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A STUDY OF THE CLASSICAL AND QUANTUM MECHANICAL EQUATIONS OF MOTION FOR THE RADIATING NONRELATIVISTIC ELECTRON

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CHAPTER 1

INTRODUCTION

The electron was identified by Thomson almost one hundred years ago [1]. Since then the electron has been thoroughly studied and mathematically described. Even though it is a fundamental particle, a mathematical description of the electron's motion contains a high level of complexity. It was found early on that when radiation effects are included in the theory, as they must be for a complete theoretical description, the results are complicated and not favorable.

Radiation effects come into play for any charged particle. Classical electromagnetism predicts that when a charged particle accelerates, it radiates. The radiation thus produced carries away with it energy and momentum from the electron, and consequently alters the electron's motion. The effect of the motion due to radiation is termed *radiation reaction*, or the reaction of the electron's motion due to its own radiation. The equation of motion for the electron including radiation reaction was first derived by Lorentz [2]. His theory of the electron yielded correct results within the domain of classical mechanics, but was not without principle problems.

Kramers spent much of his career preoccupied by the problems of the Abraham-Lorentz theory [3]. But even though he and many others attempted to resolve the problems of the electron theory within the classical domain, they were unsuccessful. With the birth of quantum mechanics, however, interest in the classical theory waned. It was hoped that the problems inherent to the classical theory could be resolved with the new quantum mechanics. But, it was found that a quantum mechanical treatment of radiation reaction was difficult to obtain, and seemed to involve even more elaborate problems than the classical theory.

Since the electron is a fundamental charged particle, a correct description of the motion of electrons in an electromagnetic theory lies at the very foundation of theoretical physics. This was an important problem to solve, hence the vast attention given to the understanding of a complete theory for the electron. The classical study began again as a basis for the study of a quantum field theory for the electron. It was found that if one was clever enough, one could use Lorentz covariance and gauge invariance to circumvent the problems associated with the theory in quantum mechanics. The result was the advent of a highly succesful theory called *quantum electrodynamics* or QED. Because of the round about way QED handles the problems of infinities, one is able to use it in calculating small radiative effects in precise agreement with experiment. Despite the great success of QED, from a fundamental standpoint, a completely satisfactory theoretical model still eludes us.

This research involves the study of the equations of motion for the electron, including radiation reaction effects, in both classical and nonrelativistic quantum mechanics. Our purpose is twofold. First, we wish to better understand the model of a charged particle and its self-fields. In order to do this, we have looked at the fundamental assumptions of the classical and quantum theories as they pertain to charged particles. Second, we seek a better understanding of how quantum mechanics resolves the problems of a classical theory in the appropriate correspondence limit. In doing so we will closely follow the work of Moniz and Sharp [4], in which it is shown that a well behaved nonrelativistic quantum theory can be derived. We wish to verify their results: namely, that the classical problems can be resolved by taking the correspondence limit of the quantum theory.

In *Chapter 2* we will derive the point electron theory, and discuss the associated problems. *Chapter 3* is devoted to an extended electron, in which different models will be considered. For models with a radius greater than two-thirds the classical radius ($r_o = 2.82 \times 10^{-15}$m) it will be seen that the problems are resolved. However, any attempt to take the point limit leads to the previous problems. In *Chapter 4* we will derive the nonrelativistic quantum mechanical theory for the electron, and show that well-behaved solutions for the classical point electron are obtained in the appropriate limit.

In section 1.1, we will briefly discuss the history of radiation reaction. Because of the importance of this subject, the literature is very extensive, some of which can be found in the references. It is our intent in this chapter to cover just enough to bring the reader up to date on the important points of radiation reaction which deal specifically with this research.

1.1 A Brief History of Radiation Reaction

James Clerk Maxwell (1831-1879)

An introduction to radiation reaction must necessarily include Maxwell, who built on the foundation of electricity and magnetism left by his predecessors. Maxwell was able to simplify the theory of electromagnetism into four equations which now bear his name. In Gaussian units, the current density for a rigid charge distribution in Maxwell's equations is given by

$$\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\dot{\mathbf{R}}(t)$$

where $\dot{\mathbf{R}}(t)$ is the velocity of the charged particle, $\mathbf{R}(t)$ being the position vector of the center of charge, and $\rho(\mathbf{r}, t)$ is the charge density such that

$$\int d\mathbf{r}\rho(\mathbf{r},t) = e.$$

Notice that the time dependence in the charge density disappears after integration. This is because we are integrating over all space, in which the total charge of the distribution always adds up to e regardless of how the charge density is moving in time.

Sir Joseph John Thomson (1856-1940)

Thomson had studied the works of Maxwell, and even heard some of Maxwell's lectures. In 1881, Thomson published a paper on the electromagnetic mass of a charged sphere [5]. He assumed a spherical charge distribution of radius L and uniform velocity $\dot{\mathbf{R}}$. Thomson observed that the associated electromagnetic field of a charge e would have kinetic energy

$$K_{\text{elm}} = \frac{1}{2} f \frac{e^2}{Lc^2} \dot{\mathbf{R}}^2,$$

where f is a dimensionless constant of order 1, and depends on the type of spherical charge chosen. For instance, $f = 2/3$ for a charged shell and $f = 4/5$ for a uniform charge distribution. But if the above equation is the kinetic energy of the charge's electromagnetic field, we can compare it with the classical equation

$$K = \frac{1}{2} m v^2$$

to identify the electromagnetic mass δm as

$$\delta m = f \frac{e^2}{Lc^2}. \tag{1.1}$$

Notice that δm has the appropriate unit of mass. Using the above values for f, we see that

$$\delta m = \frac{2}{3} \frac{e^2}{Lc^2}, \qquad \text{for a spherical shell, and}$$

$$\delta m = \frac{4}{5} \frac{e^2}{Lc^2}, \qquad \text{for uniform sphere.}$$

We will derive these expressions for electromagnetic mass in *Chapter 3* using the Abraham-Lorentz theory. Notice that if the radius L goes to zero, the electromagnetic mass goes to infinity. This electromagnetic mass or self-mass can be interpreted as a self-energy which is therefore also infinite. That the self-mass is infinite is one of the well-known problems of the classical point charge theory.

Thomson associated the electromagnetic mass (1.1) with the fact that the particle has a charge. If the partical where neutral, δm would be zero, and its mass would be entirely non-electromagnetic (or mechanical) in nature. Therefore, its kinetic energy would be

$$K_{\text{mech}} = \frac{1}{2} m_o \dot{\mathbf{R}}^2,$$

where m_0 is the mechanical mass of the charge. The total kinetic energy of a charged particle could therefore be written as

$$K = \frac{1}{2}(m_o + \delta m)\dot{\mathbf{R}}^2,$$

from which Thomson identified the experimental or observed mass m as

$$m = m_o + \delta m, \tag{1.2}$$

where δm is defined in (1.1). Thomson's equation (1.2) suggested that a charged particle of mass m is the sum of the mechanical mass m_o and the electromagnetic mass δm. In other words, Thomson suggested that the mass we experience the electron to have is made up of the mass from "matter" itself plus the mass tied up in the electromagnetic fields.

The name *electron* was first suggested in 1894 by G. Johnstone Stoney [1]. But, it wasn't until 1897 that Thomson actually identified the electron [1]. He did so by verifying the corpuscular nature of cathode rays, and showing that these rays were deflected by an electrostatic field. He was able to measure the velocity $\dot{\mathbf{R}}(t)$ and the ratio of charge to mass e/m of the corpuscles, or *electrons*.

Hendrik Antoon Lorentz (1853-1928) and *Max Abraham*

At the beginning of the twentieth century, it was thought that the electron was an *atom of electricity*. Lorentz, working in Leiden, in the Netherlands, had a strong influence on the world of physics at the time. He was the first to develop a theory of charged particles interacting *via* an electromagnetic field. Lorentz's work was the link between Maxwell's generation of waves and the new physics of quanta introduced by Planck and Einstein.

Lorentz tried to set the mechanial mass m_o equal to zero in order to explain the mass of the electron purely by electromagnetism. When this assumption was combined with equations (1.1) and (1.2), he found a theoretical radius L for the electron:

$$L = f\frac{e^2}{mc^2}. \tag{1.3}$$

With $f \sim 1$, this equation can be written as

$$r_o = \frac{e^2}{mc^2} \tag{1.4}$$

where $L = r_o$ is known as the classical electron radius, and is on the order of 10^{-15} meters. At the time, there was no experimental evidence to verify the size of the electron, and r_o was accepted to be correct for the electron radius at least in order of magnitude. (Modern experiments indicate that the electron has a point-like structure down to distances of 10^{-18}m.)

One of the well known theoretical achievements of Lorentz is the equation which describes the force on a charged particle due to an electromagnetic field, namely the Lorentz force equation:

$$\mathbf{F} = \rho\left(\mathbf{E} + \frac{\dot{\mathbf{R}}}{c} \times \mathbf{B}\right).$$

However, Lorentz did much more than this. His intent was to account for all the macroscopic phenomena of electrodynamics and optics in terms of the microscopic behavior of electrons and ions. In this pursuit, he developed a classical theory of charged particles and fields.

Lorentz considered electrons as being elastically bound to atoms, and therefore acting as charged harmonic oscillators. Larmor had previously derived an equation for the rate at which such an oscillator emits radiation as a function of its acceleration. By using Larmor's power equation [6],

$$P = \frac{2}{3}\frac{e^2}{c^3}(\ddot{\mathbf{R}})^2,$$

Lorentz showed that the equation of motion for the oscillating electron including the loss of energy due to its own radiation is [2]

$$m\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + \frac{2}{3}\frac{e^2}{c^3}\frac{d\ddot{\mathbf{R}}(t)}{dt}. \tag{1.5}$$

Even though the general theory of Lorentz is quite successful in describing some observations, such as the normal Zeeman effect, it is not without formal problems. First, for a point electron, the self-energy is infinite. Second, equation (1.5) admits runaway solutions, or solutions in which the velocity of the particle increases without bounds regardless of any external force acting on it. Third, there is a way to remove the runaway solutions (called renormalization), but then preacceleration appears. This means that the particle begins to accelerate even before the external force begins to act on it. We will derive equation (1.5) in *Chapter 2*, and discuss these problems in more detail.

Lorentz obtained a more accurate description of a charged particle by taking into account its physical extension in space. For Lorentz, the electron was indeed a finite charge distribution. In this way he was able to calculate the effects of the fields created by the charge of the particle acting back on itself. He calculated this self-force using the force density equation

$$\mathbf{F}_{\text{self}} = \int d\mathbf{r}\rho\left(\mathbf{E}_{\text{self}} + \frac{\dot{\mathbf{R}}}{c} \times \mathbf{B}_{\text{self}}\right),$$

where $\mathbf{E}_{\text{self}}$ and $\mathbf{B}_{\text{self}}$ are the fields produced by the charge itself. Lorentz was the first to obtain $\mathbf{F}_{\text{self}}$ as

$$\mathbf{F}_{\text{self}} = -\frac{2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t)\int\int d\mathbf{r}d\mathbf{r}'|\mathbf{r}-\mathbf{r}'|^{n-1}\rho(\mathbf{r})\rho(\mathbf{r}'), \tag{1.6}$$

which we will rigorously derive in *Chapter 3*. We will also determine the behavior of the solutions to the the equation of motion

$$m_o\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}}, \tag{1.7}$$

where $\mathbf{F}_{\text{self}}$ is given by (1.6). We will show that the solutions to (1.7) do not suffer from runaways and preacceleration as long as the radius L is on the order of the classical electron radius, or m_o is taken to be positive [4]. Only when one assumes a negative mechanical mass do problems enter the extended charge theory.

