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Renormalization-group classification of continuous structural phase transitions induced by six-component order parameters

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The results of renormalization-group (RG) analysis in reciprocal space are reported for solid-solid phase transitions. We list the Landau-Ginzburg-Wilson Hamiltonian densities for six-component order parameters. The resulting RG recursion relations are derived by means of symmetric products of basic invariants. A list of fixed points for the densities is given and the stable fixed points are noted. Only two of the effective Hamiltonian densities possess stable fixed points. We indicate those specific structural transitions which are allowed to be continuous as determined by RG methods. We give critical exponents for the stable fixed points.

I. INTRODUCTION

Within the framework of both Landau theory[1] and renormalization-group (RG) methods[2] the description of phase transitions induced by order parameters with a large number of components ($n>3$) yields some rather distinctive results. As the dimension of the order parameter gets higher a variety of quartic potential forms are found and as a result the classification of symmetry changes by Landau theory becomes a more sophisticated job. Recently we classified continuous phase transitions for commensurate solids described by six-component order parameters ($n=6$) and obtained their phase diagrams.[3] [For such transitions the order parameters transform as basis functions of irreducible representations (irreps) of space groups.] We found a case where a single potential violated both the Ascher conjecture[4] (where the isotropy subgroup was not maximal with respect to the physical symmetry group of the high-symmetry phase), and the Michel-Radicati conjecture[5] (where the isotropy subgroup was not maximal with respect to the symmetry group of the potential). For $n=6$ we also found examples where three phases could coexist. This is not possible for $n \leq 3$.

Likewise, within the framework of RG methods certain characteristics occur which are not present for $n \leq 3$. For example, for $n \leq 3$ there is always a stable fixed point and it is isotropic.[6] For such systems fluctuations in the critical region are so strong that any anisotropy is erased. For $n \geq 4$ the isotropic fixed point is no longer stable when anisotropic quartic terms are present.[6] Moreover, for certain systems a stable fixed point does not even exist. It has been conjectured[7] that when the transition is predicted to be continuous by Landau theory, the lack of a stable fixed point signals a fluctuation-induced first-order transition. Similarly if a given set of parameters for an effective Hamiltonian falls outside of the attraction domain of the stable fixed point, the ensuing phase transition is discontinuous.

In the Landau theory one can tell whether or not an order parameter can induce a continuous phase transition by checking symmetry criteria such as the Landau condition[1] (there must not exist a third-degree invariant), the Lifshitz condition[8] (the antisymmetric product of the representation of the order parameter must not contain the vector irrep), etc. In the same spirit, Michel and Toledano[9] have recently presented a symmetry criterion for the lack of a stable fixed point. The criterion is not as simple as the above two conditions and we refer the reader to Ref. 9 for further detail. The criterion was shown to work for all types of effective Hamiltonians with four-component order parameters. We have exploited it in the case of six-component order parameters.[10]

In this paper we find the stable fixed points for the types of effective Hamiltonians, with six-component order parameters, which occur in the description of structural transitions by solving RG recursion relations obtained through the ϵ-expansion method. We have recently obtained Landau-Ginzburg-Wilson (LGW) Hamiltonian densities for commensurate structural transitions.[3] There are six distinct densities for $n=6$. They correspond to a total of 43 different space-group representations according to which the order parameters transform. We found one additional Hamiltonian density to those found by Toledano and Toledano[11] and it occurs in two different space groups.

In Sec. II we briefly indicate the methods we have used to systematically obtain the distinct Landau potential forms. We emphasize our group-theoretical approach which we recently implemented on a computer.[12,13] In this same section we reiterate the results of the minimization of the quartic potentials applied to the six potential forms for $n=6$. We will place a special emphasis upon the two forms which yield stable fixed points within RG methods. In Sec. III we derive the RG recursion relations and list the fixed points for all but one LGW Hamiltonian density. In our derivation we use the symmetric algebra and the invariant scalar product discussed in Ref. 9. In Sec. IV we indicate the attraction domain for the stable fixed point of one of the densities. As was pointed out in Ref. 14, the attraction domain is smaller than the stability region of possible continuous transition obtained from the Landau theory. We also list those specific structural transitions (driven by a six-component order parameter) which can be continuous within the RG formalism as well as in the Landau theory. In Sec. V we obtain the critical exponents for the stable fixed points.

II. LANDAU POTENTIALS AND THEIR MINIMIZATION

The Landau theory[1] of continuous phase transitions is a phenomenological theory. However, it is a very useful theory because, when extended, it provides two important pieces of information. First, it defines a framework which can be used to obtain possible lower-symmetry space groups resulting from the order-parameter orientation. Second, it suggests the construction of invariant potentials which are then to be used as the polynomial contributions in the effective LGW Hamiltonian density for RG calculations.

The Landau theory assumes the existence of a macroscopic variable $\boldsymbol{\phi}$, the order parameter, whose symmetry determines the thermodynamic potential $h(\boldsymbol{\phi})$. The potential should be a smooth function of macroscopic variables P, T, and the order-parameter components. At high temperature the potential and equilibrium state should exhibit the symmetry G_0 of the physical system and therefore $\boldsymbol{\phi}$ is to be equal to zero. As temperature is lowered, $\boldsymbol{\phi}$ becomes nonzero at the equilibrium state. The symmetry is thus spontaneously broken since the potential remains invariant under transformations of G_0. For our considerations we take G_0 to be one of the 230 space groups in three dimensions. $\boldsymbol{\phi}$ is to be an n-component vector transforming according to a real irreducible orthogonal representation of G_0.

The potential $h(\boldsymbol{\phi})$ can be expanded as a polynomial series in terms of components of $\boldsymbol{\phi}$ with each polynomial of each degree being an invariant under G_0. Since the Landau condition prohibits third-degree terms, the potential to fourth degree can be written as

$$h(\boldsymbol{\phi})=\frac{r}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi}+\frac{1}{4!}P_4(\boldsymbol{\phi}) , \qquad (1)$$

with $P_4(\boldsymbol{\phi})$ of the general form

$$P_4(\boldsymbol{\phi})=\sum_{i,j,k,l} u_{ijkl}\phi_i\phi_j\phi_k\phi_l=u_0I_0^2(\boldsymbol{\phi})+\sum_{\nu=1}^{p-1} u_\nu I_\nu(\boldsymbol{\phi}) . \qquad (2)$$

The u_{ijkl} are symmetric under interchange of subscripts. Each $I_\nu(\boldsymbol{\phi})$ is an invariant polynomial of fourth degree, $I_0^2(\boldsymbol{\phi})\equiv(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ and the u_ν are arbitrary coefficients carrying the temperature and pressure dependence of the potential.

The systematic application of group-theoretical methods[12,13,15,16] have allowed us to select the relevant irreps and to list comprehensively the possible lower symmetry phases induced from these irreps. We followed the usual process[17] for obtaining irreps of the 230 space groups. In order that the transition be continuous and commensurate, only those irreps which satisfy both the Landau[1] and the Lifshitz[8] conditions are suitable, i.e., the *active* irreps.[18] Thus only irreps constructed from $\mathbf{k}$ points of symmetry[17] are considered. The set of distinct matrices $D(G_0)\equiv\{D(g)\}$ of an irrep form a finite subgroup of $O(n)$ and this same set of matrices (up to conjugation), which is called the *image*,[19] may appear many times in the collection of irreps of the space groups. For irreps of six dimensions, only 11 distinct images occur in all the active space-group irreps corresponding to $\mathbf{k}$ points of symmetry. In Table I we list the images (column 1) and the irreps which give rise to that image (column 2). Our labeling of images is the same as that in Ref. 11. The labeling of irreps is that of Cracknell *et al.*[20] Notice, however, that we claim the existence of an additional image (and subsequently an additional potential) L_{11} not obtained in Ref. 11.

We show in Fig. 1 a lattice (tree) of group-subgroup relations for the 11 images. By an appropriate selection of bases in each irrep, an image group (the group of matrices) can be made a "direct" subgroup of another (indicated by a solid line in the figure). The "direct" subgroup is simply a subset of matrices of the higher image group. We have chosen bases such that the "direct" subgroup relationships are satisfied simultaneously for all images in the figure. The orders of the images are indicated on the left-hand side in Fig. 1. The actual forms of the polynomial invariants are usually affected by a change of basis. We have also selected the basis of each image so that it is compatible with the list of invariants of Ref. 11. The invariants are constructed by conventional projection operator methods.[18,21] For RG analysis we are primarily interested in the fourth-degree invariants. We have listed these invariants in column 3 of Table I for each image. The explicit forms of these invariants are given in Table II. These are the forms to be used in Eq. (2). In Table III we list the u_{ijkl}, also used in Eq. (2), for each of the ten quartic invariants. In Table IV we have listed the six fourth-degree potential expansions which result for six-component order parameters.

In order to obtain the phase diagram we need to minimize the potential in the entire coupling coefficient space. Kim's minimization method[22] is suitable for this purpose. In Ref. 3 we obtained the region (call it the stability region) in the coupling coefficient space corresponding to each phase (a "phase diagram") and classified continuous transitions allowed by Landau theory. For all cases with $n=6$ there is a degeneracy of two lower-symmetry phases (or coexistence of three phases at the transition from the high-symmetry phase) at fourth degree. Of particular importance to us in this paper are the potential h_1 corresponding to L_1, L_2, L_3, and L_5, and the potential h_6 corresponding to L_{11}. The effective Hamiltonian densities constructed from these two potentials will be shown to have stable fixed points.

