity given by Auer and Gardner.[15] The identity is

$$\int_{-1}^{1}(1-y^2)^\nu P_l^\nu(y)|y-x|^{-1-2\nu}dy$$
$$= \delta_{lm} A_m^\nu N_m^\nu P_m^\nu(x), \tag{13}$$

where $-\frac{1}{2}\leqslant\nu\leqslant 0$ and where $P_m^\nu(x)$ is a Gegenbauer polynomial normalized so that $P_m^\nu(1)=1$. For example, $P_0^\nu(x)=1$ and $P_1^\nu(x)=x$. Hence, if we choose $l=m=1$, $x=r/a$, $y=r'/a$, $f(r)=x(1-x^2)^\nu$, and

$$\nu=(n-3)/2, \tag{14}$$

then Eqs. (12) and (13) are identical for properly chosen ϕ_0. The coefficients A_m^ν and N_m^ν are given by

$$A_m^\nu = \frac{-\pi}{2^{1+2\nu}\sin\nu\pi} \times\left(\frac{\Gamma(m+2\nu+1)}{\Gamma(\nu+1)\Gamma(m+1)}\right)^2\frac{(2m+2\nu+1)}{\Gamma(2\nu+1)} \tag{15}$$

and

$$N_m^\nu = \frac{2^{1+2\nu}[\Gamma(\nu+1)]^2\Gamma(m+1)}{(2m+2\nu+1)\Gamma(m+2\nu+1)}. \tag{16}$$

The coefficient N_m^ν is the normalization factor for these polynomials:

$$\int_{-1}^{1}(1-x^2)^\nu P_l^\nu(x)P_m^\nu(x)dx = \delta_{lm}N_m^\nu. \tag{17}$$

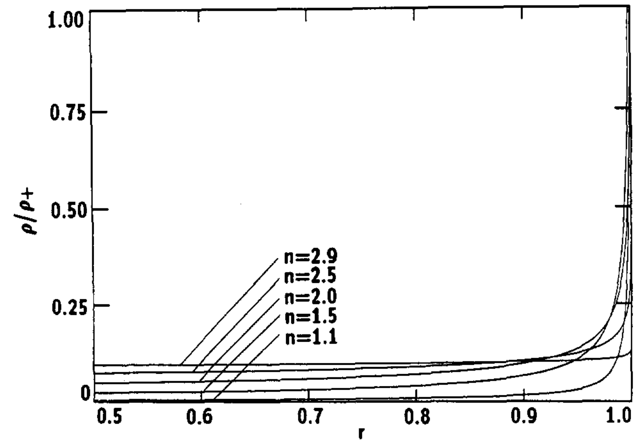

Fig. 1. The magnitude of the charge density, normalized to ρ_+, is displayed versus radius for negative excess charge and for $n=1.1$, 1.5, 2.0, 2.5, and 2.9. The charge density functions have been normalized so that the total charge is the same for each ($\Delta Q/Q_+ = 0.1$). Note that the sphere has unit radius.

Equations (13)–(17) are now combined to solve Eq. (12) for $f(r)$, which then gives the equilibrium charge density in the form

$$\rho(r) = f(r)/r$$
$$= \{\epsilon_0\phi_0\sin[(3-n)\pi/2]/\pi\}(a^2-r^2)^{(n-3)/2}. \tag{18}$$

This result is also given as an exercise in Ref. 14. Figure 1 displays $\rho(r)$ for various values of the exponent in the range $1<n<3$ with $a=1$. Note that the density is actually negative, since we are considering negative excess charge, but that the magnitude is plotted in Fig. 1. Strictly speaking, these results need not apply to the case $n=2$, since the kernel is then logarithmic. This case is discussed in Appendix A, however, where it is shown that Eq. (18) also holds for $n=2$.

Exponents $n>3$ are not considered because Eq. (12) is not integrable for such exponents. (See also Ref. 8.) The physical origin of this difficulty is that for $n>3$ the self-energy of repulsion of any continuous distribution of charge is infinite. Note that as $n\to 3$ from below, the excess electric charge density approaches a constant. This is similar to the Yukawa case when k becomes large. In the Yukawa case the constancy of the density is a result of the highly localized form of the potential at large k, and a similar localization occurs here as n approaches 3. In this limit the potential becomes localized because the strong singularity at each charge completely dominates over the behavior of the potential elsewhere.