But the extended model theory was not perfect either. For instance, the theory yielded an erroneous factor of 4/3, which was not completely made known until 1960 by Frizt Rohrlich [7]. The classical theory determined the total momentum to be

$$\mathbf{p} = \left(m_o + \frac{4}{3}\delta m\right)\dot{\mathbf{R}},$$

instead of the expected

$$\mathbf{p} = (m_o + \delta m)\dot{\mathbf{R}}. \tag{1.8}$$

Rohrlich showed that the 4/3 was inconsistent with the invarience property required by relativity. Therefore, the classical δm needed to be multiplied by the correction factor of 3/4. In this work we will not be interested in this aspect, and refer to [7] for a complete discussion of this subject.

Another problem for the classical extended model theory of Abraham and Lorentz was that some sort of cohesive forces were necessary to ensure stability. Otherwise, the particle would push itself apart. For this reason, except for its divergent self-energy, the point particle theory was more desirable. Again, for a discussion on these Poincaré stresses, see [7], page 16.

Also, one might perceive it as a problem to have an equation of motion involving *all* orders of derivatives, rather than having an equation of motion that is a second order differential equation as in Newtonian mechanics. If one uses the point charge equation of motion (1.5), one still has a third order differential equation, for which the initial position and velocity are not sufficient to determine the motion. Radiation reaction seems to imply the need for a different brand of mechanics!

After the Lorentz transformations were published in 1904, M. Abraham realized that the model of a rigid charge distribution, which was purely electromagnetic in nature, was not invariant under such transformations. In an attempt to correct this problem, Abraham was succesful in deriving a four-vector form of radiation reaction [8].

In 1938, P. A. M. Dirac derived a relativistic differential equation that a classical point charge must satisfy in which he used the Abraham four-vector of radiation reaction [9]. From this equation he was able to derive the relativistic equation of motion for the point electron. But Dirac's theory still had the problem of a divergent electromagnetic mass in the point limit. Although this problematic aspect was swept under the rug by renormalization, it was not theoretically satisfying. There were other physicists who devised extended models to solve the divergent self-mass problem. But these theories are subject to criticism because of arbitrary relativistic structure functions [10].

The relativistic treatment of radiation reaction is very complex, and to our knowledge, has not as yet been satisfactorily completed. In our work, and in the following chapters, we will discuss only nonrelativistic equations, and we will not discuss the relativistic case in any further detail.

1.2 Recent Developements in the Theory of Radiation Reaction

Before we discuss the nonrelativistic theory, we need to mention the latest developments. In 1965, Rohrlich published a new theory for a point charge [11]. This theory developed by Rohrlich was not the limiting case of an extended model theory as were previous theories. Rather, the equations of motion were obtained from a suitable Lagrangian. This particular construction had the advantage of a vanishing electromagnetic mass. In other words, the theory as developed by Rohrlich solved one of the major problems of the Abraham-Lorentz theory for the point electron: that of infinite self-energy.

As of 1965, there were basically three theories which gave conflicting results for the self-mass δm. First, there was the theory of Lorentz in which $\delta m = \infty$. Dirac's relativistic theory also suffered from the divergent self-mass problem in the point limit. When others developed relativistic extended models to avoid this problem, the result was that δm was finite, but arbitrary. And finally, Rohrlich's theory predicted that the self-mass should vanish [11].

Since the electron is a quantum particle, one needs to turn to quantum mechanics in order to resolve the conflicting results previously obtained for a classical point charge. Nine years after Rohrlich published his theory, Moniz and Sharp published a paper [4] in which they claimed to have derived a nonrelativistic quantum mechanical theory of radiation reaction for an extended charged particle. They claimed that if one takes the point limit ($L \to 0$) of this theory, and then the corresponding classical limit ($\hbar \to 0$), one obtains classical solutions for a point charge theory in which $\delta m = 0$.

Rohrlich saw the results of Moniz and Sharp as a verification of his theory. Within the same year, Rohlich published a paper [12] in which he quoted the results of Moniz and Sharp as the deciding factor to show that his theory of radiation reaction was the correct one.

In 1976, Moniz and Sharp published their results in more detail, and acknowledged receiving valuable comments from Rohrlich and others. But the following month brought with it questions and concerns. H. Grotch and E. Kazes published a paper [13] in which they noted what they considered to be discrepancies in the calculation procedures of Moniz and Sharp.

The questions raised by Grotch and Kayes remained unresolved for the next five years. However, in 1981, Grotch, Kazes, Rohrlich, and Sharp published a joint paper [14] in which they determined that the calculations of Moniz and Sharp were not only legitimate, but very important. A theory reduction from quantum mechanics not only solved the problem of a divergent self-mass, but also yielded a classical theory with no runaway solutions or preacceleration. Therefore, according to Rohrlich, the problems of the classical point electron theory including radiation reaction have been solved. "Unfortunately," Rohrlich pointed out in 1990, "this is a little known fact" [15].

We will analyze the results of Moniz and Sharp in *Chapter 4* by deriving the quantum mechanical equation of motion for a charged particle, and using different charge distributions not considered by them. We will discuss the solutions in general, as well as take the corresponding limit in order to obtain the well-behaved classical point charge theory.

Moniz and Sharp, as well as Rohrlich, are careful to point out that the above considerations are nonrelativistic. It would be most desirable to obtain a relativistic quantum mechanical equation of motion, and then take the corresponding limits to obtain the nonquantum theory. In the words of Rohrlich, "No doubt, we would learn a great deal from such a reduction" [15]. But to the present time, such a theory has remained beyond our reach, partly because of our inability to obtain exact relativistic solutions [15].

CHAPTER 2

THE CLASSICAL POINT ELECTRON

From modern scattering experiments, the electron appears to be a point particle. In modern physics, the electron is considered to be a point with no physical extension in space. Therefore, it has been an important theoretical problem to accurately describe the motion of a charged point particle. A complete description of the motion must necessarily include radiation reaction effects.

In section 2.1 of this chapter, we will derive the classical equation of motion for the point electron. This derivation closely parallels the one given by Jackson [16]. We pointed out in *Chapter 1* that the equation of motion for a classical point electron, including radiation reaction, suffers from theoretical problems. In section 2.2, we will solve the equation of motion for a point electron, and discuss the behavior of the solutions.

In section 2.3 we will discuss the solutions to the equation of motion when the electron experiences external forces. In doing so, we will review the results of Gilbert N. Plass [17]. We will see that on the level of experimentation, the equation of motion for the point electron yields correct results. It turns out that the problem of preacceleration acts over too small of a time to be measurable. The problem of finding a well behaved theory is still important, however, on the grounds that in the process, greater understanding of the microworld may be obtained.

2.1 The Equation of Motion

Lorentz never considered the electron to be a point particle. However, his first attempt of deriving an equation of motion turned out to be the point limit of the more general theory which takes into account the extension of the electron. It will be worth our while to derive the point theory before the extended theory, in order to better understand the purpose of this research; namely, to show that the associated problems can be resolved by taking the classical limit of the appropriate quantum mechanical theory of radiation reaction.

We begin by considering an electron of experimental mass m and charge e. According to Newton's equation of motion,

$$m\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}},$$

where $\mathbf{R}$ is the position vector of the point electron. However, as the electron accelerates, it radiates. We now consider the effects of the force acting back on the electron due to its own radiation. The radiation carries off some of the electron's momentum, which in turn applies a force back on the electron itself. Therefore, we can modify Newton's equation to read

$$m\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}}, \tag{2.1}$$

where $\mathbf{F}_{\text{self}}$ is the force the electron experiences as it radiates. With radiation reaction included, one must find solutions of this modified equation of motion. In order to do so, we must first find an expression for the self-force. Consider the power lost in radiation, or the rate at which radiation is emitted by an accelerating charge. This problem was worked out by Larmor in which he found that the power radiated by an accelerating charged particle can be written as [6]

$$P = \frac{2}{3}\frac{e^2}{c^3}(\ddot{\mathbf{R}})^2. \tag{2.2}$$

Notice that the power radiated, as given in (2.2), is proportional to e^2. Therefore, $\mathbf{F}_{\text{self}}$ must be proportional to e^2. We can also see this proportionality by observing that the sign of the charge must not play a role in the theory (positive, as well as negative charges radiate, and therefore experience the same radiation reaction effects).

In order to satisfy energy conservation, we require that the work done by the self-force due to radiation reaction over some time interval from 0 to t, be equal to the negative of the energy radiated in that time. The work done can be written as

$$W = \int_0^t \mathbf{F}_{\text{self}} \cdot d\mathbf{R} = \int_0^t \mathbf{F}_{\text{self}} \cdot \frac{d\mathbf{R}}{dt'}dt' = \int_0^t \mathbf{F}_{\text{self}} \cdot \dot{\mathbf{R}}dt'.$$

And since power is the rate at which energy is used, by implementing the Larmor power formula (2.2), we have that the energy lost in radiation over the time interval from zero to t is

$$E = \int_0^t Pdt' = \int_0^t \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{R}} \cdot \ddot{\mathbf{R}}dt'.$$

Equating the work done to the negative of the rate at which power is radiated, we have

$$\int_0^t \mathbf{F}_{\text{self}} \cdot \dot{\mathbf{R}}dt' = -\frac{2}{3}\frac{e^2}{c^3}\int_0^t \ddot{\mathbf{R}} \cdot \ddot{\mathbf{R}}dt'. \tag{2.3}$$

If the right hand side were in the form of $\int_0^t \{\text{something}\} \cdot \dot{\mathbf{R}} dt'$, then we could make the identification of what $\mathbf{F}_{\text{self}}$ is. Notice that the right hand side can be integrated by parts. If we let $u = \ddot{\mathbf{R}}$, and $dv = \ddot{\mathbf{R}} dt'$, we obtain

$$\int_0^t \mathbf{F}_{\text{self}} \cdot \dot{\mathbf{R}} dt' = -\frac{2}{3}\frac{e^2}{c^3}\left\{ \left[\ddot{\mathbf{R}} \cdot \dot{\mathbf{R}} \right]_0^t - \int_0^t \frac{d\ddot{\mathbf{R}}}{dt'} \cdot \dot{\mathbf{R}} dt' \right\}. \tag{2.4}$$

Lorentz wanted to account for all the macroscopic behavior of electrodynamics by considering what happens on the microscopic level of electrons and atoms. He considered the electrons to be elastically bound to the atoms. The motion of the electrons would therefore be periodic. When the time interval is taken over an integral number of periods, the first term in curly brackets of (2.4) vanishes. We then have

$$\int_0^t \mathbf{F}_{\text{self}} \cdot \dot{\mathbf{R}} dt' = \frac{2}{3}\frac{e^2}{c^3} \int_0^t \frac{d\ddot{\mathbf{R}}}{dt'} \cdot \dot{\mathbf{R}} dt',$$

and we can make the identification first obtained by Lorentz; namely that

$$\mathbf{F}_{\text{self}} = \frac{2}{3}\frac{e^2}{c^3}\frac{d\ddot{\mathbf{R}}}{dt},$$

which is the force the electron exerts back on itself as it radiates. Notice that $\mathbf{F}_{\text{self}}$ is proportional to the triple time derivative of the position vector $\mathbf{R}$, or "jerk". It will be to our advantage to define τ to be the following product of fundamental constants

$$\tau = \frac{2e^2}{3mc^3}. \tag{2.5}$$

Notice that τ has the dimensions of time. When using the definition of the classical electron radius, equation (1.4), we see that

$$\tau = \frac{2}{3}\frac{r_o}{c}.$$

In other words, τ is the time it takes for light to travel $(2/3)r_o$. Equation (2.1), the classical equation of motion for the point electron, can now be written as

$$m\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + m\tau\frac{d\ddot{\mathbf{R}}}{dt}. \tag{2.6}$$

This modified equation of motion is not a typical equation found in classical mechanics. Equation (2.6) is a third order differential equation, which requires more initial conditions to be given than the usual initial position and velocity.