The potential h_1 to fourth degree is written

$$h_1=\frac{r}{2}I_0+\frac{1}{4!}(u_0I_0^2+u_1I_1^{(4)}+u_2I_2^{(4)}) . \qquad (3)$$

In Fig. 2 we show the phase diagram for h_1. Phases $P1$, $P2$, $(P6,P7)$, and $(P11,P12)$ are stable phases in appropriate regions of the coefficients u_1 and u_2. Phases $P6$ and $P7$ are distinct (inequivalent) phases but are degenerate for a fourth-degree expansion (similarly for $P11$ and $P12$). However, the degeneracy between inequivalent phases is lifted at higher degree.[3] Which specific symmetry (isotropy subgroup) corresponds to a particular phase, $P1$, for example, is determined by the specific space-group representation being considered. The elements of the isotropy subgroup corresponding to $P1$ are those space-group elements whose matrix representatives leave the order parameter $(a,0,0,0,0,0)$ invariant. Similarly, we obtain isotropy subgroups with the correspondences $P2 \sim (a,a,0,0,0,0)$, $P6 \sim (a,0,a,0,a,0)$, $P7 \sim (0,a,0,a,0,a)$, $P11 \sim (a,a,a,a,a,a)$, $\sim P12 \sim (a,a,a,a,a,-a)$.

The potential h_6 to fourth degree is written

$$h_6=\frac{r}{2}I_0+\frac{1}{4!}(u_0I_0^2+u_1I_1^{(4)}+u_2I_2^{(4)}+u_7I_7^{(4)}) . \tag{4}$$

In Tables V and VI we indicate the stability region for each phase of h_6. We have defined $\tan\xi=|B/u_1|$ where $B^2=u_2^2+u_7^2$ and $\tan\theta=u_7/u_2$. The symmetries of the phases $C1$ and $C23$ are those space-group elements which respectively leave invariant the order parameters $(a,b,0,0,0,0)$ and (a,b,a,b,a,b). To illustrate how to use the tables consider the region $B>0$ and $u_1>0$ with $\theta=45°$. For $\xi>\xi_L\equiv 82.6°$ the phase $C1$ is stable. For $\xi<\xi_L$ the phase $C23$ is stable. ξ_R is to be used for $B<0$ and $u_1>0$.

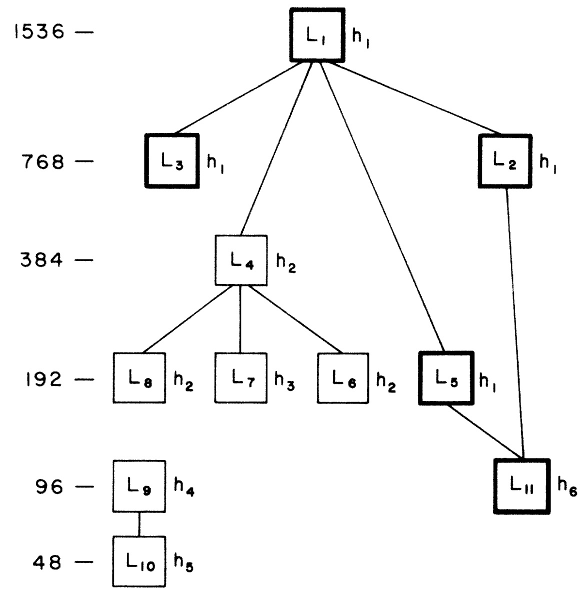

FIG. 1. "Direct" subgroup relationships for the six-dimensional images $(L_1, L_2, \ldots, L_{11})$ of structural transitions. The orders of the images are indicated on the left-hand side. The fourth-degree invariant potentials $(h_1, h_2, \ldots, h_6)$ which arise from each image are indicated. Only the potentials h_1 and h_6 yield Hamiltonian densities which have stable fixed points. An image associated with a continuous transition within RG analysis is indicated by a bold box.

TABLE I. Images of active space-group representations and their invariants of fourth degree. The space group, its international number, and the irreps which give rise to each image are indicated.

Image	Space-group irrep	Fourth-degree invariants
L_1	W_1,W_2,W_3,W_4 of O_h^5 (No. 225)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$
L_2	W_1,W_2 of O^3 (No. 209)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$
L_3	W_1,W_2,W_3,W_4 of T_d^2 (No. 216)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$
L_4	X_3,X_4 of O_h^4 (No. 224) N_1^-,N_2^-,N_3^-,N_4^- of O_h^9 (No. 229)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$
L_5	N_2,N_4 of O^8 (No. 214)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$
L_6	N_2,N_4 of O^5 (No. 211) N_2,N_3 of T_d^3 (No. 217) M_2 of O_h^2 (No. 222) M_4 of O_h^4 (No. 224) X_4 of O_h^7 (No. 227) X_4 of O_h^8 (No. 228) N_2^+,N_4^+ of O_h^9 (No. 229)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$
L_7	X_5 of T_d^1 (No. 215) X_5^+, X_5^-, M_5^- of O_h^1 (No. 221) M_5^- of O_h^3 (No. 223) X_5^- of O_h^5 (No. 225) X_5^- of O_h^6 (No. 226)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$, $I_6^{(4)}$
L_8	N_1^-,N_2^- of T_h^5 (No. 204)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$
L_9	X_3,X_4 of O_h^3 (No. 223)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$, $I_4^{(4)}$, $I_5^{(4)}$
L_{10}	$X_1\oplus X_2$, $X_3\oplus X_4$ of T_d^4 (No. 218)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_3^{(4)}$, $I_4^{(4)}$, $I_5^{(4)}$, $I_7^{(4)}$, $I_8^{(4)}$, $I_9^{(4)}$
L_{11}	$M_2\oplus M_3$ of O^6 (No. 212) $M_2\oplus M_3$ of O^7 (No. 213)	I_0^2, $I_1^{(4)}$, $I_2^{(4)}$, $I_7^{(4)}$

TABLE II. Fourth-degree invariant polynomials used in Table I.

$I_0^2=(\eta_1^2+\zeta_1^2+\eta_2^2+\zeta_2^2+\eta_3^2+\zeta_3^2)^2$
$I_1^{(4)}=\eta_1^4+\zeta_1^4+\eta_2^4+\zeta_2^4+\eta_3^4+\zeta_3^4$
$I_2^{(4)}=\eta_1^2\zeta_1^2+\eta_2^2\zeta_2^2+\eta_3^2\zeta_3^2$
$I_3^{(4)}=\eta_1\zeta_1\eta_2\zeta_2+\eta_2\zeta_2\eta_3\zeta_3+\eta_3\zeta_3\eta_1\zeta_1$
$I_4^{(4)}=\eta_1^2\eta_2^2+\zeta_1^2\zeta_2^2+\eta_2^2\eta_3^2+\zeta_2^2\zeta_3^2+\eta_3^2\eta_1^2+\zeta_3^2\zeta_1^2$
$I_5^{(4)}=\eta_1^2\zeta_2^2+\eta_2^2\zeta_3^2+\eta_3^2\zeta_1^2$
$I_6^{(4)}=\eta_1\zeta_1(\eta_2^2+\zeta_2^2-\eta_3^2-\zeta_3^2)+\eta_2\zeta_2(\eta_3^2+\zeta_3^2-\eta_1^2-\zeta_1^2)$
$\quad+\eta_3\zeta_3(\eta_1^2+\zeta_1^2-\eta_2^2-\zeta_2^2)$
$I_7^{(4)}=\eta_1\zeta_1(\eta_1^2-\zeta_1^2)+\eta_2\zeta_2(\eta_2^2-\zeta_2^2)+\eta_3\zeta_3(\eta_3^2-\zeta_3^2)$
$I_8^{(4)}=\eta_1\zeta_1(\eta_2^2-\zeta_3^2)+\eta_2\zeta_2(\eta_3^2-\zeta_1^2)+\eta_3\zeta_3(\eta_1^2-\zeta_2^2)$
$I_9^{(4)}=\eta_1\zeta_1(\zeta_2^2-\eta_3^2)+\eta_2\zeta_2(\zeta_3^2-\eta_1^2)+\eta_3\zeta_3(\zeta_1^2-\eta_2^2)$

TABLE III. Nonzero symmetric coefficients defining the fourth-degree basic invariant polynomials.

Invariant	Coefficients
L_0^2	$u_{iiii}=u_0$ $u_{iijj}=u_0/3, \quad i\neq j$
$I_1^{(4)}$	$u_{iiii}=u_1$
$I_2^{(4)}$	$u_{1122}=u_{3344}=u_{5566}=u_2/6$
$I_3^{(4)}$	$u_{1234}=u_{3456}=u_{5612}=u_3/24$
$I_4^{(4)}$	$u_{i,i,i+2,i+2}=u_4/6$
$I_5^{(4)}$	$u_{1144}=u_{3366}=u_{5522}=u_5/6$
$I_6^{(4)}$	$u_{1233}=u_{1244}=-u_{1255}=-u_{1266}=u_6/12$ $u_{3455}=u_{3466}=-u_{3411}=-u_{3422}=u_6/12$ $u_{5611}=u_{5622}=-u_{5633}=-u_{5644}=u_6/12$
$I_7^{(4)}$	$u_{1211}=-u_{1222}=u_7/4$ $u_{3433}=-u_{3444}=u_7/4$ $u_{5655}=-u_{5666}=u_7/4$
$I_8^{(4)}$	$u_{1233}=-u_{1266}=u_8/12$ $u_{3455}=-u_{3422}=u_8/12$ $u_{5611}=-u_{5644}=u_8/12$
$I_9^{(4)}$	$u_{1244}=-u_{1255}=u_9/12$ $u_{3466}=-u_{3411}=u_9/12$ $u_{5622}=-u_{5633}=u_9/12$

TABLE IV. Fourth-degree polynomials associated with the six-dimensional images. I_0^2 is assumed present in all potentials.

