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Real-time dissipation of optical pulses in passive dielectrics

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We discuss the inevitable dissipation of energy that must accompany the creation of a pulse-medium excitation state in a passive dielectric. We show that there is a minimum amount of energy that an optical pulse must deposit in the medium to create a given pulse-medium state and that energy deposited beyond this minimum value must be dissipated in the medium. We compare this notion of dissipation to a related concept found by determining the fraction of energy stored in a medium that is irrecoverable by future fields. These two notions of dissipation are model-independent and form upper and lower bounds for real-time loss. Any model-dependent notion of loss that falls outside these bounds has serious conceptual difficulties. We show that a traditional notion of real-time loss based on a multiple-Lorentz oscillator model fails to give reasonable results in classes of passive linear media near EIT, while the notions of loss we introduce give sensible results for all passive media.

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I. INTRODUCTION

As an optical pulse propagates in a dielectric, energy from the field is continuously transferred into the medium while some of the previously deposited energy is returned to the field. The dynamics of this energy exchange controls the temporal reshaping of a pulse [1]. A delay between depositing energy and returning it to the field can result in "slow" propagation of the field envelope through the medium [2]. When there is very little energy returned to the field at later times, the location of the field envelope can travel "fast" as the early portions of the pulse deposit a smaller fraction of the field energy in the medium than the later parts [1,3–7].

The methods for choosing a pulse-medium combination to display a certain pulse-reshaping behavior (slow light, fast light, etc.) are well developed. Typically, it is most convenient to use the group delay function to predict overall behavior. While this method focuses on the spectral representation of the pulse and medium, the analysis implicitly contains information about the temporal energy shuffling that occurs as they interact. In this paper, we study this temporal energy exchange more directly.

To simplify our analysis, we focus on passive media where the medium never supplies more energy to the field than it has previously received from it. In this type of medium, some of the energy transferred into the medium is dissipated and cannot be returned to the field. This "stuck" energy can no longer participate in the energy exchange process. It is informative to consider pulse reshaping in a framework where the energy in the medium is separated into dissipated and undissipated portions at each time as the medium experiences the pulse.

In a previous paper [8], we detailed a formalism that describes the amount of previously deposited energy that is irrecoverable from the medium by any future field. We denote this fraction of the energy density as the irrecoverable energy. This irrecoverable energy includes energy that has completed the absorption process as well as energy that could be classified as kinetic or potential energy. Our previous analysis of the pulse-medium system is future looking in the sense that the past history of a field is considered a given, and one determines the fraction of the energy density stored in the medium that could be extracted by an appropriately chosen future field and the fraction that is irretrievably lodged in the medium. It turns out that the fraction of energy that remains recoverable is a function only of the current pulse-medium state, irrespective of the field history.

In this paper we consider a complimentary question: what is the most energy-efficient way to create a given pulse-medium state? This notion is past looking in the sense that the current state of the pulse-medium excitation is considered fixed, and one looks at various past fields that could create that state and finds the field history that does so with the least amount of energy density being deposited in the medium. Any energy deposited in the medium beyond this minimum value is "wasted" in creating the field-medium state in a nonoptimal fashion and can be unambiguously classified as loss. We designate this fraction as the waste energy.

In general, it is not possible to unambiguously identify a single value as *the* correct amount of dissipation up to a given time, since many microscopic models give different estimations of loss while describing the same macroscopic behavior. Rather, the irrecoverable energy described in our previous paper and the waste energy introduced in this paper describe upper and lower bounds on dissipation up to a given time [9–15]. This range is natural, since energy density does not switch from a state of coherent interaction between a pulse and a medium to random thermal energy of the medium at a precise instant.

We show that any notion of real-time dissipation that does not fall between the bounds provided by the waste energy

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and the irrecoverable energy leads to serious conceptual difficulties. As an example of a model-specific notion of dissipation, we discuss a framework introduced by Barash and Ginzburg [16] that is specific to a multiple-Lorentz-oscillator model of a medium. While the Barash and Ginzburg notion of loss often leads to reasonable results (within the bounds provided by the irrecoverable and waste energies), in Sec. V we discuss a type of medium for which it falls outside these bounds and fails to provide sensible results.

II. BACKGROUND

We restrict our analysis to passive, homogeneous, isotropic linear dielectrics (nonmagnetic) without spatial dispersion and use the Lorentz-Heaviside system of units. In this setting, the pulse-medium state at a time t and a given spatial point is related to the scalar amplitudes of the electric field and the medium polarization, given by $E(t)$ and $P(t)$, respectively. These fields are connected in the frequency domain by the susceptibility $\chi(\omega)$ through

$$P(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi(\omega)\hat{E}(\omega)e^{-i\omega t}d\omega, \quad (1)$$

where $\hat{E}(\omega)$ is the Fourier transform of $E(t)$. (In our notation hats distinguish Fourier transforms of quantities.) The polarization $P(t)$ at a time t depends on $E(t)$ at other times, since this information is required to compute the spectrum $\hat{E}(\omega)$. More precisely, causality requires that only current and past values of $E(t)$ influence the current value of $P(t)$.

We can see the requirements of causality more explicitly in the temporal domain, where $P(t)$ can be determined using the impulse response function $G(t)$,

$$P(t) = \int_{-\infty}^{+\infty} G(t-\tau)E(\tau)d\tau = \int_{-\infty}^{t} G(t-\tau)E(\tau)d\tau, \quad (2)$$

where $G(t)$ is absolutely integrable and related to $\chi(\omega)$ via

$$\chi(\omega) = \int_{-\infty}^{+\infty} G(t')e^{i\omega t'}dt' = \int_{0}^{+\infty} G(t')e^{i\omega t'}dt'. \quad (3)$$

Causality is enforced by requiring $G(t')$ to be zero for $t'<0$, which results in the second forms of Eqs. (2) and (3). A straightforward analysis of Eq. (3) yields a number of properties for $\chi(\omega)$ that will be useful in our analysis [17]. First, $\chi(\omega)$ will be an analytic function of ω in the upper half of the complex plane. Second, for a passive medium (i.e., one for which $0 \le \omega\,\mathrm{Im}[\chi(\omega)]$) the reciprocal of $\chi(\omega)$ must also be analytic in the upper half-plane. This requirement ensures that E and P are mutually causal in the sense that oscillation in P cannot precede oscillations of E and oscillations of E cannot precede oscillations of P. Third, since $G(t')$ is real we have

$$\chi(-\omega) = \chi^*(\omega^*). \quad (4)$$

The notions of energy that we use arise from Poynting's energy conservation theorem. Disregarding a factor of 4π, this theorem is given by

$$\nabla\cdot\mathbf{S}(t) + \frac{\partial u(t)}{\partial t} = 0, \quad u(t) = u_{\mathrm{field}}(t) + u_{\mathrm{int}}(t), \quad (5)$$

$$u_{\mathrm{field}}(t) = \frac{1}{2}E^2(t) + \frac{1}{2}H^2(t), \quad (6)$$

$$u_{\mathrm{int}}(t) := \int_{-\infty}^{t} E(\tau)\dot{P}(\tau)d\tau. \quad (7)$$

where H is the magnetic field and the Poynting vector $\mathbf{S}(t) := \mathbf{E}(t)\times\mathbf{H}(t)$ describes energy flow. The dot notation in Eq. (7) indicates a time derivative. The total energy density $u(t)$ is composed of two parts, $u_{\mathrm{field}}(t)$ and $u_{\mathrm{int}}(t)$. The field energy density $u_{\mathrm{field}}(t)$ describes energy density stored in the electric and magnetic fields and the interaction energy density $u_{\mathrm{int}}(t)$ describes the accumulation of energy density transferred into the medium (at the point of consideration) from the beginning of the pulse-medium interaction until time t. When necessary, we use the notation $u_{\mathrm{int}}[E](t)$ to denote the dependence of u_{int} on both the field E and the time t. For example, because only past fields can influence the current state, we have $u_{\mathrm{int}}[E](t)=u_{\mathrm{int}}[E_t^-](t)$, where E_t^- represents the past field up to time t (with the future set to zero).