In order to compare (2.6) with the extended electron equation of motion, it will be useful to rewrite it in a different form. To do so we use the mass equation $m = m_o + \delta m$ as given in (1.2), as well as (2.5), to rewrite (2.6) as

$$m_o \ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} - \delta m \ddot{\mathbf{R}} + \frac{2}{3} \frac{e^2}{c^3} \frac{d\ddot{\mathbf{R}}}{dt}. \tag{2.7}$$

In this form, we are able to observe the self-mass δm explicitly. Since the self-mass is infinite for a point electron, this term is generally swept under the rug by leaving it combined with the mechanical mass m_o on the left hand side of the equation. However, in the form of (2.7), we will be able to show that the last two terms on the right hand side correspond to the first two terms in an infinite series of the extended model theory.

2.2 **Problems of the Point Theory**

The problems associated with (2.7) were thoroughly studied, but remained unresolved for the better part of this century. There is a vast amount of literature on this subject because of its fundamental importance in trying to describe the motion of elementary charged particles. In this section, we will examine the origin of the problems of (2.7), and discuss their implications.

The equation of motion (2.6) or (2.7) for the point electron suffers from three major defects. First, for a point electron, the electrostatic mass δm is infinite. We will show in section 3.2 of the next chapter that δm, which is the mass associated with the electrostatic self-energy, is

$$\delta m = \int \int d\mathbf{r} d\mathbf{r}' \frac{\rho(r)\rho(r')}{|\mathbf{r} - \mathbf{r}'|},$$

where $\mathbf{r}$ and $\mathbf{r}'$ are two different positions in an extended charge distribution. In the point charge limit, $|\mathbf{r} - \mathbf{r}'| \to 0$, the integrand goes to infinity, and for our choices of $\rho(r)$, $\delta m \to \infty$.

The second problem appears in the solution to (2.6) or (2.7). To obtain a solution, we divide both sides of (2.6) by τm,

$$\frac{d}{dt}\ddot{\mathbf{R}} - \frac{1}{\tau}\ddot{\mathbf{R}}(t) = -\frac{1}{\tau m}\mathbf{F}_{\text{ext}}(t), \tag{2.8}$$

and consider the following expression

$$\begin{aligned}\frac{d}{dt}\left[e^{-t/\tau}\ddot{\mathbf{R}}(t)\right] &= -\frac{1}{\tau}e^{-t/\tau}\ddot{\mathbf{R}}(t) + e^{-t/\tau}\frac{d\ddot{\mathbf{R}}(t)}{dt} \\ &= e^{-t/\tau}\left[\frac{d}{dt}\ddot{\mathbf{R}}(t) - \frac{1}{\tau}\ddot{\mathbf{R}}(t)\right].\end{aligned}$$

Notice that the term in brackets on the right hand side is the same as the left hand side of (2.8). Therefore we can rewrite (2.8) as

$$\frac{d}{dt}\left[e^{-t/\tau}\ddot{\mathbf{R}}\right] = -\frac{1}{\tau m}e^{-t/\tau}\mathbf{F}_{\text{ext}}(t).$$

By integration, we obtain

$$e^{-t/\tau}\ddot{\mathbf{R}}(t) = -\frac{1}{\tau m}\int_0^t e^{-t'/\tau}\mathbf{F}_{\text{ext}}(t')dt' + C'.$$

If we let $\mathbf{F}_{\text{ext}}(0) = 0$ at $t = 0$, the integration constant C' is equal to $\ddot{\mathbf{R}}(0)$. Therefore the solution for the equation of motion (2.6) is

$$\ddot{\mathbf{R}}(t) = e^{t/\tau}\left[\ddot{\mathbf{R}}(0) - \frac{1}{\tau m}\int_0^t e^{-t'/\tau}\mathbf{F}(t')dt'\right]. \tag{2.9}$$

Notice that the right hand side of (2.9) is multiplied by $e^{t/\tau}$. Therefore, as $t \to \infty$, $\ddot{\mathbf{R}}(t) \to \infty$. In other words, the acceleration of the particle increases exponentially regardless of the external force $\mathbf{F}$! Such a condition is obviously unphysical, and is called having *runaway solutions*. It was found that the runaway solutions could be suppressed by choosing the appropriate initial condition

$$\ddot{\mathbf{R}}(0) = \frac{1}{\tau m}\lim_{t\to\infty}\int_0^t dt' e^{-t'/\tau}\mathbf{F}(t').$$

By choosing this as our initial condition, the term in the brackets of (2.9) goes to zero as t goes to infinity. Therefore the acceleration $\ddot{\mathbf{R}}(t)$ goes to zero, and we have managed to suppress the runaway behavior. However, this is not without a price: a third problem arises. Notice that (2.9), with the previous condition inserted, becomes

$$\begin{aligned}
\ddot{\mathbf{R}}(t) &= \frac{e^{t/\tau}}{\tau m}\left[\int_0^\infty dt' e^{-t'/\tau}\mathbf{F}(t') - \int_0^t dt' e^{-t'/\tau}\mathbf{F}(t')\right] \\
&= \frac{e^{t/\tau}}{\tau m}\left[\int_t^\infty dt' e^{-t'/\tau}\mathbf{F}(t')\right] \\
&= \frac{1}{\tau m}\int_t^\infty dt' e^{-(t'-t)/\tau}\mathbf{F}(t').
\end{aligned}$$

If we introduce a change of variables, $s = (t' - t)/\tau$, we obtain the solution

$$\ddot{\mathbf{R}}(t) = \frac{1}{m}\int_0^\infty e^{-s}\mathbf{F}(t + \tau s)ds. \tag{2.10}$$

With the solution in this form, it is clear that the particle begins to accelerate in the future at some time τs before the actual force $\mathbf{F}$ is even applied! This is called *acausal* behavior, or *preacceleration*. In other words the particle moves according to a force that is going to be acting on it at a time τs in the future. Obviously, this is a physically unreasonable result, and undesirable. However, as we discussed in *Chapter 1*, attempts to resolve this and the other problems have been unsuccesful within classical physics. In *Chapter 4* we will see whether these problems carry over in one particular version of nonrelativistic quantum mechanics.

2.3 Solutions With External Forces

If equation (2.6) has the problems mentioned in section 2.2, why has it been so succesful? Why has it been used so much to describe classical radiation reaction? The answer lies in the ability of Lorentz's theory of the electron accurately to describe realistic experimental situations.

The validity of (2.6) in the classical domain was summed up by Gilbert N. Plass about 60 years after it was derived [17]. Plass argued that (2.6) provides an exact classical description of a radiating body. He considered a solution to be physical when the charged particle does not acquire more energy than it obtains from the physical forces which act upon it. Therefore, a nonphysical solution is one in which the particle acquires more acceleration, or more energy, than is put into it by the external forces.

The problem of preacceleration turned out to occur on such a small time scale ($\sim 10^{-23}$s) that it was experimentally undetectable. Wheeler and Feynman showed that the acausal behaviour does not violate any physical laws [18]. Such small times were out of the classical domain anyway, and must be treated quantum mechanically.

Plass considered a number of forces for which analytic solutions could be found (see Table 1 and Table 2). He considered both the nonrelativistic and relativistic cases. He obtained physical solutions for all of the forces listed.

Table 1

External Time-Dependent Forces

Force	Type
$f(t) = k\delta(t - t_o)$	pulse of radiation
$f(t) = mk$	constant force
$f(t) = \begin{cases} 0, & 0 < t < t_o \\ mk, & t_o < t < t_1 \\ 0, & t > t_1 \end{cases}$	constant force for specific time
$f(t) = mkt^n, \quad t > 0$	powerlike force
$f(t) = mk \sin \omega t$	periodic force
$f(t) = \frac{km}{(2\pi\sigma^2)^{1/2}} \exp\left[\frac{-(t-t_o)^2}{2\sigma^2}\right]$	Gaussian force
$f(t) = kme^{ct}$	exponential force
$f(t) = \frac{km}{\lvert t-t_o \rvert^n}$	inverse power force

Table 2

External Space-Dependent Forces

Force	Type
$f(x) = -\alpha mx$	harmonic oscillator
$f(x) = \begin{cases} 0, & x < 0 \\ -mk, & x > 0 \end{cases}$	linear potential wall
$f(x) = \begin{cases} 0, & x < 0 \\ mk, & 0 < x < x_o \\ 0, & x > x_o \end{cases}$	constant force in bounded region
$f(x) = \begin{cases} km, & x < 0 \\ -mk, & x > 0 \end{cases}$	field of thin infinite charged plate

Plass concluded that equation (2.6), although it has theoretical problems should be accepted as the equation of motion for the electron including radiation reaction. His reasoning was that since physical solutions can be found for relevant forces, and any preacceleration effects are not experimentally measurable, this equation was legitimate.

From the work of Plass, we see why equation (2.6) has been so successful. However, the problems inherent in (2.6) are uncomfortable. Perhaps we could gain a greater understanding of particles and their fields by solving these problems. Perhaps by so doing, new physics would be discovered.

CHAPTER 3

THE CLASSICAL EXTENDED ELECTRON

In *Chapter 2* we saw that the equation of motion for the classical point electron has the problems of runaway solutions and preacceleration. We want to see if these problems can be resolved by turning to a classical extended electron. It will also be to our advantage to have the extended electron theory for comparison with the quantum theory developed in *Chapter 4*.

Lorentz never considered the electron to be a point particle. To him, the electron had a physical extension in space. He wanted to explain the mass of this extended particle by electromagnetism. Abraham and Lorentz derived from classical electromagnetic theory the equation of motion for the extended electron including radiation reaction effects [2]. Although successful in deriving the appropriate classical equation, Lorentz was unsuccessful in explaining the mass as he had wished.

We will begin this chapter by closely following Abraham and Lorentz's derivation of the equation of motion including radiation reaction as given in the textbook by Jackson [19]. This derivation is not straightforward. Jackson chooses to leave out many details. Because of the theoretical importance of the Abraham-Lorentz equation, we will consider the details of its derivation in section 3.1.

After deriving the equation of motion, we will investigate its solutions in order to see if the aforementioned problems of the point electron can be resolved by a classical extended model theory. In order to do so, it will be necessary to use some specific examples of charge distributions. In the equation of motion, there are coefficients γ_n which depend on the structure of the model chosen. These coefficients will be derived in section 3.2 for a spherical shell, for a uniform sphere, and for a Yukawa charge distribution.