Potential	Fourth-degree invariants
h_1	$I_1^{(4)},\ I_2^{(4)}$
h_2	$I_1^{(4)},\ I_2^{(4)},\ I_3^{(4)}$
h_3	$I_1^{(4)},\ I_2^{(4)},\ I_3^{(4)},\ I_6^{(4)}$
h_4	$I_1^{(4)},\ I_2^{(4)},\ I_3^{(4)},\ I_4^{(4)},\ I_5^{(4)}$
h_5	$I_1^{(4)},\ I_2^{(4)},\ I_3^{(4)},\ I_4^{(4)},\ I_5^{(4)},\ I_7^{(4)},\ I_8^{(4)},\ I_9^{(4)}$
h_6	$I_1^{(4)},\ I_2^{(4)},\ I_7^{(4)}$

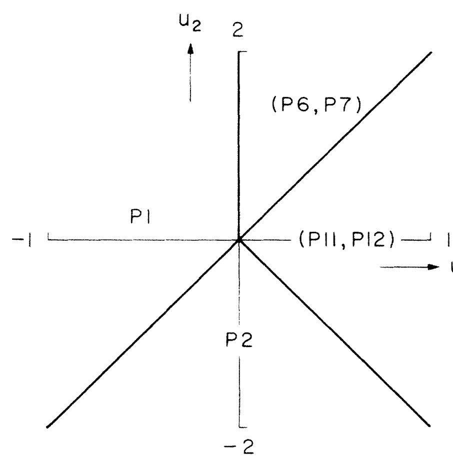

FIG. 2. Phase diagram of h_1 obtained from the quartic Landau potential. Continuous transitions to $P1$, $P2$, $(P6,P7)$, and $(P11,P12)$ can occur in the regions of u_1,u_2 as shown. Note that the line separating phases $(P6,P7)$, and $(P11,P12)$ is of slope 2.

TABLE V. Phase diagram for h_6. See Table VI for angles $\xi_L(\theta)$ and $\xi_R(\theta)$.

Region	Phase
$B<0$ $u_1>0$	$C1$: $\xi>\xi_R$ $C23$: $\xi<\xi_R$
$B<0$ $u_1<0$	$C1$: everywhere
$B>0$ $u_1<0$	$C1$: everywhere
$B>0$ $u_1>0$	$C1$: $\xi>\xi_L$ $C23$: $\xi<\xi_L$

TABLE VI. The limiting angles $\xi_L(\theta)$ and $\xi_R(\theta)$ to be used in Table V.

θ (deg)	$\xi_L(\theta)$ (deg)	$\xi_R(\theta)$ (deg)
5	89.8912	63.4351
10	89.5680	63.4376
15	89.0400	63.4478
20	88.3220	63.4743
25	87.4330	63.5273
30	86.3950	63.6179
35	85.2316	63.7570
40	83.9673	63.9551
45	82.6264	64.2212
50	81.2326	64.5633
55	79.8085	64.9880
60	78.3753	65.5003
65	76.9524	66.1042
70	75.5572	66.8018
75	74.2050	67.5943
80	72.9091	68.4812
85	71.6807	69.4605
90	70.5288	70.5288

III. RENORMALIZATION-GROUP RECURSION RELATIONS AND FIXED POINTS

Although the Landau theory provides a unique framework to explain symmetry changes occurring in phase transitions, it does not take fluctuations into account. The order parameter in the Landau theory takes a uniform value throughout space. Thus it is bound to fail in the critical region where fluctuations are dominant. Consequently, it does not predict correct critical exponents in cases where the dimension $d<4$, though its prediction is close to correct values when $d>4$. The region of validity for the Landau theory is roughly characterized by the Ginzburg criterion,[1,23]

$$u \sim (\Delta T)^{(4-d)/2} ,$$

where u is the coefficient of the quartic term and $\Delta T = |T-T_c|$. The Landau theory is valid outside ΔT. In the critical region the assumption that the free energy is a smooth function of the macroscopic variables is no longer valid, i.e., the transition point is a singular point of the free energy.

The renormalization-group idea[2] provides a powerful method for handling fluctuations. The effective Hamiltonian from which macroscopic properties are derived is an extension of the Landau potential. However, in order to derive macroscopic quantities, instead of using the mean-field approximation which would bring us back to the Landau theory, we start with the block spin method.[24] We eliminate short-wavelength microscopic degrees of freedom by functionally integrating them out (rescaling and renormalization). Then we require that the resulting lattice configuration stay the same at the price of admitting additional high-degree interactions. Due to rescaling and renormalization, we are left with the same Hamiltonian but with a different set of coefficients up to the fourth degree plus high-degree interactions. The change of coefficients under the renormalization-group transformation is described by a set of first-order differential equations (the RG recursion relations). At a fixed point the RG transformation does not change the effective Hamiltonian, which can only be realized when the correlation length is zero or infinity. Since the correlation length diverges at the critical point, the critical behavior is intimately related to the properties of a fixed point. In fact, different fixed points yield different critical exponents.

Imposing the Landau and Lifshitz condition (necessary for a commensurate continuous transition) and allowing only isotropic gradient contributions, the effective LGW Hamiltonian density is

$$H(\mathbf{x}) = \sum_{i=1}^{n} (\nabla \phi_i)^2 + \frac{r}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{1}{4!}P_4(\boldsymbol{\phi}) , \tag{5}$$

where $\phi_i(\mathbf{x})$ are the local order-parameter components, and $P_4(\boldsymbol{\phi})$ is identical to the Landau potential defined in Eqs. (1) and (2). In Eq. (5) $\mathbf{x}$ and ∇ are physical dimensional vectors while $\boldsymbol{\phi}$ is the order-parameter vector in the irrep space. Our analysis is restricted to the six effective Hamiltonian densities $H_1, H_2, \ldots, H_6$ with quartic interactions $h_1, \ldots, h_6$ occurring in structural transitions (see Table IV). The set of invariant quartic polynomials for a given H generates a p-dimensional vector space. The most general vector of this space is of the form of Eq. (2), i.e.,

$$P_4 = \sum_{i,j,k,l} u_{ijkl}\phi_i\phi_j\phi_k\phi_l = u_0 I_0^2(\boldsymbol{\phi}) + \sum_{v=1}^{p-1} u_v I_v(\boldsymbol{\phi}) ,$$

where u_v are now considered as components of the vector and the I_v as the bases. The RG method associates to a selected initial vector in P_4 a flow of vectors with the same invariants $I_v(\boldsymbol{\phi})$ but with varying coefficients. The preservation of the invariants $I_v(\boldsymbol{\phi})$ results from the covariance of the RG recursion relations under $O(6)$. Thus, the set of symmetry transformations [in $O(6)$] of the potential cannot decrease along the trajectory and can only increase at a fixed point.[25]

The flow of the coefficients u_v is governed by recursion relations of the general form

$$\frac{du_v}{d(\ln\lambda)} = \beta_v(u_0, u_1, \ldots, u_{p-1}) . \tag{6}$$

Critical behavior at a continuous transition resulting from the Hamiltonian specified by the u_v is determined by the properties of the stable fixed points as $\lambda \to 0$. This corresponds to the scale invariance of the system at its critical point. By definition, the left-hand side of Eq. (6) becomes zero at a fixed point (denoted by $u_0^*, u_1^*, \ldots, u_{p-1}^*$) and thus

$$\beta_v(u_0^*, u_1^*, \ldots, u_{p-1}^*) = 0 . \tag{7}$$

A fixed point is stable if the trajectories of Eq. (6) (originating from the neighborhood of the fixed point) flow towards the fixed point as $\lambda \to 0$. Thus the eigenvalues $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ of the matrix

$$M_{vv'} = \frac{\partial \beta_v}{\partial u_{v'}} \tag{8}$$

evaluated at the fixed point must have positive real parts.

The right-hand side of Eq. (6) can be written as a double series expansion in $\epsilon = 4-d$ and u_v (Ref. 6):

$$\beta_{ijkl} = -\epsilon u_{ijkl} + \frac{1}{2}\sum_{m,n} (u_{ijmn}u_{mnkl} + u_{ikmn}u_{mnjl} + u_{ilmn}u_{mnjk}) \tag{9}$$

to one-loop order. Michel has shown recently[26] that the functions β_v of Eq. (6) can be written in a condensed form by using two mathematical operations. If we let $P_4^{(1)}$ and $P_4^{(2)}$ be two polynomials of the form of Eq. (2) then the scalar product is defined by

$$(P_4^{(1)}, P_4^{(2)}) = \sum_{i,j,k,l} u_{ijkl}^{(1)} u_{ijkl}^{(2)} , \tag{10}$$

and the symmetric product is defined by

$$P_4^{(1)} \wedge P_4^{(2)} = P_4^{(3)} , \tag{11}$$

where

$$P_4^{(3)} = \frac{1}{144}\left[\sum_{i,j} \frac{\partial^2 P_4^{(1)}}{\partial\phi_i\partial\phi_j}\frac{\partial^2 P_4^{(2)}}{\partial\phi_i\partial\phi_j}\right] . \tag{12}$$

TABLE VII. Scalar and symmetric products of fourth-degree basic invariants.