In our previous work [8], we showed that $u_{\mathrm{int}}(t)$ can be divided as

$$u_{\mathrm{int}}(t) = u_{\mathrm{rec}}(t) + u_{\mathrm{irrec}}(t), \quad (8)$$

where the recoverable energy density u_{rec} describes the portion of $u_{\mathrm{int}}(t)$ that could possibly be returned to the field (given an appropriate future field) and the irrecoverable energy density u_{irrec} refers to the portion of $u_{\mathrm{int}}(t)$ that must remain in the medium regardless of what future field is chosen.

In this work, we divide $u_{\mathrm{int}}(t)$ as

$$u_{\mathrm{int}}(t) = u_{\mathrm{opt}}(t) + u_{\mathrm{waste}}(t) \quad (9)$$

where u_{opt} represents the portion of u_{int} that an optimally efficient pulse is required to deposit in the medium to arrive at the current pulse-medium excitation state, and u_{waste} represents the portion of u_{int} in excess of this minimum requirement that an actual field has deposited into the medium (due to a "nonoptimal" choice of the past field).

We will compare the two notions of dissipation represented by u_{irrec} and u_{waste} to a third, model-specific notion first derived by Barash and Ginzburg [16]. The Barash and Ginzburg notion is specific to a Lorentz oscillator medium where $\chi(\omega)$ is assumed to have the form

$$\chi(\omega) = \sum_{n=1}^{N}\chi_n(\omega) = \sum_{n=1}^{N}\frac{f_n\omega_{p_n}^2}{\omega_n^2 - i\gamma_n\omega - \omega^2}. \quad (10)$$

The parameters f_n, ω_{p_n}, ω_n, and γ_n are the oscillator strength, plasma frequency, resonant frequency, and damping rate of the n^{th} Lorentz oscillator. When Eq. (10) is inserted into Eq. (7) and expanded, one finds that u_{int} can be divided as

$$u_{\mathrm{int}}(t) = u_{\mathrm{e}}(t) + u_{\ell}(t), \quad (11)$$

where

$$u_e(t) = \sum_{n=1}^{N} \frac{1}{2f_n\omega_{p_n}^2}\dot{P}_n^2(t) + \frac{\omega_n^2}{2f_n\omega_{p_n}^2}P_n^2(t), \tag{12}$$

$$u_\ell(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} \frac{\gamma_n}{f_n\omega_{p_n}^2}\dot{P}_n^2(\tau)d\tau, \tag{13}$$

and

$$P_n(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi_n(\omega)\hat{E}(\omega)d\omega. \tag{14}$$

The quantity u_e contains the collected kinetic and potential energy terms of each individual oscillator. The dissipation term u_ℓ contains the usual viscous or frictional losses of each oscillator. This model of loss is explicitly model dependent. As shown by Barash and Ginzburg, a measured $\chi(\omega)$ does not have a unique representation described by Eq. (10) and two equally accurate representation of $\chi(\omega)$ can have different values for dissipation depending on the parameters used. While the framework that we develop in this paper is independent of the model used for a medium, the Barash and Ginzburg approach provides an instructive comparison for the concepts we develop.

III. OPTIMUM CREATION AND WASTE ENERGY DENSITIES

We consider pulse-medium interactions where the point under consideration in the medium begins and ends in the quiescent state [i.e., that $P(t)=0$ for $t\to\pm\infty$]. In this context, any energy density present at the point arrives via the Poynting flux and any energy density lodged in the medium as $t\to+\infty$ can be unambiguously identified as having been dissipated. Mathematically, this is ensured by requiring $E(t)$ to be absolutely integrable.

The formal definition of the energy density u_{opt} can then be written as

$$u_{opt}[E](t) := \inf_{F_t^-}\{u_{int}[F_t^-](t)\}. \tag{15}$$

The infimum is found by considering a set of past fields represented by F_t^- and finding the lower limit for the value of u_{int} at time t. Each hypothetical field history in F_t^- is required to result in the same state as the actual field history E_t^-. More precisely, to be considered a valid F_t^-, all future behavior of the medium polarization P after time t must be unaffected if the actual pulse history E_t^- is replaced by any of the hypothetical field histories F_t^-. (Like E_t^-, each F_t^- is zero after time t.) We use an infimum rather than a minimum in Eq. (15) because the lower bound cannot actually be reached, but only approached in the sense of a limit. Once u_{opt} is known, u_{waste} can be found using Eq. (9) with Eq. (7).

Definition (15) represents a well-defined quantity, but it is not in a form that can be readily computed. In the remainder of this section, we derive an algorithm to conveniently calculate u_{opt} for a given pulse-medium combination. In an effort to clearly show the main flow of the derivation, we present a general outline of the derivation in this section and defer many of the details to Sec. IV.

We first manipulate u_{int} into a form that allows the infimum to be carried out more conveniently. In Refs. [1,7] we showed that u_{int} can be written in the frequency domain as

$$u_{int}[E](t) = \int_{-\infty}^{+\infty} \omega\,\mathrm{Im}[\chi(\omega)]|\hat{E}_t^-(\omega)|^2 d\omega. \tag{16}$$

Here the instantaneous spectrum $\hat{E}_t^-(\omega)$ is the spectrum that a point in the medium experiences up until the time t,

$$\hat{E}_t^-(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E_t^-(t)e^{i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} E(t)e^{i\omega t}dt. \tag{17}$$

Through further manipulating, Eq. (16) can be written in the form

$$u_{int}(E)(t) = \frac{\gamma_\infty}{\omega_p^2}\int_{-\infty}^{+\infty}[-i\omega\tilde{\chi}_{eff}(\omega)\hat{E}_t^-(\omega)] \times[+i\omega\tilde{\chi}_{eff}(-\omega)E_t^-(-\omega)]d\omega. \tag{18}$$

where ω_p^2 is the plasma frequency, γ_∞ is a constant related to dissipation as $\omega\to\infty$, and $\tilde{\chi}_{eff}(\omega)$ is a function whose poles are those of $\chi(\omega)$ but whose zeros are the zeros of $\omega\,\mathrm{Im}[\chi(\omega)]$ that fall in the upper half-plane. The procedure for constructing $\tilde{\chi}_{eff}(\omega)$ from $\chi(\omega)$ is described more explicitly in the discussion surrounding Eqs. (28)–(32) in Sec. IV.