It is now convenient to choose a new normalization for the density corresponding to a fixed amount of excess charge. This new normalization makes it easier to compare density functions for differing exponents, and is used in Fig. 1. If ΔQ is the magnitude of the excess charge, then

$$\Delta Q \propto 2\pi\int_{-a}^{a}(a^2-r^2)^{(n-3)/2}r^2\,dr.$$

This is seen to be an instance of the normalization integral, Eq. (17) (with $l = m = 1$), so the density function is now given by

$$\rho(r) = (\Delta Q/2\pi N_1^\nu a^n)(a^2 - r^2)^{(n-3)/2}. \qquad (19)$$

Note that if $n < 1$, then the singularity is no longer integrable, and this solution is invalid. This case will be discussed in Sec. IV.

Consider now what happens as n approaches 1 (Coulomb's law) from above. In this limit $\nu = (n-3)/2 \to -1$, so we have, from Eqs. (16) and (19),

$$\rho(r) \approx (\Delta Q/4\pi a)[(n-1)/(a^2 - r^2)^{1-(n-1)/2}], \qquad (20)$$

which is a representation of $\delta(a - r)$ as $n \to 1$. (Actually, it is a one-sided delta function.) Thus Eq. (19) describes the approach to the familiar Coulomb result that all of the excess charge resides on the surface of the sphere. Equation (20) also shows that the inner density is of order

$$\rho_{\text{inner}} \cong \Delta Q(n-1)/4\pi a^3, \qquad (21)$$

for r not near a.

Note that the inner charge is proportional to $n - 1$, the same dependence that occurs for two concentric conducting spheres,[3] so, as in the Yukawa case, solid spheres do not offer any obvious improvement in the method of testing how closely Coulomb's law is obeyed.

IV. INVERSE-POWER LAW: THE OTHER CASES

The inverse-power-law cases, other than negative excess charge with $n \geqslant 1$, are too complicated to discuss in detail here. They are, however, quite interesting and will be qualitatively discussed with the aid of Fig. 2. The densities represented in this figure are the results of equilibrium calculations using the method of Auer and Gardner.[15] Six frames are shown, each corresponding to either positive or negative excess charge and to one of three choices for the exponent n:$n > 1$, $n = 1$ (Coulomb's law), and $n < 1$. Note that the sphere has unit radius.

Frames (b) and (e) show the usual Coulomb's law result. If the excess charge is negative, all of it resides at the surface [see frame (e)]. If positive, it resides in a thin layer at the outer edge from which all free negative charge has been removed [see frame (b)]. Note that in all of the positive-excess-charge cases the width of this outer region has been chosen to be much larger than is reasonable for clarity of presentation.

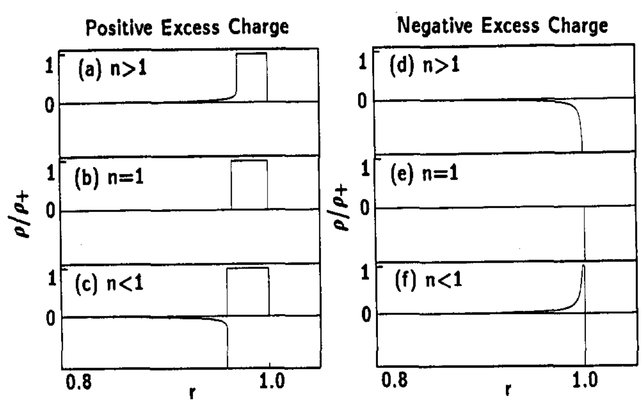

Fig. 2. Charge density distributions for a sphere of unit radius are displayed for three exponents, $n = 1.1$, $n = 1.0$ (Coulomb's law), and $n = 0.9$, and for two excess charge ratios, $\Delta Q/Q_+ = +0.1$ [(a), (b), and (c)] and $\Delta Q/Q_+ = -0.1$ [(d), (e), and (f)]. The vertical lines in (e) and (f) represent the negative delta-function shells at $r = 1$; the delta function in (f) is stronger than that in (e) by the factor 1.34.

The case discussed in Sec. III, negative excess charge with $n > 1$, is shown in frame (d). The corresponding case, positive excess charge with $n > 1$, is shown in frame (a). The charge density cannot be singular here because it cannot exceed the density of the fixed positive background. Instead, the density rises sharply (it has a singular derivative) and then remains constant out to the edge of the sphere. As the excess charge becomes a very small fraction of the total positive background charge, the density approaches the singular shape of frame (d), as might be expected.