Basic invariants	Scalar product	Symmetric product
I_0^2,I_0^2	16	$\frac{14}{9}I_0^2$
$I_0^2,I_1^{(4)}$	6	$\frac{1}{3}I_0^2+\frac{2}{3}I_1^{(4)}$
$I_0^2,I_2^{(4)}$	1	$\frac{1}{18}I_0^2+\frac{2}{3}I_2^{(4)}$
$I_0^2,I_3^{(4)}$	0	$\frac{2}{3}I_3^{(4)}$
$I_0^2,I_4^{(4)}$	2	$\frac{1}{9}I_0^2+\frac{2}{3}I_4^{(4)}$
$I_0^2,I_5^{(4)}$	1	$\frac{1}{18}I_0^2+\frac{2}{3}I_5^{(4)}$
$I_0^2,I_6^{(4)}$	0	$\frac{2}{3}I_6^{(4)}$
$I_0^2,I_7^{(4)}$	0	$\frac{2}{3}I_7^{(4)}$
$I_0^2,I_8^{(4)}$	0	$\frac{2}{3}I_8^{(4)}$
$I_0^2,I_9^{(4)}$	0	$\frac{2}{3}I_9^{(4)}$
$I_1^{(4)},I_1^{(4)}$	6	$I_1^{(4)}$
$I_1^{(4)},I_2^{(4)}$	0	$\frac{1}{3}I_2^{(4)}$
$I_1^{(4)},I_3^{(4)}$	0	0
$I_1^{(4)},I_4^{(4)}$	0	$\frac{1}{3}I_4^{(4)}$
$I_1^{(4)},I_5^{(4)}$	0	$\frac{1}{3}I_5^{(4)}$
$I_1^{(4)},I_6^{(4)}$	0	$\frac{1}{6}I_6^{(4)}$
$I_1^{(4)},I_7^{(4)}$	0	$\frac{1}{2}I_7^{(4)}$
$I_1^{(4)},I_8^{(4)}$	0	$\frac{1}{6}I_8^{(4)}$
$I_1^{(4)},I_9^{(4)}$	0	$\frac{1}{6}I_9^{(4)}$
$I_2^{(4)},I_2^{(4)}$	$\frac{1}{2}$	$\frac{2}{9}I_2^{(4)}+\frac{1}{36}I_1^{(4)}$
$I_2^{(4)},I_3^{(4)}$	0	$\frac{1}{9}I_3^{(4)}$
$I_2^{(4)},I_4^{(4)}$	0	$\frac{1}{36}I_0^2-\frac{1}{36}I_1^{(4)}-\frac{1}{18}I_2^{(4)}-\frac{1}{18}I_4^{(4)}$
$I_2^{(4)},I_5^{(4)}$	0	$\frac{1}{36}I_4^{(4)}$
$I_2^{(4)},I_6^{(4)}$	0	$\frac{1}{12}I_6^{(4)}$
$I_2^{(4)},I_7^{(4)}$	0	$\frac{1}{12}I_7^{(4)}$
$I_2^{(4)},I_8^{(4)}$	0	$\frac{1}{18}I_8^{(4)}+\frac{1}{36}I_9^{(4)}$
$I_2^{(4)},I_9^{(4)}$	0	$\frac{1}{36}I_8^{(4)}+\frac{1}{18}I_9^{(4)}$
$I_3^{(4)},I_3^{(4)}$	$\frac{1}{8}$	$\frac{1}{144}I_0^2-\frac{1}{144}I_1^{(4)}+\frac{1}{72}I_2^{(4)}+\frac{1}{36}I_3^{(4)}$
$I_3^{(4)},I_4^{(4)}$	0	$\frac{1}{9}I_3^{(4)}$
$I_3^{(4)},I_5^{(4)}$	0	$\frac{1}{18}I_3^{(4)}$
$I_3^{(4)},I_6^{(4)}$	0	$-\frac{5}{72}I_6^{(4)}$
$I_3^{(4)},I_7^{(4)}$	0	$\frac{1}{24}I_8^{(4)}-\frac{1}{24}I_9^{(4)}$
$I_3^{(4)},I_8^{(4)}$	0	$\frac{1}{72}I_7^{(4)}-\frac{1}{36}I_8^{(4)}-\frac{1}{24}I_9^{(4)}$
$I_3^{(4)},I_9^{(4)}$	0	$-\frac{1}{72}I_7^{(4)}-\frac{1}{24}I_8^{(4)}-\frac{1}{36}I_9^{(4)}$
$I_4^{(4)},I_4^{(4)}$	1	$\frac{1}{18}I_1^{(4)}+\frac{5}{18}I_4^{(4)}$
$I_4^{(4)},I_5^{(4)}$	0	$\frac{1}{36}I_0^2-\frac{1}{36}I_1^{(4)}-\frac{1}{18}I_4^{(4)}-\frac{1}{18}I_5^{(4)}$
$I_4^{(4)},I_6^{(4)}$	0	$\frac{1}{12}I_6^{(4)}$
$I_4^{(4)},I_7^{(4)}$	0	$\frac{1}{12}I_8^{(4)}-\frac{1}{12}I_9^{(4)}$
$I_4^{(4)},I_8^{(4)}$	0	$\frac{1}{36}I_7^{(4)}+\frac{1}{9}I_8^{(4)}-\frac{1}{36}I_9^{(4)}$
$I_4^{(4)},I_9^{(4)}$	0	$-\frac{1}{36}I_7^{(4)}-\frac{1}{36}I_8^{(4)}+\frac{1}{9}I_9^{(4)}$
$I_5^{(4)},I_5^{(4)}$	$\frac{1}{2}$	$\frac{1}{36}I_1^{(4)}+\frac{2}{9}I_5^{(4)}$
$I_5^{(4)},I_6^{(4)}$	0	$\frac{1}{36}I_7^{(4)}+\frac{1}{9}I_6^{(4)}-\frac{5}{36}I_8^{(4)}$
$I_5^{(4)},I_7^{(4)}$	0	$\frac{1}{12}I_9^{(4)}$

TABLE VII. (*Continued*).

Basic invariants	Scalar product	Symmetric product
$I_5^{(4)},I_8^{(4)}$	0	$-\frac{1}{36}I_8^{(4)}$
$I_5^{(4)},I_9^{(4)}$	0	$\frac{1}{36}I_7^{(4)}+\frac{1}{9}I_9^{(4)}$
$I_6^{(4)},I_6^{(4)}$	1	$\frac{1}{24}I_0^2-\frac{1}{72}I_1^{(4)}+\frac{1}{12}I_2^{(4)}-\frac{5}{9}I_3^{(4)}$
$I_6^{(4)},I_7^{(4)}$	0	$-\frac{1}{24}I_0^2+\frac{1}{24}I_1^{(4)}+\frac{1}{12}I_2^{(4)}+\frac{1}{12}I_4^{(4)}+\frac{1}{6}I_5^{(4)}$
$I_6^{(4)},I_8^{(4)}$	$\frac{1}{2}$	$\frac{1}{18}I_0^2-\frac{1}{24}I_1^{(4)}-\frac{1}{36}I_2^{(4)}-\frac{5}{72}I_4^{(4)}-\frac{5}{36}I_5^{(4)}-\frac{5}{18}I_3^{(4)}$
$I_6^{(4)},I_9^{(4)}$	$\frac{1}{2}$	$-\frac{1}{72}I_0^2+\frac{1}{36}I_1^{(4)}+\frac{1}{9}I_2^{(4)}-\frac{5}{18}I_3^{(4)}+\frac{5}{72}I_4^{(4)}+\frac{5}{36}I_5^{(4)}$
$I_7^{(4)},I_7^{(4)}$	$\frac{3}{2}$	$\frac{1}{8}I_1^{(4)}+\frac{1}{4}I_2^{(4)}$
$I_7^{(4)},I_8^{(4)}$	0	$-\frac{1}{24}I_0^2+\frac{1}{24}I_1^{(4)}+\frac{1}{12}I_2^{(4)}+\frac{1}{6}I_3^{(4)}+\frac{1}{8}I_4^{(4)}+\frac{1}{12}I_5^{(4)}$
$I_7^{(4)},I_9^{(4)}$	0	$-\frac{1}{6}I_3^{(4)}-\frac{1}{24}I_4^{(4)}+\frac{1}{12}I_5^{(4)}$
$I_8^{(4)},I_8^{(4)}$	$\frac{1}{2}$	$\frac{1}{18}I_0^2-\frac{1}{24}I_1^{(4)}-\frac{1}{18}I_2^{(4)}-\frac{1}{18}I_4^{(4)}-\frac{10}{72}I_5^{(4)}-\frac{1}{9}I_3^{(4)}$
$I_8^{(4)},I_9^{(4)}$	0	$\frac{1}{36}I_2^{(4)}-\frac{1}{6}I_3^{(4)}-\frac{1}{72}I_4^{(4)}$
$I_9^{(4)},I_9^{(4)}$	$\frac{1}{2}$	$-\frac{1}{72}I_0^2+\frac{1}{36}I_1^{(4)}+\frac{1}{12}I_2^{(4)}+\frac{1}{12}I_4^{(4)}-\frac{1}{9}I_3^{(4)}+\frac{10}{72}I_5^{(4)}$