In section 3.3 we will derive and discuss the solutions for the spherical shell. In section 3.4, we will derive the equation of motion for an arbitrary spherically symmetric charge distribution, and consider the uniform sphere charge distribution as a specific example. It will be seen that the equation of motion for a finite spherical charge distribution in general takes the form of a differential difference equation. The intent in analyzing different models is not to argue that one model is better than another, but to observe the generality of the theory.

Finally, in section 3.5, we will see that a well-behaved extended electron theory cannot produce a well behaved point electron theory in the appropriate limit.

3.1 The Equation of Motion

The electron is the simplest charged particle we know of. And yet, the equation of motion, when radiation reaction effects are included, is intricate to derive and relatively complicated in its final form. Since we are seeking a more satisfying theory of radiation reaction, it will be worth our while to go through the details of the classical derivation.

To derive the equation of motion for the extended electron including radiation reaction, we follow the treatment of Jackson which is itself based on the original calculations of Abraham and Lorentz. We start with the equation of motion for the electron as

$$m\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}} \tag{3.1}$$

where $\mathbf{R}$ is the position vector for the center of charge, $\mathbf{F}_{\text{self}}$ is the self-force back on the electron due to its own field, and $\mathbf{F}_{\text{ext}}$ is every other force acting on the electron, such as an external electric field.

The self-force can be found by adding the contribution of the Lorentz force over the extension of the charge distribution itself:

$$\mathbf{F}_{\text{self}} = \int (\rho \mathbf{E}_{\text{self}} + \frac{1}{c}\mathbf{J} \times \mathbf{B}_{\text{self}}) d\mathbf{r}. \qquad (3.2),$$

where $\mathbf{E}_{\text{self}}$ and $\mathbf{B}_{\text{self}}$ are the electric and magnetic fields created by the charge itself, and $\mathbf{J}$ is the current density as defined in *Chapter 2* above.

In order to evaluate (3.2), it is necessary to say something about the model of the electron. We will make the assumptions that: (a) the particle is instantaneously at rest, and (b) the charge distribution is rigid and spherically symmetric. These assumptions are valid for our slow moving electron, and will greatly simplify the calculations.

Assumption (a) allows us to neglect the self-magnetic terms. Assumption (b) restricts the electron to nonrelativistic motion since it does not allow invariance under Lorentz transfromations. Using these assumptions, and the fact that the electric field can be related to the potentials as

$$\mathbf{E} = -\nabla\phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t),$$

equation (3.2) becomes

$$\mathbf{F}_{\text{self}} = -\int \rho(\mathbf{r},t)\left[\nabla\phi(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right] d\mathbf{r}.$$

The self-potentials can be written in the 4-vector form $A^\alpha = (\phi, \mathbf{A})$, and related back to the charge and current densities:

$$A^\alpha(\mathbf{r},t) = \frac{1}{c}\int \frac{[J^\alpha(\mathbf{r}',t')]_{\text{ret}}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'. \tag{3.3}$$

Notice that the 4-current density components $J^\alpha = (c\rho, \mathbf{J})$ are, as is customary, evaluated at the retarded time t'. This is because we are considering the action of the field created by one point of the extended charge distribution acting on another point in the same charge. Since electromagnetic fields travel at the finite speed of light, we see that the retarded time is related to the time the field was actually emitted by the relation

$$t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c},$$

where the second term on the right hand side is the time it takes the field to travel from one part of the charge distribution located at $\mathbf{r}$ to another part located at $\mathbf{r}'$.

We want to write the current density in terms of t rather than the retarded time t'. Since the time $|\mathbf{r}-\mathbf{r}'|/c$ is on the order of 10^{-23}s or smaller (by assuming that our extended model is on the order of the classical electron radius 3×10^{-15}m and the speed of light is on the order of 3×10^{8}m/s), we can use the Taylor series expansion about some time t

$$\begin{aligned}
[J^\alpha(\mathbf{r}',t')]_{\text{ret}} &= \left[J^\alpha\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)^n \frac{\partial^n}{\partial t'^n}J^\alpha(\mathbf{r}',t')\bigg|_{t'=t} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)^n \frac{\partial^n}{\partial t^n}J^\alpha(\mathbf{r}',t)
\end{aligned}$$

Interchanging the integral and the convergent summation, equation (3.3) now becomes

$$A^\alpha(\mathbf{r},t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int \frac{|\mathbf{r}-\mathbf{r}'|^{n-1}}{c^{n+1}}\frac{\partial^n}{\partial t^n}J^\alpha(\mathbf{r}',t)d\mathbf{r}'.$$

Therefore we can write $\mathbf{F}_{\text{self}}$ as

$$\mathbf{F}_{\text{self}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^n} \int\int d\mathbf{r}d\mathbf{r}' \rho(\mathbf{r},t) \frac{\partial^n}{\partial t^n} \left\{ \nabla \left[\rho(\mathbf{r}',t) |\mathbf{r}-\mathbf{r}'|^{n-1} \right] + \frac{|\mathbf{r}-\mathbf{r}'|^{n-1}}{c^2} \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}',t) \right\}. \tag{3.4}$$

The first term in the brackets of (3.4) is the scalar potential part and can be written as $\rho(\mathbf{r}',t)\nabla|\mathbf{r}-\mathbf{r}'|^{n-1}$ since ρ is independent of $\mathbf{r}$. Looking at the scalar potential term for $n = 0$, we obtain the electrostatic self-force

$$-\int\int d\mathbf{r}d\mathbf{r}' \rho(\mathbf{r},t)\rho(\mathbf{r}',t)\nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right), \tag{3.5}$$

which vanishes for spherically symmetric charge distributions. That (3.5) equals zero can be seen by symmetry. This is an expression for a force, or a vector quantity. However, equation (3.5) has no direction associated with it, because after integration, there is no quantity left to give the direction of the vector $\nabla(1/|\mathbf{r}-\mathbf{r}'|)$. This is because both $\mathbf{r}$ and $\mathbf{r}'$ have been integrated out, leaving no preferred direction. Therefore, (3.5) must be zero.

The $n = 1$ term of the scalar potential part is identically zero since $\nabla|\mathbf{r}-\mathbf{r}'|^0$ is equal to $\nabla 1 = 0$. The sum for the first term in the brackets of equation (3.4) can now be written as

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n!c^n} \int\int d\mathbf{r}d\mathbf{r}' \rho(\mathbf{r},t) \frac{\partial^n}{\partial t^n} \left[\rho(\mathbf{r}',t)\nabla|\mathbf{r}-\mathbf{r}'|^{n-1} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!c^{n+2}} \int\int d\mathbf{r}d\mathbf{r}' \rho(\mathbf{r},t) \frac{\partial^{n+2}}{\partial t^{n+2}} \left[\rho(\mathbf{r}',t)\nabla|\mathbf{r}-\mathbf{r}'|^{n+1} \right]. \tag{3.6}$$

We leave the second term in the brackets of (3.4) (the vector potential) unchanged so that the powers of c will be of the same power in both terms. Using (3.6), equation (3.4) becomes

$$\mathbf{F}_{\text{self}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)\frac{\partial^{n+1}}{\partial t^{n+1}}$$

$$\times \left\{ \frac{\partial\rho(\mathbf{r}',t)}{\partial t}\frac{\nabla|\mathbf{r}-\mathbf{r}'|^{n+1}}{(n+1)(n+2)} + |\mathbf{r}-\mathbf{r}'|^{n-1}\mathbf{J}(\mathbf{r}',t) \right\},$$

or

$$\mathbf{F}_{\text{self}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)|\mathbf{r}-\mathbf{r}'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}$$

$$\times \left\{ \mathbf{J}(\mathbf{r}',t) + \frac{\partial\rho(\mathbf{r}',t)}{\partial t}\frac{\nabla|\mathbf{r}-\mathbf{r}'|^{n+1}}{(n+1)(n+2)|\mathbf{r}-\mathbf{r}'|^{n-1}} \right\}. \tag{3.7}$$

The idea is to find the simplest form possible for $\mathbf{F}_{\text{self}}$. To do so, we use the continuity equation

$$\frac{\partial\rho(\mathbf{r}',t)}{\partial t} + \nabla'\cdot\mathbf{J}(\mathbf{r}',t) = 0.$$

Then, the term in brackets from equation (3.7) can be written as

$$\{\quad\} = \mathbf{J}(\mathbf{r}',t) - \nabla'\cdot\mathbf{J}(\mathbf{r}',t)\frac{\nabla|\mathbf{r}-\mathbf{r}'|^{n+1}}{(n+1)(n+2)|\mathbf{r}-\mathbf{r}'|^{n-1}}.$$

Now,

$$\nabla|\mathbf{r}-\mathbf{r}'|^{n+1} = (n+1)|\mathbf{r}-\mathbf{r}'|^n\nabla|\mathbf{r}-\mathbf{r}'| = (n+1)|\mathbf{r}-\mathbf{r}'|^n\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.$$

If we insert this result into the above expression for the curly brackets, we obtain

$$\{\quad\} = \mathbf{J}(\mathbf{r}',t) - \frac{\mathbf{r}-\mathbf{r}'}{(n+2)}\nabla'\cdot\mathbf{J}(\mathbf{r}',t).$$

Then equation (3.7) takes the form

$$\mathbf{F}_{\mathrm{self}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)|\mathbf{r}-\mathbf{r}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}}$$

$$\times \left\{ \mathbf{J}(\mathbf{r}',t) - \frac{(\mathbf{r}-\mathbf{r}')}{(n+2)} \nabla' \cdot \mathbf{J}(\mathbf{r}',t) \right\}. \tag{3.8}$$

In order to simplify this expression, we can perform "integration by parts" on the second term in brackets above. If we take $\mathbf{u} = |\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')$, and $\mathbf{v} = \mathbf{J}(\mathbf{r}',t)$, and use

$$\int_{\mathrm{V}} \mathbf{u}(\nabla \cdot \mathbf{v}) dv = \int_{\mathrm{S}} \mathbf{u}(\mathbf{v} \cdot \hat{\mathbf{n}}) da - \int_{\mathrm{V}} (\mathbf{v} \cdot \nabla) \mathbf{u} dv,$$

we obtain

$$\int d\mathbf{r}' |\mathbf{r}-\mathbf{r}'|^{n-1} \frac{\mathbf{r}-\mathbf{r}'}{(n+2)} \nabla' \cdot \mathbf{J}(\mathbf{r}',t) =$$

$$\frac{1}{(n+2)} \int_{\mathrm{S}} |\mathbf{r}-\mathbf{r}'|^{n-1} (\mathbf{r}-\mathbf{r}')[\mathbf{J}(\mathbf{r}',t) \cdot \hat{\mathbf{n}}] da$$

$$-\frac{1}{(n+2)} \int d\mathbf{r}' (\mathbf{J} \cdot \nabla') |\mathbf{r}-\mathbf{r}'|^{n-1} (\mathbf{r}-\mathbf{r}'). \tag{3.9}$$

Since we are integrating over all space, the boundary surface S is at infinity. Therefore, the first term on the right hand side goes to zero since $\mathbf{J}(\mathbf{r}',t)$ equals zero at the boundary surface. The integrand in the second term can be rewritten in component form as

$$\begin{aligned}
\left[(\mathbf{J} \cdot \nabla') |\mathbf{r}-\mathbf{r}'|^{n-1} (\mathbf{r}-\mathbf{r}') \right]_j &= J_i \partial_i' (|\mathbf{r}-\mathbf{r}'|^{n-1} (\mathbf{r}-\mathbf{r}')_j) \\
&= J_i (\partial_i' |\mathbf{r}-\mathbf{r}'|^{n-1}) (\mathbf{r}-\mathbf{r}')_j \\
&\quad + J_i |\mathbf{r}-\mathbf{r}'|^{n-1} \partial_i' (\mathbf{r}-\mathbf{r}')_j.
\end{aligned}$$