When $n < 1$, the solutions become much more complicated. The reason is that when $n < 1$, a shell of charge repels other like charges that are outside the shell, but attracts like charges within it. Hence, an equilibrium with charge all of one sign is impossible (see Ref. 8). The equilibria now are as shown in frames (c) and (f). If the excess charge is positive, a positive outer shell forms containing more positive charge than the amount of the excess, and within this outer shell a singular distribution of negative charge forms [see frame (c)]. If the excess charge is negative, then all of the excess negative charge and some of the free negative charge move to the outer edge of the sphere, leaving behind a positive shell and a sharply rising positive distribution with a singular derivative [see frame (f)]. This case is the strangest of them all, as discussed in Appendix B.

In spite of the complicated details, however, there is one common feature shared by all of the cases with $n \neq 1$: The inner charge density is approximately given by Eq. (21) as $n \to 1$.

Finally, it must be pointed out that there is an important difference between the inverse-square-law potential and the Yukawa potential, namely, that there is no differential equation corresponding to Proca's equation. As Shaw points out, this means that there is no uniqueness theorem to assure us that these equilibria are the only possible ones.[8] In particular, there may be others that are nonspherically symmetric. Furthermore, when the stability of these equilibria are considered, there are no differential equations to linearize in the usual way. This makes a whole set of interesting questions abut equilibrium and stability, especially for the peculiar case $n < 1$, very difficult to approach.

V. CONCLUSION

If the electrostatic force law were not inverse square, then the excess charge on a conducting sphere would not all reside on its surface. For the cases of the Yukawa potential

$$\phi(r) \propto q e^{-kr}/r,$$

and the inverse-power-law potential,

$$\phi(r) \propto q/r^n,$$

the equilibrium charge distributions are somewhat similar, in that for parameter choices near Coulomb's law, i.e., for n near 1 or for k near 0, most of the excess charge ΔQ is concentrated near the outer edge of the sphere, but the interior of the sphere is also filled with excess charge. In the Yukawa case this density is constant and is given by

$$\rho_{\text{inner}} = \Delta Q\, k^2/4\pi a.$$

In the case of the inverse-power-law the inner density is not constant, but its magnitude is approximately given by

$$\rho_{\text{inner}} \approx \Delta Q(n-1)/4\pi a^3.$$

The manner in which the charge is concentrated near the outer edge is even quite similar for the Yukawa case and for the inverse-power law with $n > 1$. In the inverse-power-law case with $n < 1$, however, the excess charge distribution near the outer edge is more complicated, and the equilibrium problem need not even have a unique solution (see Appendix B).

The appearance of all of this diversity as the electrostatic force law is varied slightly from the inverse-square law is quite striking. Of course, experiments show that any possible deviation is extremely small, but even small deviations introduce oddities into electrostatic theory. The inverse-square law is very special indeed.

ACKNOWLEDGMENTS

The author thanks Grant Hart, Grant Mason, and Neil Rasband for stimulating discussions, and gives special thanks to the College of Physical and Mathematical Sciences at Brigham Young University for supporting this work. Special thanks are also due to referees of this paper whose comments have been most helpful.

APPENDIX A: NEGATIVE EXCESS CHARGE WITH $n = 2$

For the special case where $n = 2$ and where the excess charge is negative, it is possible to give an elementary proof that the equilibrium density is of the form

$$\rho(r) = A/\sqrt{a^2 - r^2}, \tag{A1}$$

with A a constant. Substitution into the equilibrium equation for this case, Eq. (12), with the power-law kernel replaced by the logarithmic kernel from Eq. (11) gives the proposed identity

$$\int_0^a \ln\left[\left(\frac{r+r'}{r-r'}\right)^2\right] \frac{r'\,dr'}{\sqrt{a^2 - r'^2}} = Br, \tag{A2}$$

with B another constant, namely, $B = 2\epsilon_0\phi_0/A$. We set $r' = a\sin\theta$ to obtain

$$\int_0^{\pi/2} \ln\left[\left(\frac{r/a + \sin\theta}{r/a - \sin\theta}\right)^2\right] \sin\theta\, d\theta = B\frac{r}{a}.$$

which, because of the special form of the integrand, can be written as

$$I(r) = \int_0^{2\pi} \ln\left[\left(\frac{r/a + \sin\theta}{r/a - \sin\theta}\right)^2\right] \sin\theta\, d\theta = 4B\frac{r}{a}. \tag{A3}$$