The two factors in the integral of Eq. (18) can be thought of as the time derivatives of the Fourier transform of a function $\tilde{P}_{eff}$, defined by

$$\tilde{P}_{eff}[E_t^-](\tau) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{\chi}_{eff}(\omega)\hat{E}_t^-(\omega)e^{-i\omega\tau}dw. \tag{19}$$

The mathematical form of Eq. (19) is reminiscent of the constitutive relation (1) for P, and hence the notation [18]. However, there are important differences in the behavior of P and $\tilde{P}_{eff}$ because of the differences in $\chi(\omega)$ and $\tilde{\chi}_{eff}(\omega)$. Using definition (19) and Parseval's theorem, we finish our manipulation of u_{int} by writing Eq. (18) in the time domain as

$$u_{int}[E](t) = \frac{\gamma_\infty}{\omega_p^2}\int_{-\infty}^{+\infty}\dot{\tilde{P}}_{eff}^2[E_t^-](\tau)d\tau, \tag{20}$$

where the dot notation again indicates a time derivative.

We are now ready to find the infimum. Inserting Eq. (20) into Eq. (15) and splitting the integral at the defining time t, we have

$$u_{opt}[E](t) = \inf_{F_t^-}\left\{\frac{\gamma_\infty}{\omega_p^2}\int_{-\infty}^{t}\dot{\tilde{P}}_{eff}[F_t^-]^2(\tau)d\tau + \frac{\gamma_\infty}{\omega_p^2}\int_{t}^{+\infty}\dot{\tilde{P}}_{eff}^2[F_t^-](\tau)d\tau\right\} \tag{21}$$

As stated previously, the requirement for F_t^- to be an acceptable past field is that the future behavior of the medium (i.e., the polarization P) is unchanged if the actual past field E_t^- is interchanged with the hypothetical past field F_t^-. In Sec. IV we show that this requirement ensures that the future behavior of $\tilde{P}_{\text{eff}}$ is also unchanged when F_t^- is replaced by E_t^-. As a result, the second integral in Eq. (21) is invariant with respect to F_t^- (since we can replace F_t^- with E_t^-, which does not vary) and can be removed from the infimum,

$$u_{\text{opt}}[E](t) = \frac{\gamma_\infty}{\omega_{\text{p}}^2}\int_t^{+\infty} \dot{\tilde{P}}_{\text{eff}}[E_t^-]^2(\tau)d\tau + \inf_{F_t^-}\left\{\frac{\gamma_\infty}{\omega_{\text{p}}^2}\int_{-\infty}^{t} \dot{\tilde{P}}_{\text{eff}}[F_t^-]^2(\tau)d\tau\right\} \qquad (22)$$

We also show in Sec. IV that there exists a sequence of fields F_t^- for which the infimum in Eq. (22) is zero. With this result, we arrive at an expression for u_{opt},

$$u_{\text{opt}}[E](t) = \frac{\gamma_\infty}{\omega_{\text{p}}^2}\int_t^{+\infty} \dot{\tilde{P}}^2_{\text{eff}}[E_t^-](\tau)d\tau. \qquad (23)$$

By comparing Eq. (23) with Eq. (20) in light of Eq. (9), we see that u_{waste} is given by

$$u_{\text{waste}}[E](t) = \frac{\gamma_\infty}{\omega_{\text{p}}^2}\int_{-\infty}^{t} \dot{\tilde{P}}_{\text{eff}}[E_t^-]^2(\tau)d\tau = \frac{\gamma_\infty}{\omega_{\text{p}}^2}\int_{-\infty}^{t} \dot{\tilde{P}}_{\text{eff}}[E]^2(\tau)d\tau. \qquad (24)$$

Note that in Eq. (24) we have indicated that it is equivalent to use either the past field E_t^- or the entire field E in calculating $\dot{\tilde{P}}_{\text{eff}}$. This is a result of the facts that the integral only goes to the current time t and that the past behavior of $\dot{\tilde{P}}_{\text{eff}}$ is unaffected by the future behavior of E [see the discussion of causality following Eq. (32) in Sec. IV]. Since the integral in Eq. (23) considers future times, it is necessary to use only the past field E_t^- when calculating u_{opt} in Eq. (23). For this reason, it is usually more computationally efficient to calculate u_{waste} using the second form in Eq. (24) and then calculate u_{opt} using Eqs. (9) and (7).

Before justifying the assertions made in the derivation above, we discuss some of the properties that are apparent from these forms for u_{waste} and u_{opt}. First, note that formula (24) for u_{waste} is a running integral of the square of a real function. This requires that u_{waste} always increase or stay the same as time progresses (i.e., it is monotonically nondecreasing). This is consistent with the notion that the portion of the energy density represented by u_{waste} is lost to the medium permanently.

We can also see from this derivation that there is not a unique value of u_{waste} associated with a given medium state. This is a result of the fact that there are many ways to create the same state in a medium (i.e., there are many admissible hypothetical field histories F_t^- that create identical future behaviors), and each field history can deposit different amounts of waste energy density into the medium. In contrast, for any given state of the medium there does exist a unique u_{opt} that specifies the minimum value of the pulse-medium interaction energy u_{int} that can be associated with that state of the medium.

A similar situation occurs when we divide the interaction energy density into recoverable and irrecoverable portions using Eq. (8) as done in [8]. In that case, the recoverable energy u_{rec} is unique to a given medium state, but any value for the irrecoverable energy density u_{irrec} can be obtained by choosing different field histories that result in the same state. The energy "inefficiency" of a medium state can be described by the difference $u_{\text{opt}}(t) - u_{\text{rec}}(t)$, which describes the minimum amount of energy density that must be irretrievably deposited in the medium in order to realize a given medium excitation: $u_{\text{opt}}(t)$ is the minimum net energy input required to create to a state and $u_{\text{rec}}(t)$ is the maximum amount that can be subsequently returned.

The two notions of dissipation represented by u_{waste} and u_{irrec} give bounds that limit the values that a model-specific notion of real-time loss can take on. The most energy density that can be said to be dissipated at a given time (i.e., the upper bound on dissipation) is given by u_{irrec}. Any fraction of the energy density in the medium beyond u_{irrec} could in principle be returned to a future field. The minimum amount of energy that must be considered dissipated (i.e., lower bound on dissipation) is given by u_{waste}. No future behavior of the pulse-medium interaction would change if the pulse-medium excitation had been produced with a past field for which $u_{\text{waste}}=0$. Thus, u_{waste} represents the minimum fraction of energy density that must be classified as no longer participating in the pulse-medium interaction.