But $\partial_i(\mathbf{r}-\mathbf{r}')_j$ equals the Kronecker delta $-\delta_{ij}$. Therefore we have

$$\left[(\mathbf{J}\cdot\nabla')|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')\right]_j = J_i(n-1)|\mathbf{r}-\mathbf{r}'|^{n-2}(\partial_i'|\mathbf{r}-\mathbf{r}'|)(\mathbf{r}-\mathbf{r}')_j$$

$$-J_j|\mathbf{r}-\mathbf{r}'|^{n-1}.$$

Now we can use $\partial_i'|\mathbf{r}-\mathbf{r}'| = -(\mathbf{r}-\mathbf{r}')_i/|\mathbf{r}-\mathbf{r}'|$ to write

$$\left[(\mathbf{J}\cdot\nabla')|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')\right]_j = |\mathbf{r}-\mathbf{r}'|^{n-1}\left[-J_j-(n-1)\frac{J_i(\mathbf{r}-\mathbf{r}')_i}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')_j\right]$$

or, in vector form

$$(\mathbf{J}\cdot\nabla')|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}') = -|\mathbf{r}-\mathbf{r}'|^{n-1}\left[\mathbf{J}+(n-1)\frac{\mathbf{J}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\right].$$

Inserting this result into equation (3.9), and then into (3.8), we obtain

$$\mathbf{F}_{self} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)|\mathbf{r}-\mathbf{r}'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\{\quad\} \tag{3.10}$$

where,

$$\begin{aligned}\{\quad\} &= \mathbf{J}(\mathbf{r}',t)-\frac{1}{n+2}\mathbf{J}(\mathbf{r}',t)-\frac{(n-1)}{(n+2)}\frac{\mathbf{J}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\\ &= \frac{(n+1)}{(n+2)}\mathbf{J}(\mathbf{r}',t)-\frac{(n-1)}{(n+2)}\frac{\mathbf{J}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{3.11}$$

The current density and charge distribution are related by

$$\mathbf{J}(\mathbf{r}',t) = \rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)$$

where $\dot{\mathbf{R}}(t)$ is the velocity of the center of charge for a rigid, spherically symmetric electron. Substituting this expression for current density into equation (3.11), we have

$$\{\quad\} = \rho(\mathbf{r}',t)\left[\left(\frac{n+1}{n+2}\right)\dot{\mathbf{R}}(t) - \left(\frac{n-1}{n+2}\right)\frac{\dot{\mathbf{R}}(t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\right]. \qquad (3.12)$$

The expression in curly brackets is now written in terms of the charge density and the velocity of the charge at a given time. We can further simplify the second term in the brackets of equation (3.12) by remembering that for a spherically symmetric charged particle, the only relevant direction after integration is that of the velocity $\dot{\mathbf{R}}(t)$. In other words, only the component of expression (3.12) which is along the velocity axis will survive the spherical integration. To incorporate this direction explicity into (3.12), one simply takes the dot product of the second term with respect to the unit vector $\dot{\mathbf{R}}(t)/\dot{R}$, where $\dot{R}$ is the magnitude of the velocity. With this in mind, we can rewrite the second term in the square brackets of (3.12) in the following way:

$$\left(\frac{n-1}{n+2}\right)\frac{\dot{\mathbf{R}}(t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\cdot\frac{\dot{\mathbf{R}}(t)}{\dot{R}}\left(\frac{\dot{\mathbf{R}}(t)}{\dot{R}}\right) = \left(\frac{n-1}{n+2}\right)\frac{[(\mathbf{r}-\mathbf{r}')\cdot\dot{\mathbf{R}}(t)]^2}{|\mathbf{r}-\mathbf{r}'|^2\dot{R}}\frac{\dot{\mathbf{R}}(t)}{\dot{R}}.$$

Now (3.12) becomes,

$$\{\quad\} = \rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)\left[\left(\frac{n+1}{n+2}\right) - \left(\frac{n-1}{n+2}\right)\frac{[(\mathbf{r}-\mathbf{r}')\cdot\dot{\mathbf{R}}(t)]^2}{|\mathbf{r}-\mathbf{r}'|^2\dot{R}^2}\right]. \qquad (3.13)$$

Remember that we are looking for the simplest possible expression for $\mathbf{F}_{\text{self}}$. Equation (3.13) can be simplified further by noting that all directions of $\mathbf{r}$ and $\mathbf{r}'$ are equally probable. We define θ to be the angle between $(\mathbf{r}-\mathbf{r}')$ and $\dot{\mathbf{R}}$:

$$(\mathbf{r}-\mathbf{r}')\cdot\dot{\mathbf{R}}(t) = |\mathbf{r}-\mathbf{r}'||\dot{\mathbf{R}}(t)|\cos\theta.$$

Using this expression we identify

$$\frac{[(\mathbf{r}-\mathbf{r}')\cdot\dot{\mathbf{R}}(t)]^2}{|\mathbf{r}-\mathbf{r}'|^2\dot{R}^2} = \cos^2\theta$$

which allows us to write (3.13) as

$$\{\quad\} = \rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)\left[\left(\frac{n+1}{n+2}\right) - \left(\frac{n-1}{n+2}\right)\cos^2\theta\right]. \qquad (3.14)$$

In order to simplify even further, we can insert equation (3.14) back into (3.10). Notice that the integral over $\mathbf{r}$ can be simplified by integrating a general function $f(\mathbf{r},t)\cos^2\theta$ over $\mathbf{r}$:

$$\int f(r,\theta,\phi,t)\cos^2\theta d\mathbf{r} = \int r^2 dr \int d\Omega f(r,\theta,\phi,t)\cos^2\theta.$$

Since our charge distribution is spherically symmetric, it does not depend on θ and ϕ. Therefore, we will look at a function $f(r,t)$, leaving us with

$$\int r^2 dr f(r,t) \int d\Omega \cos^2\theta = 2\pi \int r^2 dr f(r,t) \int_{-1}^{1} u^2 du = \frac{4\pi}{3} \int r^2 dr f(r,t) \quad (3.15)$$

where we let $u = \cos\theta$. Since $\frac{1}{4\pi}\int d\Omega = 1$, we can multiply the right hand side of (3.15) by this factor to obtain

$$\int d\mathbf{r} f(r,t)\cos^2\theta = \frac{1}{3}\int d\mathbf{r} f(r,t).$$

Applying this result to what we have when we insert equation (3.14) into (3.10), we see that equation (3.14) can be rewritten as

$$\{\quad\} = \rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)\left[\frac{n+1}{n+2} - \frac{n-1}{n+2}\left(\frac{1}{3}\right)\right].$$

But,

$$\frac{n+1}{n+2} - \frac{n-1}{n+2}\left(\frac{1}{3}\right) = \frac{3n+3-n+1}{3(n+2)} = \frac{2n+4}{3(n+2)} = \frac{2}{3}.$$

So our final expression for Eq. (3.14) is

$$\{\quad\} = \frac{2}{3}\rho(\mathbf{r}',t)\dot{\mathbf{R}}(t).$$

The complicated expression we started with in curly brackets of equation (3.4) has been reduced under the spherically symmetric integration to the simple expression of 2/3 the charge density times its velocity, or 2/3 the current density. Therefore, equation (3.10) for $\mathbf{F}_{\text{self}}$ becomes

$$\mathbf{F}_{\mathrm{self}} = -\frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)|\mathbf{r}-\mathbf{r}'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}[\rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)].$$

We can simplify $\mathbf{F}_{\mathrm{self}}$ further by noticing that the time derivatives of the current density can be written as

$$\frac{\partial^{n+1}}{\partial t^{n+1}}[\rho(\mathbf{r}',t)\dot{\mathbf{R}}(t)] = \sum_{m=0}^{n+1}\binom{n+1}{m}\rho^{(m)}\dot{\mathbf{R}}^{(n+1-m)}. \tag{3.15}$$

Since we are dealing with low velocities at all times, higher powers of the velocity and its derivatives will become negligible. Let us examine the nature of the $n = 0$ term for one component of the expression on the right hand side of (3.15). We now assume explicity rigidity and spherical symmetry for the density:

$$\rho(\mathbf{r}',t) = \rho(|\mathbf{r}' - \mathbf{R}(t)|).$$

The density depends on both space and time, but only as a function of the instantaneous distance to the center of charge. Using the repeated index summation convention, we get:

$$\begin{aligned}
\frac{\partial}{\partial t}[\rho(|\mathbf{r}-\mathbf{R}(t)|)\dot{R}_i(t)] &= \frac{\partial\rho}{\partial t}\dot{R}_i + \rho\frac{\partial\dot{R}_i}{\partial t} \\
&= \frac{\partial\rho}{\partial|\mathbf{r}'-\mathbf{R}|}\frac{\partial}{\partial R_j}|\mathbf{r}'-\mathbf{R}|\dot{R}_j\dot{R}_i + \rho\ddot{R}_i \\
&= -\frac{\partial\rho}{\partial|\mathbf{r}'-\mathbf{R}|}\frac{r'_j - R_j}{|\mathbf{r}'-\mathbf{R}|}\dot{R}_j\dot{R}_i + \rho\ddot{R}_i.
\end{aligned}$$

Notice that the first term on the right hand side is bilinear in the velocity. For $n > 0$ in (3.15) all of the terms will be proportional to the higher powers of the velocity components except for the last term. For $n = 0$, the last term was proportional to $\ddot{R}_i$. For $n = 1$ the last term will be proportional to the triple time derivative of R_i. In other words, all derivatives of ρ with respect to time are multiplied by nonlinear terms in the velocity, and are therefore negligible. Equation (3.10) can therefore be written in its final form as

$$\mathbf{F}_{\mathrm{self}} = -\frac{2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)|\mathbf{r}-\mathbf{r}'|^{n-1}\rho(\mathbf{r}',t)\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t) \tag{3.16}$$

Equation (3.16) is the self-force back on the electron due to its own radiation. In order to compare our results with those of reference [4], we will use the definition for $\rho(\mathbf{r})$ as

$$\rho(\mathbf{r}) \equiv \frac{\text{charge density}}{e}.$$

Therefore, we will add an e^2 to the coefficient in front of the infinite series of (3.16). Now $\mathbf{F}_{\text{self}}$ can be calculated for a given spherical charge distribution $\rho(\mathbf{r}, t)$. Returning to our starting point, namely

$$m_o \ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + \mathbf{F}_{\text{self}}(t)$$

we see that the final form of the equation of motion for a nonrelativistic extended electron including radiation reaction is

$$m_o \ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) - \frac{2e^2}{3c^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! c^n} \gamma_n \frac{d^{n+2}\mathbf{R}(t)}{dt^{n+2}} + (\text{nonlinear terms}), \tag{3.17}$$

where

$$\gamma_n = \int \int d\mathbf{r} d\mathbf{r}' \rho(\mathbf{r}, t) |\mathbf{r} - \mathbf{r}'|^{n-1} \rho(\mathbf{r}', t) \propto L^{n-1}, \tag{3.18}$$

where L is the effective charge radius to be discussed below. This is the equation of motion for an extended electron as derived by Abraham and Lorentz. When we include radiation reaction effects, we see that the equation of motion for the electron is not a simple equation based on the Lorentz force (as is the case when radiation reaction is neglected). Because of the self-force, it now involves an infinite series, which includes time derivatives from second order, up to infinite order (not to mention the non-linear terms).