The integral can now be evaluated as a contour integral by setting $z = e^{i\theta}$ and by integrating about the unit circle in the complex z plane. We have

$$I(r) = -\frac{1}{2}\oint \ln\left[\left(\frac{2i(r/a) + z - z^{-1}}{2i(r/a) - z + z^{-1}}\right)^2\right]\left(1 - \frac{1}{z^2}\right) dz. \tag{A4}$$

The only pole within the unit circle is at $z = 0$; expanding in small z and using the residue theorem yields

$$I(r) = 8\pi r/a; \tag{A5}$$

choosing $B = 2\pi$ then makes Eq. (A3) hold, and the density can be written

$$\rho(r) = \epsilon_0\phi_0/\pi\sqrt{a^2 - r^2}, \tag{A6}$$

in agreement with Eq. (18) for $n = 2$.

APPENDIX B: NONUNIQUENESS OF THE EQUILIBRIUM PROBLEM

It is interesting to note that even in the case of Coulomb's law, the spherical equilibrium problem has no unique solution. The reason is that equilibrium does not require that the electric field vanish throughout the conductor, but only that the electric field vanish in regions where there is free charge. In particular, for a given amount of excess charge it is possible to make a whole family of equilibria by varying the width of a positive charge region at the outer edge of the sphere.

To see how this comes about, consider the case of positive excess charge. The excess charge resides in a thin layer at the outer edge of the sphere from which all negative charge has been removed. Now imagine that some more of the free negative charge in the interior is collected and formed into a zero-thickness shell. This, of course, increases the width of the outer positive region. Now we must ask if there is an equilibrium position for the new negative shell. It is easy to see that there is such a position within the outer positive region, for the shell tends to expand under its own electric repulsion, but the positive electric field in the outer region tends to pull it back. Somewhere in the outer positive region, these forces balance. It is easy to compute this equilibrium position by using Gauss' law to find the positive electric field and by computing the effective electric field of self-repulsion from the relations

$$F_r = qE_{\text{self}} = -\frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\frac{q^2}{2C}, \tag{B1}$$

where C is the capacitance of a sphere, $C = 4\pi\epsilon_0 r$, and r is the radius of the sphere. Hence,

$$E_{\text{self}} = q/8\pi\epsilon_0 r^2. \tag{B2}$$

Note that this situation is only of mathematical importance; the removal of just one electron per atom at the outer surface of a conductor would produce an electric field in excess of 10^{11} V/m. Furthermore, it is easy to see that these additional equilibria are unstable. Imagine that just one electron is displaced inward from the extra negative shell. It no longer feels the influence of the negative shell, since it is within it, and immediately is pulled inward by the positive electric field of the outer positive region. Of course, if one electron can do this, all of them eventually will and the usual equilibrium situation will quickly be established.

The case of negative excess charge is even simpler. The excess charge resides in a thin layer at the outer edge of the sphere. To make a new equilibrium with the same total excess charge, we move some of the interior free negative charge to the outside, opening up a positive region containing no free charge just inside the outer negative shell. The outer layer is kept from being pulled inward by its self-repulsion as long as the extra charge transferred to the outside does not exceed the amount of the excess charge. [This may easily by shown by using Eq. (B2).] Once again a whole range of equilibria is possible, and once again the new equilibria are all unstable. Individual electrons will be pulled inward from the outer shell and the positive region will be neutralized.

Because of the instability, this entire discussion is of no importance whatever for the case of Coulomb's law. Instability also occurs in nearly all of the non-Coulomb's law cases discussed in this article. There is, however, a single exception: negative excess charge with $n < 1$. In this case, the outer negative charge attracts any electrons that are displaced inward, pulling them back into the shell in opposition to the inward pull of the positive region. If too much interior charge is transferred to the outer edge, instability once again occurs, but there is a small range of stable equilibria if $n < 1$. This behavior has been verified with computer simulations in which many negative shells were allowed to move in response to each other and in response to a fixed positive background. A range of possible equilibria was observed, with the amount of extra charge that can be transferred to the outside before instability once again occurs being roughly proportional to $1 - n$. (Note that it is still necessary to have some force that keeps the outer negative shell from leaving the sphere.)