Within the bounds provided by u_{waste} and u_{irrec} it is possible to consider other model-specific notions of loss that satisfy various properties. For instance, the Barash and Ginzburg notion of loss comes from treating a medium as a collection of mechanical oscillators with various resonance frequencies. In this model, the "kinetic" and "potential" energies of the oscillators represent undissipated energy, and the accumulated "frictional" losses of the oscillators are summed to find u_ℓ in Eq. (11). The Barash and Ginzburg notion of dissipation obeys

$$u_{\text{waste}} \le u_\ell \le u_{\text{irrec}}, \qquad (25)$$

when $f_n > 0$ for all resonance. However, Eq. (25) can fail in a medium where $f_n < 0$ for one or more of the resonances even when the other oscillators in the medium combine with the negative oscillator strengths to make the medium completely passive (see Sec. V).

To illustrate the behavior of these quantities, we have calculated the various notions of loss for a Gaussian pulse given by

$$E(t) = E_0 e^{-t^2/T^2}\cos(\bar{\omega}t). \qquad (26)$$

propagating in a multiple-Lorentz oscillator medium, described by Eq. (10) with $N=2$. For this example, we use the parameters

$$\gamma_1 = \gamma_2 = \gamma,$$

$$\omega_1 = 49\gamma,$$

$$\omega_2 = 51\gamma,$$

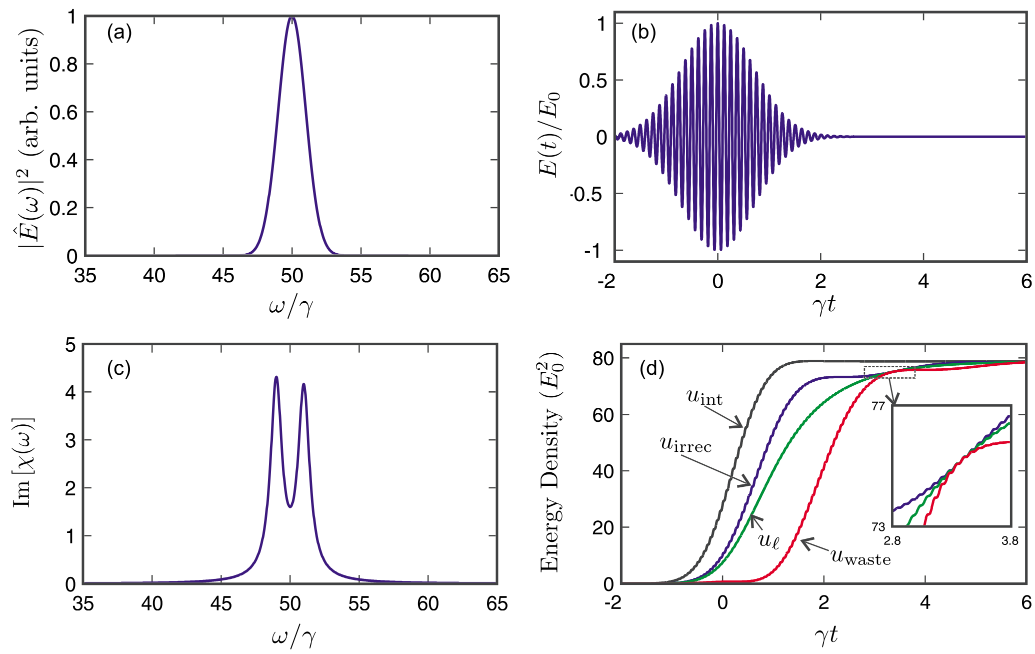

FIG. 1. (Color online) Quantities related to the parameters in Eq. (27). (a) The normalized spectrum $|\hat{E}(\omega)|^2$ of the pulse field. (b) The time evolution of the electric field. (c) $\mathrm{Im}[\chi(\omega)]$. (d) Energy densities in the medium at the point experiencing the pulse in (a).

$$f_1\omega_{p_1}^2 = f_2\omega_{p_2}^2 = 200\gamma^2,$$

$$\bar{\omega} = 50\gamma,$$

$$T = 1/\gamma. \tag{27}$$

This represents an optical pulse with its spectrum encompassing two closely spaced absorbing resonances [see Figs. 1(a) and 1(c)]. This situation results in a large amount of dissipation and apparent "superluminal" propagation to a nearby point since the trailing portion of the pulse experiences greater attenuation than the leading portion. Figure 1(d) shows the evolution of the three notions of loss: u_{irrec}, u_ℓ, and u_{waste}. Each notion of loss increases with time, although u_{waste} and u_{irrec} periodically flatten out (this is a signature of the fact that u_ℓ is not an extremal notion of loss, while the other two are). The inset in Fig. 1(d) shows that inequality (25) holds even when the three notions of dissipation are very close as the two Lorentz oscillators come into phase. Note that u_{int} reaches its maximum value during the main portion of the field oscillations in Fig. 1(b), but a non-negligible fraction of this energy remains undissipated for some time after the field oscillations have essentially ceased. Eventually, the dissipation of the energy is finalized, and all notions of loss converge to the same value as $t \to +\infty$.

Both u_{irrec} and u_{waste} are independent of the model used for $\chi(\omega)$. However, u_ℓ given by Eq. (13) is explicitly model dependent and it is possible to represent the same $\chi(\omega)$ with more than one set of model parameters in Eq. (10). Different sets of parameters representing the same $\chi(\omega)$ can have different evolutions for u_ℓ, but these evolutions will always fall within the bounds set by u_{irrec} and u_{waste} as long as all the individual oscillator strengths satisfy $f_n > 0$. This behavior indicates that there is not a unique quantity that unambiguously represents the evolved dissipation up to a given time, but that the dissipation depends on microscopic details of the medium. Nevertheless, u_{irrec} and u_{waste} always define a range of energy densities that can be said to meaningfully represent dissipation.

IV. DERIVATION DETAILS AND THE OPTIMAL PAST FIELD

We now return to the derivation and justify the several assertions made above. We begin by writing $\chi(\omega)$ in an explicit form where the zeros and poles are evident. For a general passive dielectric, $\chi(\omega)$ can be written as a rational function of the form

$$\chi(\omega) = -\frac{\omega_{\mathrm{p}}^2(\omega - \Omega_1)(\omega + \Omega_1^*)\ldots(\omega - \Omega_N)(\omega + \Omega_N^*)}{(\omega - \omega_1)(\omega + \omega_1^*)\ldots(\omega - \omega_{N+1})(\omega + \omega_{N+1}^*)} \tag{28}$$

where the plasma frequency ω_{p} is given by

$$\omega_{\mathrm{p}}^2 = -\lim_{\omega\to\infty} \omega^2\chi(\omega) \tag{29}$$

and the complex frequencies ω_j and Ω_j represent the poles and zeros of $\chi(\omega)$, respectively. Each subscript appears twice in Eq. (28) because the symmetry (4) requires poles and zeros that are not on the imaginary axis to come in pairs, arranged symmetrically across the imaginary axis (e.g., ω_1 and its negative complex conjugate $-\omega_1^*$ are both poles). Note that there are $2N$ zeros while there are $2N+2$ poles. (This difference in number is associated with the fact that the charge carriers in a medium have inertia [17].) As discussed in the introduction, all of the poles of Eq. (28) must be in the lower half-plane because of causality, and the zeros must also be in the lower half-plane because the medium is passive [17].