The operation $(P_4^{(1)},P_4^{(2)})$ has the properties of an $O(n)$ invariant scalar product in the polynomial space, and $P_4^{(1)}\wedge P_4^{(2)}$ defines a symmetric nonassociative algebra. (See Ref. 26 for more detail.) In terms of the symmetric product, Eq. (9) can be written in the compact form

$$\beta_\nu(u)=-\epsilon u_\nu+\tfrac{3}{2}(u\wedge u)_\nu\,. \tag{13}$$

Because of the linear dependence of $P_4(\boldsymbol{\phi})$ on $I_\nu(\boldsymbol{\phi})$, any scalar product in Eq. (10) or symmetric product in Eq. (11) can be obtained from the products of the basic invariants, i.e., (I_μ,I_ν) or $I_\mu\wedge I_\nu$. We have tabulated these products in Table VII for all pairs of basic invariants occurring in the six densities under consideration here.

The following general results have been obtained[6,7,9] for

TABLE VIII. Nonlinear contributions to the recursion relations of Eq. (6).

Density	u_ν	$\frac{3}{2}(u\wedge u)_\nu$
H_1	u_0	$\frac{7}{3}u_0^2+u_0u_1+u_0u_2/6$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+u_2^2/24$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2$
H_2	u_0	$\frac{7}{3}u_0^2+u_0u_1+u_0u_2/6+u_3^2/96$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+u_2^2/24-u_3^2/96$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2+u_3^2/48$
	u_3	$2u_0u_3+u_2u_3/3+u_3^2/24$
H_3	u_0	$\frac{7}{3}u_0^2+u_0u_1+\frac{1}{6}u_0u_2+u_3^2/96+\frac{1}{16}u_6^2$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+\frac{1}{24}u_2^2-\frac{1}{96}u_3^2-\frac{1}{48}u_6^2$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2+\frac{1}{48}u_3^2+\frac{1}{8}u_6^2$
	u_3	$2u_0u_3+u_2u_3/3+\frac{1}{24}u_3^2-\frac{5}{6}u_6^2$
	u_6	$2u_0u_6+\frac{1}{2}u_1u_6+\frac{1}{4}u_2u_6-\frac{5}{24}u_3u_6$
H_4	u_0	$\frac{7}{3}u_0^2+u_0u_1+\frac{1}{6}u_0u_2+\frac{1}{3}u_0u_4+\frac{1}{6}u_0u_5+\frac{1}{12}u_2u_4+\frac{1}{96}u_3^2+\frac{1}{12}u_4u_5$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+\frac{1}{24}u_2^2-\frac{1}{12}u_2u_4-\frac{1}{96}u_3^2+\frac{1}{12}u_4^2-\frac{1}{12}u_4u_5+\frac{1}{24}u_5^2$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2-\frac{1}{6}u_2u_4+\frac{1}{48}u_3^2$
	u_3	$2u_0u_3+\frac{1}{3}u_2u_3+\frac{1}{24}u_3^2+\frac{1}{3}u_3u_4+\frac{1}{6}u_3u_5$
	u_4	$2u_0u_4+u_1u_4-\frac{1}{6}u_2u_4+\frac{1}{12}u_2u_5+\frac{5}{12}u_4^2-\frac{1}{6}u_4u_5$
	u_5	$2u_0u_5+u_1u_5-\frac{1}{6}u_4u_5+\frac{1}{3}u_5^2$
H_5	u_0	$\frac{7}{3}u_0^2+u_0u_1+\frac{1}{6}u_0u_2+\frac{1}{3}u_0u_4+\frac{1}{6}u_0u_5+\frac{1}{12}u_2u_4+\frac{1}{96}u_3^2+\frac{1}{12}u_4u_5$ $-\frac{1}{8}u_7u_8+\frac{1}{12}u_8^2-\frac{1}{48}u_9^2$

TABLE VIII. (*Continued*).

Density	u_ν	$\frac{3}{2}(u\wedge u)_\nu$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+\frac{1}{24}u_2^2-\frac{1}{12}u_2u_4-\frac{1}{96}u_3^2+\frac{1}{12}u_4^2-\frac{1}{12}u_4u_5+\frac{1}{24}u_5^2$ $+\frac{3}{16}u_7^2+\frac{1}{8}u_7u_8-\frac{1}{16}u_8^2+\frac{1}{24}u_9^2$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2-\frac{1}{6}u_2u_4+\frac{1}{48}u_3^2+\frac{3}{8}u_7^2+\frac{1}{4}u_7u_8-\frac{1}{12}u_8^2$ $+\frac{1}{12}u_8u_9+\frac{1}{8}u_9^2$
	u_3	$2u_0u_3+\frac{1}{3}u_2u_3+\frac{1}{24}u_3^2+\frac{1}{3}u_3u_4+\frac{1}{6}u_3u_5+\frac{1}{2}u_7u_8-\frac{1}{2}u_7u_9-\frac{1}{6}u_8^2$ $-\frac{1}{2}u_8u_9-\frac{1}{6}u_9^2$
	u_4	$2u_0u_4+u_1u_4-\frac{1}{6}u_2u_4+\frac{1}{12}u_2u_5+\frac{5}{12}u_4^2-\frac{1}{6}u_4u_5+\frac{3}{8}u_7u_8-\frac{1}{8}u_7u_9$ $-\frac{1}{12}u_8^2-\frac{1}{24}u_8u_9+\frac{1}{8}u_9^2$
	u_5	$2u_0u_5+u_1u_5-\frac{1}{6}u_4u_5+\frac{1}{3}u_5^2+\frac{1}{4}u_7u_8+\frac{1}{4}u_7u_9-\frac{5}{24}u_8^2+\frac{5}{24}u_9^2$
	u_7	$2u_0u_7+\frac{3}{2}u_1u_7+\frac{1}{4}u_2u_7+\frac{1}{24}u_3u_8-\frac{1}{24}u_3u_9$ $+\frac{1}{12}u_4u_8-\frac{1}{12}u_4u_9+\frac{1}{12}u_5u_9$
	u_8	$2u_0u_8+\frac{1}{2}u_1u_8+\frac{1}{6}u_2u_8+\frac{1}{12}u_2u_9+\frac{1}{8}u_3u_7-\frac{1}{12}u_3u_8-\frac{1}{8}u_3u_9$ $+\frac{1}{4}u_4u_7+\frac{1}{3}u_4u_8-\frac{1}{12}u_4u_9-\frac{1}{12}u_5u_8$
	u_9	$2u_0u_9+\frac{1}{2}u_1u_9+\frac{1}{12}u_2u_8+\frac{1}{6}u_2u_9-\frac{1}{8}u_3u_7-\frac{1}{8}u_3u_8-\frac{1}{12}u_3u_9$ $-\frac{1}{4}u_4u_7-\frac{1}{12}u_4u_8+\frac{1}{3}u_4u_9+\frac{1}{4}u_5u_7+\frac{1}{3}u_5u_9$
H_6	u_0	$\frac{7}{3}u_0^2+u_0u_1+\frac{1}{6}u_0u_2$
	u_1	$2u_0u_1+\frac{3}{2}u_1^2+\frac{1}{24}u_2^2+\frac{3}{16}u_7^2$
	u_2	$2u_0u_2+u_1u_2+\frac{1}{3}u_2^2+\frac{3}{8}u_7^2$
	u_7	$2u_0u_7+\frac{3}{2}u_1u_7+\frac{1}{4}u_2u_7$

the flow of the coefficients u_ν.

(i) For $n\geq 5$ the isotropic fixed point is unstable for any density with anisotropic quartic invariants (every density of our consideration here).

(ii) There is at most one stable fixed point for each H_i, and all other fixed points are on the boundary of its attraction basin.

Using the symmetric products of the basic invariants, we have obtained the recursion relations [Eq. (6)] for each density $H_1, H_2, \ldots, H_6$. In Table VIII, column 3, we list $\frac{3}{2}(u\wedge u)_\nu$, the second term of the right-hand side of Eq. (13). We solved the RG fixed point equations, Eq. (7), numerically.[27] In Tables IX and X we list the fixed points of Eq. (6) and the eigenvalues of the matrix of Eq. (8) at each fixed point. In Table IX we also list the eigendirection for each eigenvalue obtained from H_1. We have indicated by an asterisk (*) a stable fixed point. Notice only two densities allow stable fixed points. Both stable fixed points have the same location in the space of P_4. *Thus only H_1 and H_6 allow continuous transitions and they both exhibit the same critical phenomena.*

These results are, in general, consistent with results obtained by other authors for the case $n=6$. Jarić and Birman[28] applied RG methods to the $R(4)$, $X(3)$, and $X(4)$ irreps of O_h^3. These irreps correspond to the Hamiltonian density H_4. They found no stable fixed point, which is consistent with our results. Toledano and Meimarkis[29] considered the density arising from several W point irreps of O^3. This density corresponds to our H_1. They found a stable fixed point as we did. Using the symmetry criterion of Ref. 9 we studied[10] the fixed points of the densities $H_1, \ldots, H_6$. The results we obtained there are consistent for all densities with the ϵ-expansion results obtained here. Independent of our work, Meimarkis and

TABLE IX. Fixed points (F) in terms of (u_0,u_1,u_2), eigenvalues, and eigendirections for the Hamiltonian density H_1. Stable fixed points are indicated by an asterisk. See Table X for additional densities.