L is an important parameter that comes into the equations for any spherically symmetric model we choose. For the spherical shell model, L is the radius of the shell. For the uniform spherical charge distribution, L is the radius of the sphere beyond which the charge density is zero. It will turn out that the ratio of L to the classical electron radius will be the critical parameter in order to determine whether the solutions will be well behaved or not.

Since γ_n is proportional to L^{n-1}, we can consider the point charge limit by letting L approach zero. For $n = 2$ and greater, all of the terms in the infinite series would approach zero. The first term ($n = 0$) is proportional to $1/L$, and this is the reason for the infinite self-mass as we will see shortly. If we keep only the first two terms in the infinite series of (3.17), we obtain

$$m_o \ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) - \frac{2e^2}{3c^2}\gamma_o \ddot{\mathbf{R}}(t) + \frac{2e^2}{3c^3}\gamma_1 \frac{d}{dt}\ddot{\mathbf{R}}(t). \tag{3.19}$$

Comparing with the point electron equation (2.7), we see that the electromagnetic mass δm is given by the interaction of the charge density with itself:

$$\delta m = \frac{2e^2}{3c^2}\int\int d\mathbf{r}d\mathbf{r}'\frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}.$$

This is the expression for the self-mass that we mentioned in *Chapter 2*. As we already pointed out, $\delta m \sim 1/|\mathbf{r}-\mathbf{r}'|$. In the point limit, $\mathbf{r}$ and $\mathbf{r}'$ become equal, and $\delta m \to \infty$. However, δm remains finite for any finite extension of the electron. As in the point electron case, we can add the electromagnetic mass, or the mass coming from the self-energy, to the mechanical mass m_o to obtain the physical mass m. In other words, we have found that radiation reaction adds to, or accounts for some of the mass of the electron, and thus for some of the electron's inertia.

Notice that the γ_n in (3.17) are coefficients in the infinite series and are structure dependent as can be seen by equation (3.18). It is through these coefficients only that our choice of an extended model plays a role. Notice also that they are proportional to L^{n-1} independent of the particular model used. This can be seen by looking at the integral (3.18) for γ_n. In order for γ_n to converge, our model must either have a finite extension, or at least drop off to zero as $r \to \infty$. We also see that $\mathbf{r}$ and $\mathbf{r}'$ are integrated out. Now, by looking at the dimensions, we are left with a length to the power $n-1$. Since we are integrating over the extension of the model, it is obvious that γ_n is proportional to L^{n-1}.

We will confine ourselves to spherically symmetric charge distributions. In other words, the charge density will be angular independent. With this in mind, we can rewrite (3.18):

$$\gamma_n = \int\int d\mathbf{r}d\mathbf{r}'\rho(r)|\mathbf{r}-\mathbf{r}'|^{n-1}\rho(r'), \tag{3.20}$$

where

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}.$$

The $\cos\gamma$ under the radical is the angle between $\mathbf{r}$ and $\mathbf{r}'$, and can be written as a function of the spherical coordinates of $\mathbf{r}$ and $\mathbf{r}'$ with respect to an arbitrary orientation:

$$\begin{aligned}\cos\gamma &= \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi' + \cos\theta\cos\theta' \\ &= \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi') + \cos\theta\cos\theta' \\ &= \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'.\end{aligned}$$

Therefore

$$|\mathbf{r}-\mathbf{r}'|^{n-1} = \left\{r^2 + r'^2 - 2rr'\left[\cos(\phi-\phi')\sin\theta\sin\theta' + \cos\theta\cos\theta'\right]\right\}^{(n-1)/2}.$$

We have written the expression $|\mathbf{r}-\mathbf{r}'|$ in terms of r, θ, and ϕ in order to calculate the integrals over the angles in equation (3.20). Now (3.20) can be written as

$$\begin{aligned}\gamma_n = &\int_0^\infty r^2 dr\rho(r)\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi\int_0^\infty r'^2 dr'\rho(r')\int_0^\pi \sin\theta' d\theta'\int_0^{2\pi} d\phi' \\ &\times\left[r^2 + r'^2 - 2rr'\left(\cos(\phi-\phi')\sin\theta\sin\theta' + \cos\theta\cos\theta'\right)\right]^{(n-1)/2}. \qquad (3.21)\end{aligned}$$

Since we are dealing with a spherically symmetric charge distribution, we can simplify (3.21) by letting $\theta' = 0$ and $\phi' = 0$, and multiplying by a factor of 4π. This is because we are integrating over the two arbitrary directions $\mathbf{r}$ and $\mathbf{r}'$, both over the spherically symmetric charge distribution. We are essentially averaging over the angles of the vector $\mathbf{r}'$, obtaining the factor of 4π. We also get a factor of 2π from the integral over ϕ. Therefore, equation (3.21) becomes

$$\gamma_n = 8\pi^2\int_0^\infty r^2 dr\rho(r)\int_0^\pi \sin\theta d\theta\int_0^\infty r'^2 dr'\rho(r')\left[r^2 + r'^2 - 2rr'\cos\theta\right]^{(n-1)/2}. \qquad (3.22)$$

Performing the integration over θ first (called I for convenience) we have

$$I = \int_0^\pi \sin\theta\left[r^2 + r'^2 - 2rr'\cos\theta\right]^{(n-1)/2} d\theta.$$

Now defining $a = r^2 + r'^2$ and $b = 2rr'$,

$$
\begin{aligned}
I &= \int_0^{\pi} \sin\theta[a - b\cos\theta]^{(n-1)/2} d\theta \\
&= \left(\frac{2}{n+1}\right)\frac{1}{b}\left[(a+b)^{(n+1)/2} - (a-b)^{(n+1)/2}\right] \\
&= \left(\frac{2}{n+1}\right)\frac{1}{2rr'}\left[\left(r^2 + r'^2 + 2rr'\right)^{(n+1)/2} - \left(r^2 + r'^2 - 2rr'\right)^{(n+1)/2}\right].
\end{aligned}
$$

We must be very careful with the second term in brackets of the last line. We can factor it as

$$
(r^2 + r'^2 - 2rr')^{(n+1)/2} = \left[(r - r')^2\right]^{(n+1)/2} = |r - r'|^{n+1}.
$$

The last equality follows because the middle term is always positive since $(r - r')^2$ is always positive. If the last term had been written as $(r - r')^{n+1}$, it could be positive or negative. Therefore, the absolute value signs are necessary, and our expression for I becomes

$$
I = \frac{1}{(n+1)rr'}\left[(r + r')^{n+1} - |r - r'|^{n+1}\right].
$$

Notice that in equation (3.17) the sum over n goes from 0 to ∞. Therefore n is never negative, and there are no points in which the integrand blows up. With the integration I over θ inserted, equation (3.22) becomes

$$
\gamma_n = \frac{8\pi^2}{(n+1)}\int_0^{\infty} r dr \rho(r) \int_0^{\infty} r' dr' \rho(r') \left[(r + r')^{n+1} - |r - r'|^{n+1}\right]. \qquad (3.23)
$$

In order to obtain the coefficients γ_n, we will need to choose specific charge distributions. In the next section we will derive the coefficients γ_n for different types of charge distributions. Then, in section 3.3, we will examine the equations of motion for the spherical shell model, and determine the behavior of its solutions.

3.2 Coefficients γ_n for Specific Models

3.2.1 *Coefficients γ_n for the Spherical Shell*

We now want to turn our attention to a specific model. The spherical shell turns out to be a simple model to consider for the electron. The spherical shell is the model chosen by Moniz and Sharp [4] for their work with the classical theory.

We take the radius to be L of the spherical shell on which all of the charge is evenly distributed. Since the spherical shell model is rigid, the charge density ρ is independent of time. Because of the spherical symmetry, ρ is also independent of θ and ϕ. All of the charge resides on the spherical shell, and the charge density takes the form

$$\rho(\mathbf{r}) = \rho(r, \theta, \phi) = \rho(r) \propto \delta(r - L),$$

We require the charge distribution to be normalized so that the charge distribution corresponds to exactly one electron. Therefore

$$\int \rho(\mathbf{r}) d\mathbf{r} = 1,$$

where we use $\rho(\mathbf{r})$ to be

$$\rho(\mathbf{r}) \equiv \frac{\text{charge density}}{e}.$$

Therefore, our normalized spherical charge density takes the form

$$\rho(r) = \frac{\delta(r - L)}{4\pi L^2},$$

which is graphed in *Figure 3.1*.

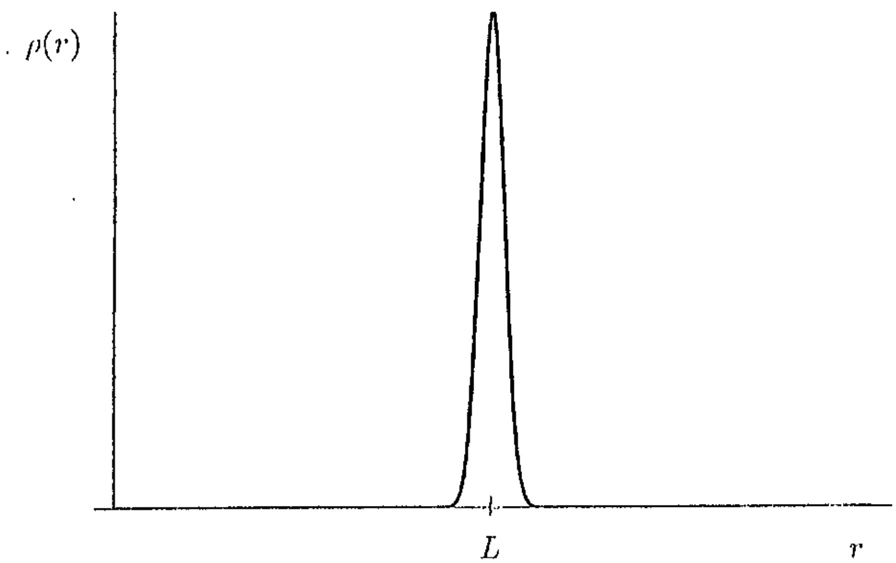

Figure 3.1: *Spherical Shell Charge Distribution.*

Our goal is to find the equation of motion, including radiation reaction effects, for the spherical shell model of the electron. Once this equation has been found, we can investigate its solutions to see if the problems inherent to the point electron theory can be resolved by using an extended model theory. The coefficients γ_n can be found by inserting the charge density for the spherical shell into (3.23), which becomes

$$\gamma_n = \frac{8\pi^2}{(n+1)} \int_0^\infty r dr \frac{\delta(r-L)}{4\pi L^2} \int_0^\infty r' dr' \frac{\delta(r'-L)}{4\pi L^2} \left[(r+r')^{n+1} - |r-r'|^{n+1} \right].$$

The delta functions make the integrals over r and r' trivial, leaving

$$\gamma_n = \frac{2(2L)^{n-1}}{n+1}. \tag{3.24}$$

Notice that this result is proportional to L^{n-1} as we previously anticipated. Also, since the first term in the series of (3.17) is $\delta m \ddot{\mathbf{R}}$, where δm is the mass coming from the electrostatic self-energy, we see that

$$\delta m = \frac{2e^2}{3Lc^2}, \tag{3.25}$$

as was mentioned in *Chapter 1*. Once again, we note that δm is finite except in the point limit $L \to 0$. The experimental mass m is

$$m = m_o + \frac{2e^2}{3Lc^2}, \tag{3.26}$$

where m_o is the mechanical mass, or the mass whose origin is not electromagnetic. This expression for the experimental mass derived by Abraham and Lorentz is exactly the same as (1.2) derived by Thomson, where $f = 2/3$ for the spherical shell.