We can use this representation of $\chi(\omega)$ to find the quantity $\tilde{\chi}_{\text{eff}}(\omega)$ introduced in Eq. (18). If the form (28) is inserted into the integrand of Eq. (16), some straightforward analysis shows that the integrand can be written in the form

$$\omega\,\mathrm{Im}[\chi(\omega)] = \frac{-i\omega\chi(\omega)+i\omega\chi(-\omega)}{2} = \frac{\gamma_\infty}{\omega_p^2}\left(-i\omega\frac{-\omega_p^2(\omega+\nu_1)(\omega-\nu_1^*)\cdots(\omega+\nu_N)(\omega-\nu_N^*)}{(\omega-\omega_1)(\omega+\omega_1^*)\cdots(\omega-\omega_{N+1})(\omega+\omega_{N+1}^*)}\right)\left(i\omega\frac{-\omega_p^2(-\omega+\nu_1)(-\omega-\nu_1^*)\cdots(-\omega+\nu_N)(-\omega-\nu_N^*)}{(-\omega-\omega_1)(-\omega+\omega_1^*)\cdots(-\omega-\omega_{N+1})(-\omega+\omega_{N+1}^*)}\right) \tag{30}$$

where the high frequency dissipation γ_∞ is given by

$$\gamma_\infty = \lim_{\omega\to\infty}\frac{\omega^3}{\omega_\mathrm{p}^2}\mathrm{Im}[\chi(\omega)] \tag{31}$$

and the frequencies ν_n in Eq. (30) identify the complex zeros of $\omega\,\mathrm{Im}[\chi(\omega)]$ that are in the lower half-plane. With the definition

$$\tilde{\chi}_{\text{eff}}(\omega) \coloneqq \frac{-\omega_p^2(\omega+\nu_1)(\omega-\nu_1^*)\ldots(\omega+\nu_N)(\omega-\nu_N^*)}{(\omega-\omega_1)(\omega+\omega_1^*)\ldots(\omega-\omega_{N+1})(\omega+\omega_{N+1}^*)} \tag{32}$$

in Eq. (30) we arrive at the form of the integrand presented in Eq. (18).

The quantity $\tilde{\chi}_{\text{eff}}(\omega)$ plays the same role for $\tilde{P}_{\text{eff}}$ in Eq. (19) as $\chi(\omega)$ plays for P in Eq. (1). The differences in behavior between P and $\tilde{P}_{\text{eff}}$ arise due to the construction of $\tilde{\chi}_{\text{eff}}(\omega)$. Note that the poles of $\tilde{\chi}_{\text{eff}}$ are the same as the poles of $\chi(\omega)$. These poles are in the lower half-plane, which ensures that $\tilde{P}_{\text{eff}}$ defined by Eqs. (19) and (32) is causal in the sense that there can be no oscillations of $\tilde{P}_{\text{eff}}$ before E begins to oscillate. However, the zeros of $\tilde{\chi}_{\text{eff}}$ are in the *upper* half-plane so that the inverse of $\tilde{\chi}_{\text{eff}}$ is not analytic in the upper half-plane. This indicates that E and $\tilde{P}_{\text{eff}}$ are not mutually causal. Specifically, we can have oscillations of E without causing oscillations in $\tilde{P}_{\text{eff}}$. This is an important point, since we wish to show that

$$\inf_{F_t^-}\left\{\frac{\gamma_\infty}{\omega_p^2}\int_{-\infty}^{t}\dot{\tilde{P}}_{\text{eff}}^2[F_t^-](\tau)d\tau\right\} = 0. \tag{33}$$

This can only be true if there exists a nontrivial past field F_t^- that oscillates before t but does not cause $\tilde{P}_{\text{eff}}$ to oscillate.

In preparation for showing that Eq. (33) holds, we also need to prove the claim used to move from Eqs. (21) and (22), specifically that for each F_t^-

$$\int_t^{+\infty}\dot{\tilde{P}}_{\text{eff}}[F_t^-]^2(\tau)d\tau = \int_t^{+\infty}\dot{\tilde{P}}_{\text{eff}}[E_t^-]^2(\tau)d\tau. \tag{34}$$

In essence, Eq. (34) claims that any acceptable F_t^- produces the very same "ringing" of $\tilde{P}_{\text{eff}}$ after the field ceases at time t as the actual past field E_t^-. If we were to replace $\tilde{P}_{\text{eff}}$ with P in Eq. (34), the statement would be true by definition of what constitutes a valid F_t^-, namely, that the future behavior of the medium is the same for E_t^- and F_t^-, i.e.,

$$P[E_t^-](\tau) = P[F_t^-](\tau) \quad (\tau > t) \tag{35}$$

must be satisfied for all future times ($\tau > t$). Requirement (35), combined with the fact that we are considering a linear medium, effectively defines the state of the medium.

We can rewrite the left-hand side of Eq. (35) in a useful form by calculating

$$P(E_t^-)(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi(\omega)\hat{E}_t^-(\omega)e^{-i\omega\tau}d\omega = \sqrt{2\pi}\sum_{j=1}^{N+1}[a_j\hat{E}_t^-(\omega_j)e^{-i\omega_j\tau} + a_j^*\hat{E}_t^-(-\omega_j^*)e^{i\omega_j^*\tau}], \tag{36}$$

where the a_j and a_j^* coefficients are the residues connected to the poles of $\chi(\omega)$ at ω_j and $-\omega_j^*$. Since the frequencies ω_1, $-\omega_1^*,\ldots,\omega_{N+1},-\omega_{N+1}^*$ are distinct, requirement (35) holds if, and only if

$$\hat{E}_t^-(\omega_j) = \hat{F}_t^-(\omega_j), \quad \hat{E}_t^-(-\omega_j^*) = \hat{F}_t^-(-\omega_j^*), \quad (j=1,\ldots,N+1). \tag{37}$$

If we perform the analogous integral to Eq. (36) for $\tilde{P}_{\text{eff}}$, we obtain

$$\tilde{P}_{\text{eff}}(E_t^-)(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{\chi}_{\text{eff}}(\omega)\hat{E}_t^-(\omega)e^{-i\omega\tau}d\omega$$
$$= \sqrt{2\pi}\sum_{j=1}^{N+1} [\tilde{a}_j\hat{E}_t^-(\omega_j)e^{-i\omega_j\tau} + \tilde{a}_j^*\hat{E}_t^-(-\omega_j^*)e^{i\omega_j^*\tau}] \quad (38)$$

The residues $\tilde{a}$ and $\tilde{a}^*$ are different from those in Eq. (36), but since $\tilde{\chi}_{\text{eff}}(\omega)$ has the same poles as $\chi(\omega)$ the complex frequencies ω_j are the same in Eqs. (36) and (38). If a past field F_t^- that satisfies the conditions (37) is used in Eq. (38) instead of E_t^-, we find

$$\tilde{P}_{\text{eff}}[F_t^-](\tau) = \tilde{P}_{\text{eff}}[E_t^-](\tau) \quad (\tau \geq t), \quad (39)$$

from which assertion Eq. (34) directly follows.