Fixed points (u_0,u_1,u_2)	Eigenvalues	Eigendirections
$F1$: $\epsilon(0,0,0)$	$-\epsilon(1,1,1)$	$\pm(1,0,0)$
		$\pm(0,1,0)$
		$\pm(0,0,1)$
$F2$: $\epsilon(0,\frac{2}{3},0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3})$	$\pm(0,1,0)$
		$\pm(0,0,1)$
		$\pm(1,-1,0)$
$F3$: $\epsilon(\frac{1}{3},\frac{2}{9},0)$	$-\epsilon(-1,\frac{1}{9},-\frac{1}{9})$	$\pm(3,2,0)$
		$\pm(-1,1,10)$
		$\pm(1,-2,0)$
$F4$: $\epsilon(\frac{3}{7},0,0)$	$-\epsilon(-1,\frac{1}{7},\frac{1}{7})$	$\pm(1,0,0)$
		$\pm(-1,0,16)$
		$\pm(-3,8,0)$
$F5$: $\epsilon(0,\frac{3}{5},\frac{6}{5})$	$-\epsilon(-1,\frac{1}{5},-\frac{1}{5})$	$\pm(0,1,2)$
		$\pm(1,-1,-2)$
		$\pm(0,-1,6)$
$F6$: $\epsilon(0,\frac{1}{3},2)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3})$	$\pm(0,1,6)$
		$\pm(-1,0,4)$
		$\pm(1,-2,0)$
$F7$: $\epsilon(\frac{1}{3},\frac{1}{9},\frac{2}{3})$	$-\epsilon(-1,\frac{1}{9},-\frac{1}{9})$	$\pm(3,1,6)$
		$\pm(1,-3,2)$
		$\pm(-1,1,6)$
$F8$: $\epsilon(\frac{3}{11},\frac{3}{11},\frac{6}{11})$*	$-\epsilon(-1,-\frac{1}{11},-\frac{1}{11})$	$\pm(1,1,2)$
		$\pm(1,0,0)$
		$\pm(0,1,0)$

Toledano[30] performed RG analysis of structural and magnetic transitions for $n=6$. They obtained the same densities but they do not allow H_6 as a possible density for structural transitions. Their RG results, however, do agree with ours.

By using the scalar products of basic invariants listed in Table VII, we obtained the length $(u^*,u^*)=\frac{522}{121}$ for the stable fixed point. It is easily seen that this length is greater than those of the other fixed points. This is consistent with the general result stated in Ref. 26.

TABLE X. Fixed points and eigenvalues for four of the six Hamiltonian densities. Stable fixed points are indicated by an asterisk. See Table IX for density H_1.

Density	Fixed points	Eigenvalues
H_2	(u_0,u_1,u_2,u_3)	
	$\epsilon(0,0,0,0)$	$-\epsilon(1,1,1,1)$
	$\epsilon(0,\frac{2}{3},0,0)$	$-\epsilon(1,-1,\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{2}{9},0,0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{7},0,0,0)$	$-\epsilon(-1,\frac{1}{7},\frac{1}{7},\frac{1}{7})$
	$\epsilon(0,\frac{3}{5},\frac{6}{5},0)$	$-\epsilon(-1,\frac{3}{5},\frac{1}{5},-\frac{1}{5})$
	$\epsilon(0,\frac{1}{3},2,0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{1}{9},\frac{2}{3},0)$	$-\epsilon(-1,\frac{1}{9},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{11},\frac{3}{11},\frac{6}{11},0)$	$-\epsilon(-1,\frac{3}{11},-\frac{1}{11},-\frac{1}{11})$
	$\epsilon(\frac{6}{17},0,\frac{12}{17},\frac{24}{17})$	$-\epsilon(-1,\frac{7}{17},-\frac{1}{17},-\frac{1}{17})$
	$\epsilon(\frac{3}{11},0,\frac{12}{11},\frac{24}{11})$	$-\epsilon(-1,\frac{7}{11},\frac{1}{11},-\frac{1}{11})$
	$\epsilon(\frac{1}{3},-\frac{1}{9},\frac{2}{3},\frac{8}{3})$	$-\epsilon(-1,\frac{1}{9},\frac{7}{9},\frac{1}{9})$
	$\epsilon(\frac{2}{5},-\frac{1}{15},\frac{2}{5},\frac{8}{5})$	$-\epsilon(-1,\frac{7}{15},\frac{1}{15},-\frac{1}{15})$
H_3	(u_0,u_1,u_2,u_3,u_6)	
	$\epsilon(0,0,0,0,0)$	$-\epsilon(1,1,1,1,1)$
	$\epsilon(0,\frac{2}{3},0,0,0)$	$-\epsilon(1,-1,\frac{2}{3},\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{2}{9},0,0,0)$	$-\epsilon(-1,\frac{1}{3},\frac{2}{9},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{7},0,0,0,0)$	$-\epsilon(-1,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$
	$\epsilon(0,\frac{3}{5},\frac{6}{5},0,0)$	$-\epsilon(-1,\frac{3}{5},\frac{2}{5},\frac{1}{5},-\frac{1}{5})$
	$\epsilon(0,\frac{1}{3},2,0,0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{1}{9},\frac{2}{3},0,0)$	$-\epsilon(-1,\frac{1}{9},\frac{1}{9},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{11},\frac{3}{11},\frac{6}{11},0,0)$	$-\epsilon(-1,\frac{3}{11},\frac{2}{11},-\frac{1}{11},-\frac{1}{11})$
	$\epsilon(\frac{6}{17},0,\frac{12}{17},\frac{24}{17},0)$	$-\epsilon(-1,\frac{7}{17},\frac{7}{17},-\frac{1}{17},-\frac{1}{17})$
	$\epsilon(\frac{3}{11},0,\frac{12}{11},\frac{24}{11,}0)$	$-\epsilon(-1,\frac{7}{11},\frac{7}{11},\frac{1}{11},-\frac{1}{11})$
	$\epsilon(\frac{1}{3},-\frac{1}{9},\frac{2}{3},\frac{8}{3},0)$	$-\epsilon(-1,\frac{7}{9},\frac{7}{9},\frac{1}{9},\frac{1}{9})$
	$\epsilon(\frac{2}{5},-\frac{1}{15},\frac{2}{5},\frac{8}{5},0)$	$-\epsilon(-1,\frac{7}{15},\frac{7}{15},\frac{1}{15},-\frac{1}{15})$
	$\epsilon(\frac{1}{4},-\frac{1}{12},\frac{1}{2}-2,1)$	$-\epsilon(1,-1,\frac{2}{3},\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{4},-\frac{1}{12},\frac{1}{2},-2,-1)$	$-\epsilon(1,-1,\frac{2}{3},\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{5}{12},-\frac{1}{36},\frac{1}{6},-\frac{2}{3},\frac{1}{3})$	$-\epsilon(-1,\frac{1}{3},\frac{2}{9},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{5}{12},-\frac{1}{36},\frac{1}{6},-\frac{2}{3},-\frac{1}{3})$	$-\epsilon(-1,\frac{1}{3},\frac{2}{9},\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{20},\frac{3}{20},\frac{3}{2},-\frac{6}{5},\frac{3}{5})$	$-\epsilon(-1,\frac{3}{5},\frac{2}{5},\frac{1}{5},-\frac{1}{5})$
	$\epsilon(\frac{3}{20},\frac{3}{20},\frac{3}{2},-\frac{6}{5},-\frac{3}{5})$	$-\epsilon(-1,\frac{3}{5},\frac{2}{5},\frac{1}{5},-\frac{1}{5})$
	$\epsilon(\frac{15}{44},\frac{3}{44},\frac{15}{22},-\frac{12}{22},\frac{3}{11})$	$-\epsilon(-1,\frac{3}{11},\frac{2}{11},-\frac{1}{11},-\frac{1}{11})$
	$\epsilon(\frac{15}{44},\frac{3}{44},\frac{15}{22},-\frac{12}{22},-\frac{3}{11})$	$-\epsilon(-1,\frac{3}{11},\frac{2}{11},-\frac{1}{11},-\frac{1}{11})$
H_4	$(u_0,u_1,u_2,u_3,u_4,u_5)$	
	$\epsilon(0,0,0,0,0,0)$	$-\epsilon(1,1,1,1,1,1)$
	$\epsilon(\frac{3}{7},0,0,0,0,0)$	$-\epsilon(-1,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$

TABLE X. (*Continued*).