Hence, for negative excess charge and $n < 1$, the idealized equilibrium model discussed in this article admits a continuum of stable charge distributions for a given excess charge. Hence, there is no unique stable solution to the problem of how excess negative charge is distributed within a solid conducting sphere if $n < 1$.

[1]J. Priestley, *The History and Present State of Electricity*, 3rd ed., London, 1775 (Johnson Reprint Corporation, New York, 1966), Vol. 2, p. 374.

[2]H. Cavendish, in *Electrical Researches*, edited by J. C. Maxwell (Cambridge U.P., Cambridge, 1879), p. 104.

[3]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Dover, New York, 1954, reprint of the third edition, 1891), Vol. 1, Art. 74.

[4]S. Plimpton and W. Lawson, "Very accurate test of Coulomb's law of force between charges," Phys. Rev. **50**, 1066–1071 (1936).

[5]E. R. Williams, J. E. Faller, and H. A. Hill, "New experimental test of Coulomb's law: A laboratory upper limit on the photon rest mass," Phys. Rev. Lett. **26**, 721–724 (1971).

[6]A. S. Goldhaber and M. M. Nieto, "Terrestrial and extraterrestrial limits on the photon mass," Rev. Mod. Phys. **43**, 277–296 (1971).

[7]L. P. Fulcher, "Improved result for the accuracy of Coulomb's law: A review of the Williams, Faller, and Hill experiment," Phys. Rev. A **33**, 759–761 (1986).

[8]R. Shaw, "Symmetry, uniqueness, and the Coulomb law of force," Am. J. Phys. **33**, 300–305 (1965).

[9]N. D. Lang and W. Kohn, "Theory of metal surfaces: Charge density and surface energy," Phys. Rev. B **1**, 4555–4568 (1970).

[10]N. D. Lang and W. Kohn, "Theory of metal surfaces: Work function," Phys. Rev. B **4**, 1215–1223 (1971).

[11]V. E. Kenner, R. E. Allen, and W. M. Saslow, "Screening of external fields and distribution of excess charge near a metal surface," Phys. Rev. Lett. A **38**, 255–256 (1972).

[12]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), pp. 23–24.

[13]F. G. Tricomi, *Integral Equations* (Interscience, New York, 1957).

[14]J. A. Cochran, *The Analysis of Linear Integral Equations* (McGraw-Hill, New York, 1972), pp. 305–306.

[15]P. L. Auer and C. S. Gardner, "Note on singular integral equations of the Kirkwood–Riseman type," J. Chem. Phys. **23**, 1545–1546 (1955).

Doppler effect for sound via classical and relativistic space-time diagrams

Robert E. Reynolds

Department of Physics, Reed College, Portland, Oregon 97202

(Received 6 March 1989; accepted for publication 13 June 1989)

The longitudinal classical and relativistic Doppler effects for sound are derived through use of space-time diagrams. Diagrams appropriate to Galilean and Lorentz transformations are reviewed in order to stress their simplicity and instructional value. The relativistic limiting case in which the signal is light is also considered.

I. SPACE-TIME DIAGRAMS

Diagrams of a two-dimensional x-t subspace of space-time have long been used to help develop intuitions appropriate to physics in Minkowski space.[1] The corresponding classical diagrams are particularly well suited to development of the Doppler formulas for sound. Further, experience with application of a space-time diagram in a classical context may usefully serve as an early introduction to the space-time perspective for a student.

This article will compare classical and relativistic space-time diagrams, emphasizing interpretations of relevant intervals in each, and then develop the appropriate Doppler formulas for sound (the relativistic acoustic Doppler effect has been treated recently by several authors[2–6]). Since the Minkowski diagram is likely to be more familiar, it is considered first.

II. SPECIAL RELATIVITY DIAGRAMS

The standard simplest diagram is concerned with space-time coordinates of events for two inertial observers connected by the Lorentz transformation for which two frames, S with event coordinates (x,t) and S′ with event coordinates (x',t'), have origins coincident at $t = 0 = t'$. S′ moves in the $+x$ direction with speed v. The transformation from (x,t) to (x',t') is

$$x' = \gamma(x - vt), \tag{1a}$$

$$t' = \gamma[t - (v/c^2)x]. \tag{1b}$$