Now we are in a position to verify our claim that the infimum in Eq. (33) is zero. We seek the existence of (the limit of) an optimum past field $F_{t,\text{opt}}^-$ that produces

$$\dot{\tilde{P}}_{\text{eff}}[F_{t,\text{opt}}^-](\tau) = 0 \quad (\tau < t) \quad (40)$$

so that the integrand in Eq. (33) is zero. From basic complex analysis [17], it is known that Eq. (40) is satisfied if the Fourier transform of $\dot{\tilde{P}}_{\text{eff}}[F_{t,\text{opt}}^-](\tau - t)$, given by $-i\omega\tilde{\chi}_{\text{eff}}(\omega)\hat{F}_{t,\text{opt}}^-(\omega)$, is analytic and rapidly vanishing in the upper half-plane. Since $\tilde{\chi}_{\text{eff}}(\omega)$ is analytic in the upper half-plane, only singularities in the spectrum $\hat{F}_{t,\text{opt}}^-(\omega)$ (which are necessarily located in the upper half-plane) could ruin the analyticity of $-i\omega\tilde{\chi}_{\text{eff}}(\omega)\hat{F}_{t,\text{opt}}^-(\omega)$. Thus, the requirement on analyticity can only be met by a past field whose spectrum $\hat{F}_{t,\text{opt}}^-(\omega)$ is of the form

$$\hat{F}_{t,\text{opt}}^-(\omega) = D_\infty + \frac{D_0}{i(\omega - i\epsilon)} + \frac{D_1}{i(\omega + \nu_1)} + \frac{D_1^*}{i(\omega - \nu_1^*)} + \dots + \frac{D_N}{i(\omega + \nu_N)} + \frac{D_N^*}{i(\omega - \nu_N^*)}. \quad (41)$$

Note that the form in Eq. (41) has simple poles located at precisely the same places in the complex plane as the zeros of $\tilde{\chi}_{\text{eff}}(\omega)$ in Eq. (32), so that the combination $-i\omega\tilde{\chi}_{\text{eff}}(\omega)\hat{F}_{t,\text{opt}}^-(\omega)$ is analytic in the upper half-plane.

The parameters $D_\infty, D_0, D_1, D_1^*, \dots, D_N, D_N^*$ represent the strength of each frequency component in the field. D_∞ is the strength of the components as $\omega \to \infty$ (a delta impulse in time) and D_0 is the strength of the component as $\omega \to 0$ (a dc field). The parameter ϵ in Eq. (41) is a small positive number that represents a very slow "ramp up" of the field that approximates dc forward tail as $\epsilon \to 0$. This limit is required because an optical pulse with a dc forward tail is not absolutely integrable and does not correspond to the medium originating from the quiescent state as $t \to -\infty$. (This limit is part of the reason we needed to take an infimum rather than a minimum.) The rest of the parameters (D_1, D_2, etc.) represent the strength of the complex frequency components ($-\nu_1$, $-\nu_2$, etc.) that describe exponentially growing oscillations for the past field.

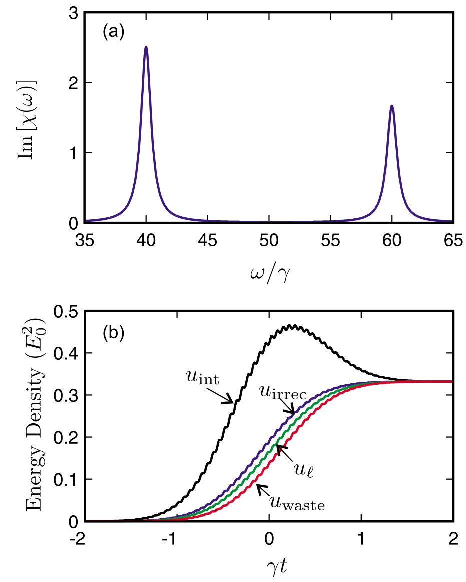

FIG. 2. (Color online) Quantities related to the parameters in Eq. (42). (a) Im[$\chi(\omega)$]. (b) Energy densities associated with the field shown in Fig. 1(b) for this medium.

The values for the $2N+2$ parameters D_j in Eq. (41) are not arbitrary, but must be chosen so that the $2N+2$ constraints in Eq. (37) are satisfied. This ensures that $F_{t,\text{opt}}^-$ produces the same state as E_t^- by time t. The linear independence of the functions indicated in Eq. (41) shows that the Eqs. (37) will have a unique solution for $\hat{F}_{t,\text{opt}}^-(\omega)$ (for each value of ϵ). All other past fields will give rise to an oscillation in $\tilde{P}_{\text{eff}}[F_t^-](\tau)$ since $\tilde{\chi}_{\text{eff}}(\omega)\hat{F}_t^-(\omega)$ will not be analytic in the upper half-plane if $\hat{F}_t^-(\omega)$ has poles at places other than the zeroes of $\tilde{\chi}_{\text{eff}}(\omega)$ (or has other types of singularities).

To illustrate the behavior of the optimal creation field, we consider a second numerical example. We use the same Gaussian pulse as in Fig. 1, but change the medium parameters to

$$\gamma_1 = \gamma_2 = \gamma,$$
$$\omega_1 = 40\gamma,$$
$$\omega_2 = 60\gamma,$$
$$f_1\omega_{p_1}^2 = f_2\omega_{p_2}^2 = 100\gamma^2. \quad (42)$$

This represents an optical pulse with its spectrum centered in a low-absorption region between two absorbing resonances [see Fig. 2(a)]. The pulse deposits energy in the medium during the leading portion of the pulse and the medium returns a part of this energy to the latter portion of the pulse, resulting in modestly subluminal propagation to nearby spatial points. Note that the energy densities have the same ordering as specified by Eq. (25) since f_1 and f_2 are positive.

In Fig. 3 we have replaced the portion of the field before $t=0$ with the optimal field $F_{t,\text{opt}}^-$ calculated at $t=0$. Note that the optimal creation field shown in Fig. 3(a) has the three general features discussed above: a dc portion, a delta spike

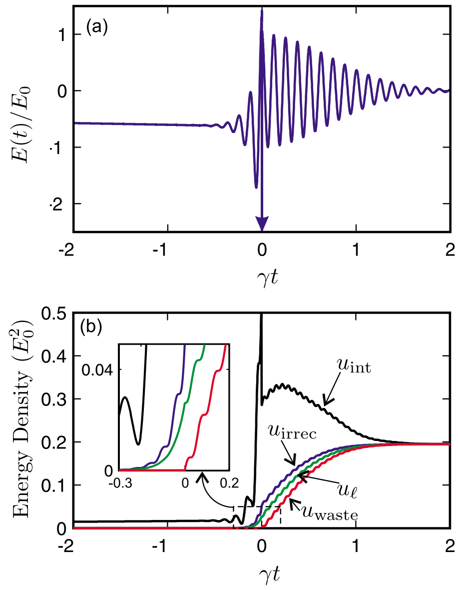

FIG. 3. (Color online) (a) The electric field composed of the optimum creation field before $t=0$ and the same Gaussian field as Fig. 1(b) after $t=0$. (b) Energy densities associated with the field shown in (a).