Density	Fixed points	Eigenvalues
	$\epsilon(0,\frac{2}{3},0,0,0,0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},1,\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{2}{9},0,0,0,0)$	$-\epsilon(-1,-\frac{1}{9},\frac{1}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9})$
	$\epsilon(0,\frac{1}{3},0,0,0,2)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},1,\frac{2}{3},\frac{2}{3})$
	$\epsilon(0,\frac{1}{3},2,0,0,0)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3},1,\frac{2}{3})$
	$\epsilon(0,\frac{3}{5},\frac{6}{5},0,0,0)$	$-\epsilon(-\frac{1}{5},-1,\frac{1}{5},\frac{3}{5},\frac{3}{5},\frac{2}{5})$
	$\epsilon(0,\frac{6}{11},0,0,\frac{12}{11},0)$	$-\epsilon(-\frac{1}{11},-1,-\frac{1}{11},\frac{7}{11},\frac{7}{11},\frac{7}{11})$
	$\epsilon(0,\frac{4}{9},0,0,\frac{4}{3},0)$	$-\epsilon(-1,\frac{1}{9},\frac{1}{9},\frac{7}{9},\frac{5}{9},\frac{7}{9})$
	$\epsilon(\frac{3}{11},\frac{3}{11},\frac{6}{11},0,0,0)$	$-\epsilon(-1,-\frac{1}{11},-\frac{1}{11},\frac{3}{11},\frac{3}{11},\frac{2}{11})$
	$\epsilon(0,\frac{3}{5},0,0,0,\frac{6}{5})$	$-\epsilon(-\frac{1}{5},-1,\frac{1}{5},\frac{3}{5},\frac{2}{5},\frac{4}{5})$
	$\epsilon(\frac{1}{3},\frac{1}{9},\frac{2}{3},0,0,0)$	$-\epsilon(-1,-\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{3},\frac{2}{9})$
	$\epsilon(\frac{3}{17},\frac{6}{17},0,0,\frac{12}{17},0)$	$-\epsilon(-\frac{1}{17},-1,-\frac{1}{17},\frac{7}{17},\frac{7}{17},\frac{7}{17})$
	$\epsilon(\frac{1}{5},\frac{4}{15},0,0\frac{4}{5},0)$	$-\epsilon(-1,-\frac{1}{15},\frac{1}{15},\frac{7}{15},\frac{1}{3},\frac{7}{15})$
	$\epsilon(\frac{3}{11},\frac{3}{11},0,0,0,\frac{6}{11})$	$-\epsilon(-1,-\frac{1}{11},-\frac{1}{11},\frac{3}{11},\frac{2}{11},\frac{4}{11})$
	$\epsilon(\frac{1}{3},\frac{1}{9},0,0,0,\frac{2}{3})$	$-\epsilon(-1,-\frac{1}{9},\frac{1}{9},\frac{1}{3},\frac{2}{9},\frac{2}{9})$
	$\epsilon(\frac{6}{17},0,\frac{12}{17},\frac{24}{17},0,0)$	$-\epsilon(-1,-\frac{1}{17},\frac{7}{17},-\frac{1}{17},\frac{7}{17},\frac{5}{17})$
	$\epsilon(\frac{3}{11},0,\frac{12}{11},\frac{24}{11},0,0)$	$-\epsilon(\frac{1}{11},-1,\frac{7}{11},-\frac{1}{11},\frac{7}{11},\frac{5}{11})$
	$\epsilon(\frac{1}{3},-\frac{1}{9},\frac{2}{3},\frac{8}{3},0,0)$	$-\epsilon(-1,\frac{7}{9},\frac{1}{9},\frac{1}{9},\frac{5}{9},\frac{4}{9})$
	$\epsilon(\frac{2}{5},-\frac{1}{15},\frac{2}{5},\frac{8}{5},0,0)$	$-\epsilon(-1,-\frac{1}{15},\frac{7}{15},\frac{1}{15},\frac{1}{3},\frac{4}{15})$
	$\epsilon(\frac{6}{11},0,-\frac{6}{11},0,-\frac{6}{11},-\frac{6}{11})$	$-\epsilon(-1,-\frac{1}{11},\frac{3}{11},\frac{2}{11},-\frac{1}{11},\frac{4}{11})$
	$\epsilon(\frac{3}{5},0,-\frac{6}{5},0,-\frac{6}{5},-\frac{6}{5})$	$-\epsilon(\frac{1}{5},\frac{3}{5},\frac{2}{5},-1,-\frac{1}{5},\frac{4}{5})$
	$\epsilon(\frac{2}{3},-\frac{2}{9},-\frac{2}{3},0,-\frac{2}{3},-\frac{2}{3})$	$-\epsilon(-1,-\frac{1}{9},\frac{1}{9},\frac{1}{3},\frac{2}{9},\frac{2}{9})$
	$\epsilon(1,-\frac{2}{3},-2,0,-2,-2)$	$-\epsilon(-1,\frac{1}{3},\frac{1}{3},1,\frac{2}{3},\frac{2}{3})$
H_6	(u_0,u_1,u_2,u_7)	
	$\epsilon(0,0,0,0)$	$-\epsilon(1,1,1,1)$
	$\epsilon(0,\frac{2}{3},0,0)$	$-\epsilon(0,-1,\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{2}{9},0,0)$	$-\epsilon(0,-1,\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{7},0,0,0)$	$-\epsilon(-1,\frac{1}{7},\frac{1}{7},\frac{1}{7})$
	$\epsilon(0,\frac{3}{5},\frac{6}{5},0)$	$-\epsilon(-1,\frac{1}{5},-\frac{1}{5},-\frac{1}{5})$
	$\epsilon(0,\frac{1}{3},2,0)$	$-\epsilon(0,-1,\frac{1}{3},\frac{1}{3})$
	$\epsilon(\frac{1}{3},\frac{1}{9},\frac{2}{3},0)$	$-\epsilon(0,-1,\frac{1}{9},-\frac{1}{9})$
	$\epsilon(\frac{3}{11},\frac{3}{11},\frac{6}{11},0)^*$	$-\epsilon(-1,-\frac{1}{11},-\frac{1}{11},-\frac{1}{11})$

IV. ATTRACTION DOMAIN AND ASSOCIATED STRUCTURAL TRANSITIONS

We now wish to consider in more detail the physical system corresponding to the density H_1. A sample of flow trajectories are depicted in Fig. 3. The attraction domain of the stable fixed point is found to be

$$u_0>0,\ \ u_2>0,\ \ \text{and}\ \ u_2-6u_1<0\ . \tag{14}$$

One can find a candidate for the boundary of the attraction domain by setting one (or a linear combination) of the β_v's equal to zero. Notice that the stable fixed point for H_1 is located *on the phase boundary* between phases ($P6$,$P7$) and ($P11$,$P12$) (see Table IX and Fig. 2). In Fig. 4 we show the attraction domain. The directions of flow near each fixed point are shown and these eigendirections are also listed in Table IX. The unstable fixed points are located on the boundary of the attraction domain as was pointed out in Ref. 26.

Fixed points with the one unstable direction r [see Eq. (5)] describe physical systems with continuous transitions. For $T>T_c$, RG flow leads to an effective Landau potential whose minimum is given by $\phi=0$, since $r>0$. For $T<T_c$ RG flow leads again to an effective Landau potential whose minimum is now given by $\phi\neq0$. When $T=T_c$ the system is on the critical surface. For H_1 this is the

three-dimensional "surface" of u_0, u_1, and u_2. If the system intersects the critical surface within the attraction domain the transition is continuous. Depending upon where the physical system intersects in the attraction domain the corresponding lower phase will be determined.

For $2u_1-u_2<0$ phases ($P6,P7$) are selected, while if $2u_1-u_2>0$ phases ($P11,P12$) are selected. When the parameters u_ν are in the attraction domain, the RG flow leads to the stable fixed point $(3\epsilon/11, 3\epsilon/11, 6\epsilon/11)$ and the appropriate critical exponents describe the critical phenomena. For any point of the attraction domain the same critical exponents are obtained. Points outside of the attraction domain, but in the Landau continuous transition region, will yield fluctuation-driven first-order transitions. Indeed, all other regions of the (u_1,u_2) plane (i.e., outside of the region $u_2>0$, $u_2-6u_1<0$) correspond to continuous transitions in Landau theory and will be fluctuation-driven first-order transitions.

A similar description of the systems corresponding to H_6 can be given. The attraction domain is a four-dimensional region symmetrical above and below the $u_7=0$ surface. The attraction domain also extends symmetrically above and below $u_7=0$. Comparison with Tables V and VI indicate that continuous transitions to phases $C23$ and $C1$ are possible. Notice, for example, that the stable fixed point $(3\epsilon/11, 3\epsilon/11, 6\epsilon/11, 0)$ corresponds to $\tan\theta=u_7/u_2=0$, $\tan\xi=|B/u_1| = |u_2/u_1| = \frac{6}{3}=2$, and $B>0$. Since $\xi\approx 63°\ 27'$ the stable fixed point is in the stability region of the phase $C23$.