Since $\gamma_o = 1/L$ and $\gamma_1 = 1$, we see that equation (3.19) is indeed the point charge equation of motion. Our extended classical theory reduces to the classical point theory in the limit as L goes to zero as it should.

In section 3.4, we will insert the γ_n (3.24) into (3.17), sum the series, and determine the behavior of the solutions for the extended model equation of motion. But now, let us consider the γ_n for some other charge distributions, and compare them with the γ_n for the spherical shell.

3.2.2 *Coefficients γ_n for the Uniform Sphere*

Let us choose a uniform spherical charge distribution, and perform calculations similar to those in the last section. The reason for doing this is not necessarily to find a better model of the electron *per se*, but to check the generality of the considerations for the spherical shell. Since the charge distribution is spherically symmetric, $\rho(\mathbf{r}) = \rho(r, \theta, \phi) = \rho(r)$. And for a uniform charge

$$\rho(r) \propto \theta(L - r), \quad \text{where} \quad \theta(L - r) = \begin{cases} 0, & \text{if } r > L; \\ 1, & \text{if } 0 < r < L. \end{cases}$$

We want to normalize the total charge to one, where we again use the definition

$$\rho(r) \equiv \frac{\text{charge density}}{e}.$$

With N as our normalization constant we have

$$\int \rho(r) r^2\, dr\, d\Omega = 4\pi N \int_0^\infty \theta(L-r) r^2\, dr$$
$$= 4\pi N \int_0^L r^2\, dr$$
$$= \frac{4\pi N L^3}{3} = 1.$$

Therefore $N = 3/4\pi L^3$, which means that $\rho(r)$ for the uniform sphere is

$$\rho(r) = \frac{3\theta(L-r)}{4\pi L^3},$$

which is graphed in *Figure 3.2.*

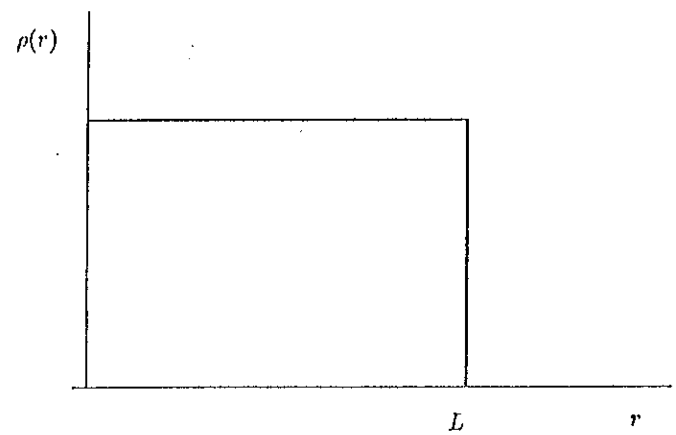

Figure 3.2: *Uniform Sphere Charge Distribution*

Substituting this charge distribution into equation (3.23), we have

$$\gamma_n = \int_0^\infty r dr \frac{3\theta(L-r)}{4\pi L^3} \int_0^\infty r' dr' \frac{3\theta(L-r')}{4\pi L^3} \left[(r+r')^{n+1} - |r-r'|^{n+1}\right]. \quad (3.27)$$

We can split the integrations over r into the two separate integrals

$$\int_0^L r(r+r')^{n+1} dr = \frac{L(L+r')^{n+2}}{(n+2)} - \frac{(L+r')^{n+3}}{(n+2)(n+3)} + \frac{r'^{n+3}}{(n+2)(n+3)}$$

and

$$\int_0^L r|r-r'|^{n+1}dr = \frac{L|L-r'|^{n+2}}{(n+2)} - \frac{|L-r'|^{n+3}}{(n+2)(n+3)} + \frac{|-r'|^{n+3}}{(n+2)(n+3)},$$

where we have used the step functions to change the limits of integration. Using these results, equation (3.27) becomes

$$\gamma_n = \frac{9}{2L^6}\frac{1}{(n+1)(n+2)}\int_0^L r'dr'\left[L(L+r')^{n+2} - \frac{(L+r')^{n+3}}{(n+3)} + \frac{r'^{n+3}}{(n+3)}\right.$$
$$\left. -L|L-r'|^{n+2} + \frac{|L-r'|^{n+3}}{(n+3)} - \frac{|-r'|^{n+3}}{(n+3)}\right]. \tag{3.28}$$

Since r' is always positive, $|-r'| = r'$, and the third and sixth terms in the brackets of (3.28) cancel. The other four integrals over r' can be done separately, yielding

$$L\int_0^L r'(L+r')^{n+2}dr' = L^{n+5}\left[\frac{2^{n+3}(n+2)}{(n+3)(n+4)} + \frac{1}{(n+3)(n+4)}\right],$$

$$-\frac{1}{(n+3)}\int_0^L r'(L+r')^{n+3}dr' = -\frac{L^{n+5}}{(n+3)}\left[\frac{2^{n+4}(n+3)}{(n+4)(n+5)} + \frac{1}{(n+4)(n+5)}\right],$$

$$-L\int_0^L r'(L-r')^{n+2}dr' = \frac{-L^{n+5}}{(n+3)(n+4)},$$

and

$$\frac{1}{(n+3)}\int_0^L r'(L-r')^{n+3}dr' = \frac{L^{n+5}}{(n+3)(n+4)(n+5)}.$$

Substituting all of these values into (3.28), we obtain

$$\gamma_n = \frac{9L^{n+5}}{2L^6(n+1)(n+2)(n+3)}\left[\frac{2^{n+3}(n+2)}{(n+4)} + \frac{1}{(n+4)} - \frac{2^{n+4}(n+3)}{(n+4)(n+5)}\right.$$
$$\left. - \frac{1}{(n+4)(n+5)} - \frac{1}{(n+4)} + \frac{1}{(n+4)(n+5)}\right].$$

All of the terms in brackets cancel each other except for the first and third. Therefore, the terms in brackets become

$$\left[\frac{2^{n+3}(n+2)}{(n+4)} - \frac{2^{n+4}(n+3)}{(n+4)(n+5)}\right] = \frac{2^{n+3}}{(n+4)}\left[n+2-\frac{2(n+3)}{(n+5)}\right] = \frac{2^{n+3}}{(n+5)}.$$

And our final expression for the coefficients is

$$\gamma_n = \frac{9L^{n-1}2^{n+2}}{(n+2)(n+3)(n+5)}. \tag{3.29}$$

This result was first obtained by Herglotz in 1903 [20], and later reaffirmed by Wildermuth [21].

Comparing (3.29) with (3.24), we see that the coefficients γ_n have become a little more complicated for the uniform sphere than for the spherical shell. Again we see that γ_n is proportional to L^{n-1}. Also, setting $n = 0$, we see that $\gamma_o = 6/5L$. We can find δm by inserting this value into the first term of the series of equation (3.17), since the first term is equal to $\delta m\ddot{\mathbf{R}}$. Doing so yields

$$\delta m = \frac{4}{5}\frac{e^2}{Lc^2} \tag{3.30}$$

for the uniform sphere. This is the value we stated in *Chapter 1*. As for the spherical shell, $\gamma_1 = 1$. Therefore, in the point charge limit, we again obtain (2.7) from (3.19). For the uniform sphere, the experimental mass m is

$$m = m_o + \frac{4e^2}{5Lc^2}. \tag{3.31}$$

Notice that even though the spherical shell and the uniform sphere models of the electron have the same charge e, the mass coming from the electrostatic self-energy is not the same. The uniform sphere has the larger self-mass, because the charge is uniformly distributed throughout the sphere. This allows one piece of the charge to interact more intensely with another piece of the same charge because they are closer together in general than if all of the charge were on the surface. The more compact the charge is, the greater δm will be. And as we have already pointed out, in the extreme case when all of the charge is concentrated at a single point, δm is infinite. This is not to say that the observed mass is different for the two different models. As the self-mass gets larger, the mechanical mass gets smaller, maintaining a constant value for the experimental mass.

3.2.3 *Coefficients γ_n for a Yukawa Distribution*

So far, we have considered the γ_n for two finite charge distributions. In this section, we will consider a charge distribution which is not finite, but which tapers off to zero rapidly enough so that the integrals in (3.23) converge. This particular model is called a Yukawa distribution. When working with quantum mechanics in *Chapter 4*, we will use a Yukawa distribution as an example to show that $\delta m = 0$ in the point charge limit. The particular distribution we will use, as written in k-space, is

$$\tilde{\rho}(\mathbf{k}) = (1 + k^2 L^2)^{-1}.$$

In order to see what results can be obtained in the classical theory for this charge distribution, it will be necessary to transform it into $\mathbf{r}$-space. We will use the Fourier transform as

$$\rho(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \tilde{\rho}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}.$$

The integrals can easily be evaluated by using the computer program "Mathematica," from which we obtain

$$\rho(r) = \frac{1}{4\pi L^2} \frac{e^{-r/L}}{r}. \tag{3.32}$$

We have graphed this charge distribution in *Figure 3.3*.

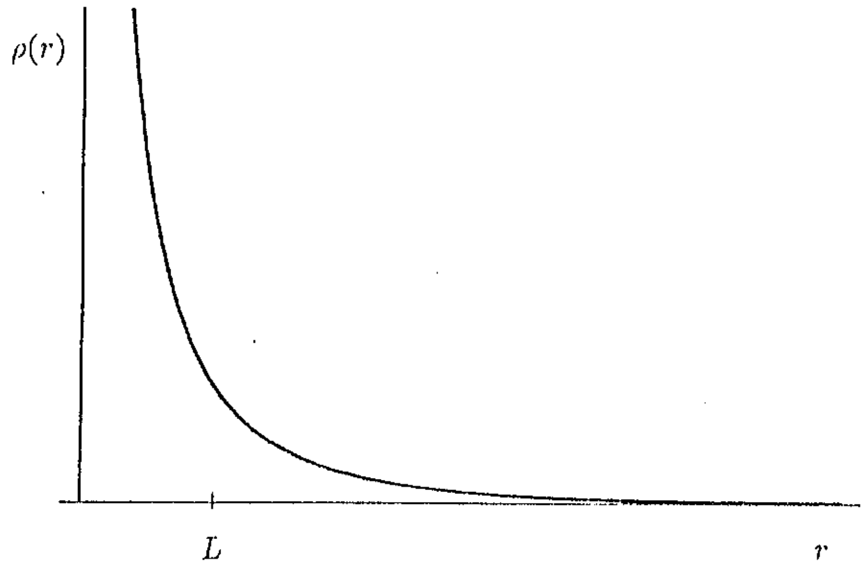

Figure 3.3 *Yukawa Charge Distribution*

Notice that our size parameter L is no longer the radius, but is used to maintain the correct dimensions. In order to obtain the γ_n for the Yukawa model, we substitute (3.32) into (3.23), which becomes

$$\gamma_n = \frac{1}{2L^4(n+1)} \int_0^\infty e^{-r/L} dr \int_0^\infty e^{-r'/L} dr' [(r+r')^{n+1} - |r-r'|^{n+1}].$$