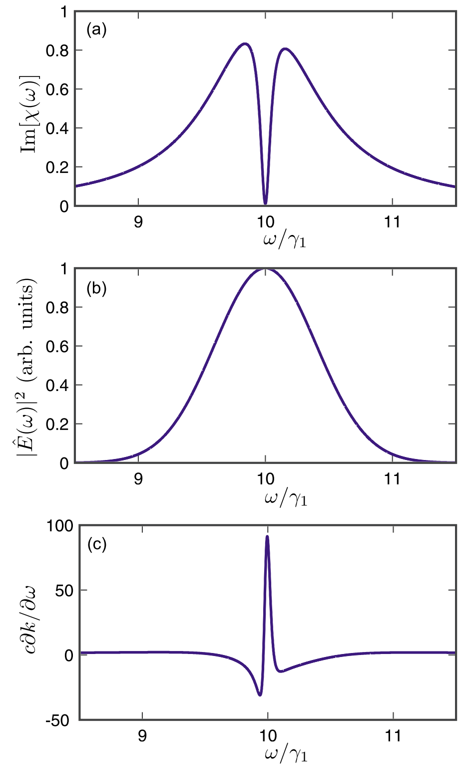

FIG. 4. (Color online) (a) $\mathrm{Im}[\chi(\omega)]$ for the material specified by the parameters in Eq. (43). (b) $\hat{E}(\omega)$ for the parameters in Eq. (44). (c) Group delay function for the material specified by the parameters in Eq. (43).

at $t=0$ (represented by the downward arrow in the figure), and an exponentially growing oscillations before our defining time $t=0$. This field creates the same behavior of $P(t)$ for $t>0$ as the original field shown in Fig. 1(b) (to within the numerical accuracy of our calculations). In Fig. 3(b), we have plotted the various dissipation energy densities. Note in the inset of Fig. 3(b) that $u_{\rm waste}(t)=0$ for all $t<0$ as required. The other notions of loss (i.e., $u_{\rm irrec}$ and u_ℓ) are nonzero before $t=0$, but $u_{\rm int}$ is the lowest possible for this pulse-medium state at $t=0$.

V. FAILURE OF A MODEL-SPECIFIC NOTION OF LOSS AND EIT

While the model-specific loss u_ℓ serves as an interesting comparison to $u_{\rm waste}$ and $u_{\rm irrec}$, it has some features that make it problematic for general use. In this section we detail these issues. First, as Barash and Ginzburg pointed out, it is possible to use many different sets of parameters in Eq. (10) to obtain the exact same $\chi(\omega)$, and each parameterization can result in a different evolution of u_ℓ. The macroscopic Maxwell equations (which involve the medium polarization P) respond only to $\chi(\omega)$, not to any microscopic details that give rise to $\chi(\omega)$. Thus, analysis based on the macroscopic Maxwell equations cannot classify one parameterization of a multiple-Lorentz oscillator representation of a medium as the "correct" one.

In contrast, the notions of loss given by $u_{\rm waste}$ and $u_{\rm irrec}$ are independent of the representation of the medium and uniquely defined within the macroscopic Maxwell description. These two concepts provide bounds for the dissipation up to a given time rather than an exact value. However, the authors find it unlikely that definitions of loss based on macroscopic quantities [such as $\chi(\omega)$ or evolved heat in the medium] can provide a more precise picture of real-time loss than is given by the limits of $u_{\rm waste}$ and $u_{\rm irrec}$. There is a fundamental "fuzziness" involved in describing the time when energy transitions from the ordered interaction of light and matter to the disordered thermal energy described by temperature.

Models of dissipation that give precise values of "loss" are derived by making assumption about the microscopic details of a medium (e.g., u_ℓ). While these microscopic models fall outside of the scope of the macroscopic Maxwell equations, we can still use the analysis from this paper to check if the values of loss predicted by the microscopic model are reasonable. The fraction of energy density represented by $u_{\rm waste}$ provides the lower limit of what energy density must be considered lost at a given time. This can be understood by noting that any pulse-medium state can be created with an optimal past field such that $u_{\rm waste}=0$, and all future behavior of the pulse-medium interaction is unaffected if the optimum past field is used rather than a nonoptimal past field. Thus, any nonzero value for $u_{\rm waste}$ represents energy density that cannot participate in future pulse-medium interactions. If one invents a notion of loss that is less than $u_{\rm waste}$, that notion has not classified a fraction of energy as "lost" even though that energy density can never influence the future pulse-medium interaction. Clearly such a notion underestimates what has been lost. Conversely, $u_{\rm irrec}$ provides the upper bound for real-time loss. Any notion of loss that exceeds $u_{\rm irrec}$ is flawed because by definition the energy density in excess of $u_{\rm irrec}$ could be restored to a future field (and thus cannot be considered "lost").

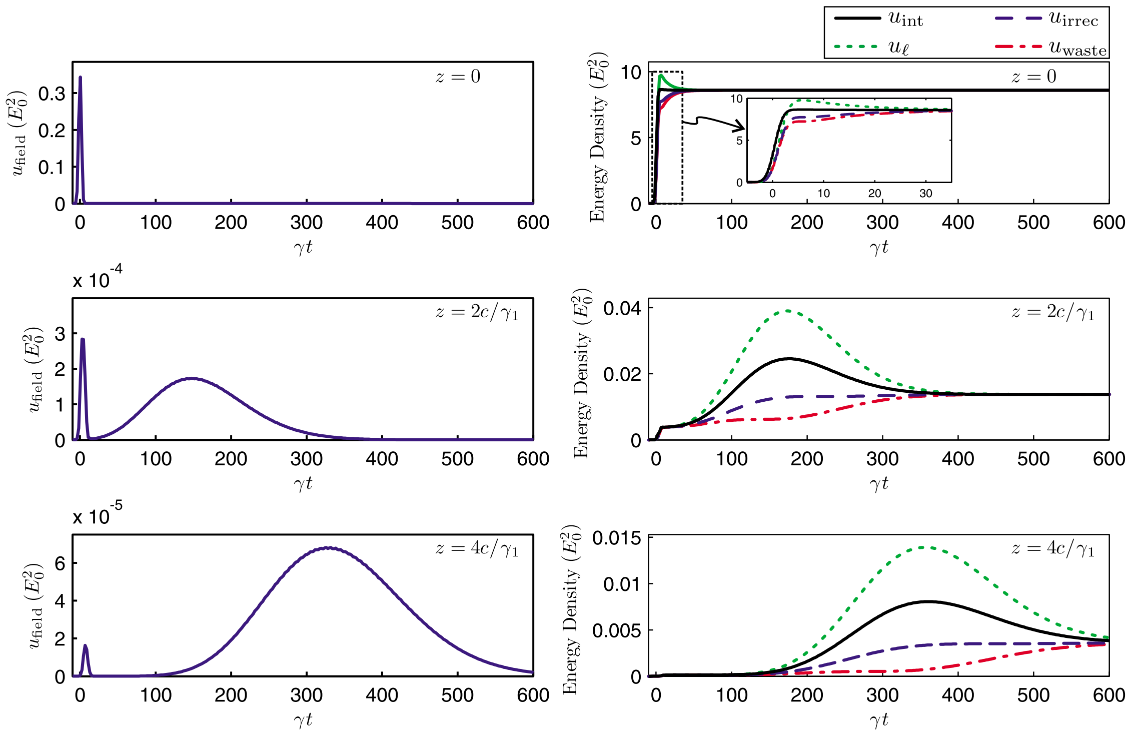

FIG. 5. (Color online) Energy densities associated with the pulse and medium specified by Eqs. (44) and (43). The left column shows field energy density and the right column shows energy densities in the medium. The top row is for no propagation (z=0). The middle row is after propagating a distance z=2c/γ_1 and the bottom row is after propagating a distance z=4c/γ_1.