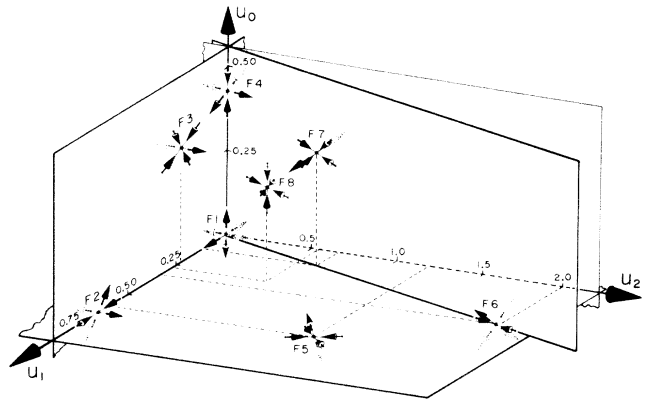

FIG. 4. Fixed points and attraction domain for the Hamiltonian density H_1. The region defined by $u_0>0$, $u_2>0$, and $u_2-6u_1<0$ forms the attraction domain. The directions of flow near each fixed point are shown. These eigendirections are given in Table IX. RG flow, given by Eq. (6), attracts any point initially within the attraction domain to the stable fixed point $F8$ while any point initially outside of the attraction domain is driven away to infinity.

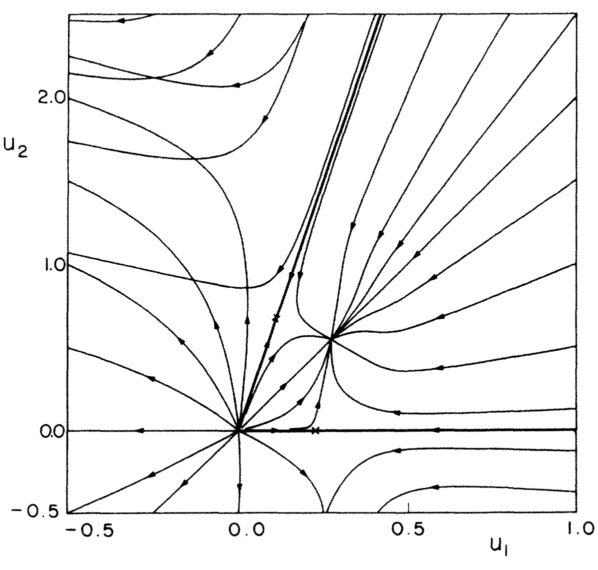

FIG. 3. A sample of the flows of quartic coefficients (u_0,u_1,u_2) determined by Eq. (6). We show a projection along the u_0 axis. Our sample is chosen in such a way that the three-dimensional curves either originate from or flow through a point of the $u_0=\frac{3}{11}$ plane. For example, curves ending at the stable fixed point originate from the plane $u_0=\frac{3}{11}$ and the ones which originate from the origin (0,0,0) end at the plane $u_0=\frac{3}{11}$. All curves (including those not shown in the figure) originating from a point within the attraction domain flow towards the stable fixed point while all others originating from outside the domain veer away to infinity.

As discussed in Sec. II, the phases $P6$, $P7$, $P11$, $P12$, $C23$, and $C1$ preserve the isotropy subgroups whose elements leave the corresponding order-parameter directions invariant. For example, the symmetry of $P6$ is the set of space-group elements whose matrix representatives leave $(a,0,a,0,a,0)$ invariant. The representation of a space group depends upon the details of the mapping of the space-group elements onto the representation matrices. Thus $P6$ will determine different subgroups for different space-group irreps even though the irreps have the same image. In Table XI we list subgroup symmetries which correspond to each of the phases $P6$, $P7$, $P11$, $P12$, $C23$, and $C1$. It is a list of continuous transitions allowed by both Landau theory and RG theory. If $P11$ and $P12$ are not both listed they determine equivalent subgroup symmetries. Notice that $P11$ and $P12$ are inequivalent only for the image L_2. We list the subgroups as well as the origin and lattice for each subgroup.

Our discussion has assumed, as is usually done, only isotropic gradient contributions to the Hamiltonian density. Recently, Felix and Hatch showed[31] that anisotropic gradient terms can destroy the stability of the fixed point. Such terms occur occasionally in structural systems with order parameter components of $n=2$, 3, and 6. In all cases where such terms are allowed by symmetry the stable fixed point becomes unstable.[32] Our results indicate that the transitions for images L_5 and L_{11}, determined to be continuous by the usual RG methods, will not possess stable fixed points when considering the allowed anisotropic gradient terms. Such anisotropic terms are allowed every time these images occur. Thus, even though listed here, the L_5 and L_{11} transitions are not expected to be continuous within the more general RG methods.

V. CRITICAL EXPONENTS

Using well-known scaling laws[33] the values of all the critical exponents can be obtained from the knowledge of just two exponents. Brézin *et al.*[6] have given general expressions for the exponent η of the pair correlation and for the exponent ν of the correlation length. The expressions take the form

$$\eta=\tfrac{1}{24}(1+\tfrac{3}{4}\epsilon)[\epsilon(1-\epsilon/2)U^{*}-(U^{*})^{2}/2] \times(1+2\epsilon/3+5U^{*}/12) \quad (15)$$

TABLE XI. Lower-symmetry phases (subgroups) which may arise from continuous transitions according to RG analysis. All subgroups arise from the Hamiltonian densities H_1 and H_6. The origin and lattice of each subgroup is given. The transitions listed for L_5 and L_{11} are not expected to be continuous due to allowed anisotropic gradient terms.

Image	Space group	Irrep	ϕ	Subgroup	New primitive axes	New origin
L_1	O_h^5	W_1	$P6$	O_h^3	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O_h^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^7	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	O_h^5	W_2	$P6$	O_h^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O_h^3	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^6	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	O_h^5	W_3	$P6$	O_h^2	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O_h^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^6	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	O_h^5	W_4	$P6$	O_h^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O_h^2	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^7	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
L_2	O^3	W_1	$P6$	O^2	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^7	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
			$P12$	O^6	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{11}{4},\frac{3}{4},\frac{7}{4})$
	O^3	W_2	$P6$	O^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	O^2	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	O^6	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
			$P12$	O^7	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{11}{4},\frac{3}{4},\frac{7}{4})$
L_3	T_d^2	W_1	$P6$	T_d^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	T_d^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	T^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	T_d^2	W_2	$P6$	T_d^1	$(-2,2,2),(2,-2,2)(2,2,-2)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
			$P7$	T_d^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
			$P11$	T^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	T_d^2	W_3	$P6$	T_d^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
			$P7$	T_d^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{3}{4},\frac{3}{4},\frac{3}{4})$
			$P11$	T^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)
	T_d^2	W_4	$P6$	T_d^1	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
			$P7$	T_d^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	$(\frac{3}{4},\frac{3}{4},\frac{3}{4})$
			$P11$	T^4	$(-2,2,2),(2,-2,2),(2,2,-2)$	(0,0,0)

TABLE XI. (*Continued*).

Image	Space group	Irrep	ϕ	Subgroup	New primitive axes	New origin
L_5	O^8	N_2	$P6$	D_3^7	(2,0,0),0,2,0),(0,0,2)	$(\frac{3}{4},\frac{3}{4},\frac{3}{4})$
			$P7$	C_3^4	$(1,1,-1),(-1,1,1),(1,-1,1)$	(0,0,0)
	O^8	N_4	$P6$	D_3^7	(2,0,0),(0,2,0),(0,0,2)	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
			$P7$	C_3^4	$(1,1,-1),(-1,1,1),(1,-1,1)$	(0,0,0)
L_{11}	O^6	$M_2 \oplus M_3$	$C1$	D_4^2	$(0,1,1),(0,-1,1),(1,0,0)$	$(\frac{3}{8},\frac{1}{2},\frac{1}{4})$
			$C23$	C_3^4	$(-1,1,1),(1,-1,1),(1,1,-1)$	(0,0,0)
	O^7	$M_2 \oplus M_3$	$C1$	D_4^2	$(0,1,1),(0,-1,1),(1,0,0)$	$(\frac{1}{8},0,\frac{1}{4})$
			$C23$	C_3^4	$(-1,1,1),(1,-1,1),(1,1,-1)$	(0,0,0)

and

$$(1/\nu-2) = -\frac{U^*}{2}(1+\epsilon/2)+\tfrac{5}{24}[\epsilon U^* - (U^*)^2/2] \ , \qquad (16)$$

with

$$U^*\delta_{kl} = \sum_i u^*_{iikl} \ . \qquad (17)$$

Here the asterisk denotes the stable fixed point values of the u_{ijkl}. For the stable fixed point $(3\epsilon/11, 3\epsilon/11, 6\epsilon/11)$ of H_1 or $(3\epsilon/11, 3\epsilon/11, 6\epsilon/11, 0)$ of H_6 we obtain $U^* = \frac{12}{11}\epsilon$. Using Eqs. (15) and (16), $\eta = \frac{5}{2}(\epsilon/11)^2$ and $\nu = \frac{1}{2}(1+\frac{3}{11}\epsilon)$. Notice that we obtain the same critical exponents for both densities and thus the same critical phenomena.

VI. CONCLUSION

We have given a complete discussion, using RG methods, of the critical properties for structural phase transitions induced by six-component order parameters. The discussion is as self-contained as practical and includes lists of space-group irreps, basic invariants, symmetric products of invariants, recursion relations, fixed points, stable fixed points, attraction domains, and critical exponents. The final list, Table XI, will be of particular interest to experimentalists since it gives specific transitions and their critical exponents (which we obtained in Sec. V) for physical systems. As far as we know this is the first published list of specific transitions which are allowed to be continuous by RG theory for $n=6$.

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