In our previous examples, the model-specific loss u_ℓ stayed within the bounds provided by u_{waste} and u_{irrec}. This may leave the impression that one can always create a Lorentz model for a medium and get a general sense for the energy loss dynamics using u_ℓ. However, this approach can give nonsensical results in cases where a Lorentz medium has one or more oscillators with $f_n<0$. (Barash and Ginzburg derived the notion of loss for the specific case of $f_n>0$.) Note that if an f_n is negative in the representation (13), u_ℓ is not necessarily monotonically increasing. This is an immediate conceptual problem, since energy cannot be said to be truly dissipated at a given time if it can later be reclassified as undissipated.

To see an example where this occurs, consider a Lorentz medium defined by Eq. (10) with

$$\gamma_2 = 0.1\gamma_1,$$

$$\omega_1 = \omega_2 = 10\gamma_1,$$

$$f_1\omega_{p_1}^2 = 10\gamma_1^2,$$

$$f_2\omega_{p_1}^2 = -0.99\gamma_1^2. \tag{43}$$

This medium is passive (i.e., $\omega\,\mathrm{Im}[\chi(\omega)]\geq 0$), but the negative f_2 creates a narrow spectral window at the resonance frequency with very little absorption [see Fig. 4(a)]. This resonance feature is similar to an absorption line with an electromagnetically induced transparency (EIT) window in the center. A standard method for creating ultraslow propagation is to propagate a narrowband pulse with its spectrum centered on such a resonance so that the entire spectrum of the pulse experiences a large group delay [see Fig. 4(c)] and little loss. For our example, we consider the case of a relatively broadband pulse with its spectrum centered on the resonance. We choose our pulse defined by Eq. (26) with

$$\bar{\omega} = 10\gamma_1,$$

$$T = 2.5/\gamma_1, \tag{44}$$

[see Fig. 4(b)]. Note in Figs. 4(b) and 4(c) that the pulse contains frequencies with both "superluminal" group delays as well as the highly subluminal delays.

Figure 5 shows the temporal reshaping of this pulse that occurs as it propagates into the medium. The top left panel shows the temporal evolution of the field energy u_{field} at a point z=0 where the field is described by Eq. (44) and the corresponding energy stored in the medium u_{int} is shown in the top right panel. The three notions of loss u_ℓ, u_{irrec}, and u_{waste} are also plotted on the right. The panels in the second row show the temporal evolution after the pulse has propagated a distance z=2c/γ_1 into the medium, and the bottom panels show the evolution at a propagation distance of z=4c/γ_1.

Notice that the Barash and Ginzburg notion of loss u_ℓ is not monotonically increasing for this pulse-medium combination. At all three points, it initially increases and later decreases. As pointed out previously, this behavior is inconsistent with the notion that dissipation describes energy that is permanently transferred into the medium. In addition to decreasing at times, u_ℓ exceeds $u_{\rm irrec}$ for periods of time, which means there exists possible future fields that could recover a portion of u_ℓ. Perhaps most striking is that u_ℓ even exceeds $u_{\rm int}$ so that this notion of loss seems to indicate that more energy has been dissipated by a given time than has even been transferred to the medium before that time. This behavior emphasizes the need to use such model-dependent notions of loss with care. The dissipation concepts represented by $u_{\rm waste}$ and $u_{\rm irrec}$ remain well defined for all overall passive media, including the one in this example. In principle, one could create a different microscopic model of loss for this system that does not have the same issues as u_ℓ (for example, using quantum dissipation theory and perturbation theory on a three-level atom). However, the dissipation given by the model must fall within the bounds of $u_{\rm waste}$ and $u_{\rm irrec}$ to be considered reasonable in the context of the macroscopic Maxwell's equations.

This example can also be used to study pulse reshaping. We first discuss this reshaping using group delay methods, and then using the energy methods developed in this paper. In the group delay context, each spectral components in the pulse is delayed according to its group delay [19]. Since there are frequency components with both "superluminal" and highly subluminal delays, this pulse breaks into two pieces as it propagates (see Fig. 5). The superluminal spectral components experience a large amount of absorption [compare Figs. 4(a) and 4(c)], so the peak of energy that arrives at early times is absorbed before reaching large propagation distances. At propagation distances longer than those shown in Fig. 5, the slow components dominate the behavior.

The notions of loss that we have introduced can be used to enhance our understanding of the temporal reshaping of this pulse. At all propagation depths shown $u_{\rm waste}$ and $u_{\rm irrec}$ both track closely with $u_{\rm int}$ during the early portion of the pulse. Thus, the energy transferred into the medium during the early portion of the pulse is largely "stuck" in the medium at that point. The absorption at $z=0$ happens asymmetrically around the peak of the pulse (at $t=0$), indicating that more energy is absorbed from the trailing edge of the pulse than the leading edge. This explains the superluminal delays predicted by the group delay method at very small propagation depths. At greater propagations depths the frequency components that experience superluminal delays have been absorbed, and the overall behavior of the pulse is dominated by the frequency components with highly subluminal delays. The energy exchange behavior for these frequency components is well illustrated by the frames for $z=2c/\gamma_1$ and $z=4c/\gamma_1$. Note that a large fraction of the energy transferred from the early portion of the "slow" part of the pulse into the medium remains undissipated at these spatial positions. The long temporal delay between the transfer of energy into the medium and returning the energy to the field results in the highly subluminal behavior.

VI. CONCLUSION

In this paper, we introduced the waste energy density $u_{\rm waste}$ and defined it in relation to the minimum energy required to create a pulse-medium excitation. We derived expressions that allow the waste energy density to be easily calculated, and compared its behavior to the irrecoverable energy density $u_{\rm irrec}$, which was introduced in a previous work. Because of the way in which $u_{\rm waste}$ is defined, it describes the portion of the energy at a given time that cannot participate in future pulse-medium interactions that shape the trailing portion of the pulse. The waste energy density and the irrecoverable energy density are model-independent and comprise bounds on the amount of dissipation that a medium has experienced up to a given time. We showed that there exist relevant example media in which model-specific notions of real-time dissipation do not fall within these bounds and demonstrated that these notions of loss have serious conceptual problems.

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