Or, rewriting without the absolute value signs, we have

$$\gamma_n = \frac{1}{2L^4(n+1)} \int_0^\infty e^{-r/L} dr \left\{ \int_0^r e^{-r'/L} dr' [(r+r')^{n+1} - (r-r')^{n+1}] \right. \\ \left. + \int_r^\infty e^{-r'/L} dr' [(r+r')^{n+1} - (r'-r)^{n+1}] \right\}. \tag{3.33}$$

We will do each of the integrals over r' in (3.33) separately, beginning with the integral

$$I_1 = \int_0^r e^{-r'/L} dr' (r+r')^{n+1}.$$

This integral can be evaluated by using the u substitution $u = r + r'$ and $du = dr'$, in which case we have

$$I_1 = e^{r/L} \int_r^{2r} u^{n+1} e^{-u/L} du.$$

With the integral in this form, we see that it can be evaluated using integration by parts $n+1$ times, which yields the result

$$I_1 = e^{r/L} \left[e^{-u/L} \sum_{R=0}^{n+1} \frac{(-1)^R (n+1)! u^{n+1-R}}{(n+1-R)!} \left(-\frac{1}{L} \right)^{R+1} \right]_r^{2r}.$$

Notice that $(-1)^R/(-1)^{R+1} = -1$. Therefore,

$$I_1 = \sum_{R=0}^{n+1} \frac{(n+1)! L^{R+1} r^{n+1-R}}{(n+1-R)!} [1 - 2^{n+1-R} e^{-r/L}].$$

Again using u substitutions, we see that the next two integrals are

$$\begin{aligned} I_2 &= -\int_0^r e^{-r'/L} (r - r')^{n+1} dr' \\ &= -e^{-r/L} \int_0^r u^{n+1} e^{u/L} du \\ &= -e^{-r/L} \left[e^{u/L} \sum_{R=0}^{n+1} \frac{(-1)^R (n+1)! u^{n+1-R}}{(n+1-R)! \left(\frac{1}{L}\right)^{R+1}} \right]_0^r \\ &= -e^{-r/L} (-1)^n (n+1)! L^{n+2} - \sum_{R=0}^{n+1} \frac{(-1)^R (n+1)! r^{n+1-R} L^{R+1}}{(n+1-R)!}, \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_r^\infty e^{-r'/L} (r + r')^{n+1} dr' \\ &= -e^{r/L} \left[e^{-u/L} \sum_{R=0}^{n+1} \frac{(n+1)! u^{n+1-R} L^{R+1}}{(n+1-R)!} \right]_{2r}^\infty. \end{aligned}$$

In the limit as $u \to \infty$, the exponential term $e^{-u/L} \to 0$. This is multiplied by $u^{n+1-R} \to \infty$ in the limit, but is killed by the exponential term going to zero. Hence,

$$I_3 = e^{-r/L} \sum_{R=0}^{n+1} \frac{(n+1)!L^{R+1}(2r)^{n+1-R}}{(n+1-R)!}.$$

The last integral over r' is

$$\begin{aligned}
I_4 &= -\int_r^\infty e^{-r'/L} dr' (r'-r)^{n+1} \\
&= -e^{-r/L} \int_0^\infty u^{n+1} e^{-u/L} du \\
&= e^{-r/L} \left[e^{-u/L} \sum_{R=0}^{n+1} \frac{(n+1)!L^{R+1} u^{n+1-R}}{(n+1-R)!} \right]_0^\infty \\
&= -e^{-r/L}(n+1)!L^{n+2}.
\end{aligned}$$

When the results I_1 through I_4 are inserted back into (3.33), γ_n becomes

$$\begin{aligned}
\gamma_n = \frac{1}{2L^4(n+1)} \int_0^\infty e^{-r/L} dr \Bigg\{ & \sum_{R=0}^{n+1} \frac{(n+1)!L^{R=1} r^{n+1-R}}{(n+1-R)!} \\
& - e^{r/L} \sum_{R=0}^{n+1} \frac{(n+1)L^{R+1}(2r)^{n+1-R}}{(n+1-R)!} - e^{-r/L}(-1)^n (n+1)!L^{n+2} \\
& - \sum_{R=0}^{n+1} \frac{(-1)^R (n+1)! r^{n+1-R} L^{R+1}}{(n+1-R)!} + e^{-r/L} \sum_{R=0}^{n+1} \frac{(n+1)!L^{R+1}(2r)^{n+1-R}}{(n+1-R)!} \\
& - e^{-r/L}(n+1)!L^{n+2} \Bigg\}.
\end{aligned}$$

The second and fifth terms in brackets cancel each other. Notice that the first and fourth terms cancel each other if R is even, and add to two times the first if R is odd. Also, the third and sixth terms cancel for n odd, and add to two times the third for n even. Thus,

$$\gamma_n = \frac{1}{L^4(n+1)} \int_0^\infty e^{-r/L} dr \left\{ \begin{array}{ll} \sum_{R=0}^{n+1} \frac{(n+1)!L^{R+1} r^{n+1-R}}{(n+1-R)!}, & R \text{ odd} \\ 0, & R \text{ even} \end{array} \right\}$$

$$- \frac{1}{L^4(n+1)} \int_0^\infty e^{-r/L} \left\{ \begin{array}{ll} e^{-r/L}(n+1)!L^{n+2}, & n \text{ even} \\ 0, & n \text{ odd} \end{array} \right\}. \quad (3.34)$$

We now have two integrations over r to perform. They boil down to

$$\begin{aligned} \int_0^\infty e^{-r/L} r^{n+1-R} dr &= -e^{-r/L} \sum_{K=0}^{n+1-R} \frac{(n+1-R)!L^{K+1} r^{n+1-R-K}}{(n+1-R-K)!} \Bigg|_0^\infty , \\ &= (n+1-R)!L^{n+2-R} \end{aligned}$$

and

$$\int_0^\infty e^{-2r/L} dr = -\frac{L}{2} e^{-2r/L} \Bigg|_0^\infty = \frac{L}{2}.$$

Substituting these results back into (3.34), we obtain

$$\gamma_n = \left\{ \begin{array}{ll} \sum_{R=0}^{n+1} n!L^{n-1}, & R \text{ odd} \\ 0, & R \text{ even} \end{array} \right\} - \left\{ \begin{array}{ll} \frac{n!L^{n-1}}{2}, & n \text{ even} \\ 0, & n \text{ odd} \end{array} \right\}.$$

Notice the summation over R in the first set of curly brackets. All of the R dependence has canceled out. Therefore,

$$\sum_{R \text{ odd}}^{n+1} = \left\{ \begin{array}{ll} \frac{n+2}{2}, & \text{for } n \text{ even;} \\ \frac{n+1}{2}, & \text{for } n \text{ odd.} \end{array} \right.$$

With this result, (3.34) becomes

$$\gamma_n = \left\{ \begin{array}{ll} n!L^{n-1} \frac{n+2}{2}, & n \text{ even} \\ n!L^{n-1} \frac{n+1}{2}, & n \text{ odd} \end{array} \right\} - \left\{ \begin{array}{ll} \frac{n!L^{n-1}}{2}, & n \text{ even} \\ 0, & n \text{ odd} \end{array} \right\}.$$

Equation (3.34) can be simplified one step further by considering the terms for n even and n odd seperately.

n even

$$\begin{aligned}\gamma_n &= n!L^{n-1}\frac{n+2}{2} - \frac{n!L^{n-1}}{2} \\ &= \frac{n!L^{n-1}}{2}(n+2-1) \\ &= \frac{(n+1)!L^{n-1}}{2}.\end{aligned}$$

n odd

$$\begin{aligned}\gamma_n &= n!L^{n-1}\frac{n+1}{2} \\ &= \frac{(n+1)!L^{n-1}}{2}.\end{aligned}$$

We obtain the same result for n even and odd. Thus we have the final expression for γ_n in the form

$$\gamma_n = \frac{(n+1)!L^{n-1}}{2}. \tag{3.35}$$

These coefficients for the Yukawa distribution were harder to obtain than the previous charge distributions considered. However, in the end, the expression for the γ_n turned out to be fairly simple. It is interesting to compare (3.35) with (3.24) and (3.29).

The coefficient $\gamma_o = 1/(2L)$ can be inserted into the sum of equation (3.17) for $n = 0$ to obtain

$$\delta m = \frac{e^2}{3Lc^2} \tag{3.36}$$

which is the self-mass of the Yukawa distribution. Notice that $\gamma_1 = 1$, and we again obtain equation (2.7) from (3.19), which is the point limit. The experimental mass for the Yukawa model of the electron is

$$m = m_o + \frac{e^2}{3Lc^2}. \tag{3.37}$$

We would expect the self-energy of the Yukawa to be less than that for the uniform sphere because the Yukawa is more spread out as can be seen in Figure 3.3. In comparing (3.36) with (3.30), we see that the self-mass, and therefore the self-energy, for the Yukawa is smaller than that for the uniform sphere. In comparing with (3.25), notice that the Yukawa self-mass is exactly one-half the spherical shell self-mass. The results for the three charge distributions considered are summarized in Table 3.

Table 3.

Distribution	$\rho(r)$	γ_n	δm
Spherical Shell	$\frac{\delta(r-L)}{4\pi L^2}$	$\frac{2(2L)^{n-1}}{n+1}$	$\frac{2}{3}\frac{e^2}{Lc^2}$
Uniform Sphere	$\frac{3\theta(L-r)}{4\pi L^3}$	$\frac{9L^{n-1}2^{n+2}}{(n+2)(n+3)(n+5)}$	$\frac{4}{5}\frac{e^2}{Lc^2}$
Yukawa	$\frac{1}{4\pi L^2}\frac{e^{-r/L}}{r}$	$\frac{(n+1)!L^{n-1}}{2}$	$\frac{1}{3}\frac{e^2}{Lc^2}$

3.3 The Equation of Motion and Behavior of the Solutions for the Spherical Shell

We now return to the spherical shell model of the electron in order to determine if a well behaved theory can be obtained. To do so, we will need to find the solutions to the equation of motion, and determine the behavior of these solutions.

In order to obtain the equation of motion, we need to insert (3.24) into (3.17). The series can be summed by the following steps:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\frac{2(2L)^{n-1}}{(n+1)}\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t) \\
&= \frac{2(2L)^{-1}}{1!}\frac{d^2}{dt^2}\mathbf{R}(t) - \frac{2}{2!c}\frac{d^3}{dt^3}\mathbf{R}(t) + \frac{2(2L)}{3!c^2}\frac{d^4}{dt^4}\mathbf{R}(t) + \cdots \\
&= \frac{c}{2L^2}\left[\frac{(2L)}{1!c}\frac{d}{dt} - \frac{(2L)^2}{2!c^2}\frac{d^2}{dt^2} + \frac{(2L)^3}{3!c^3}\frac{d^3}{dt^3} + \cdots\right]\dot{\mathbf{R}}(t) \\
&= \frac{-c}{2L^2}\left[1 - \frac{(2L)}{1!c}\frac{d}{dt} + \frac{(2L)^2}{2!c^2}\frac{d^2}{dt^2} - \frac{(2L)^3}{3!c^3}\frac{d^3}{dt^3} + \cdots - 1\right]\dot{\mathbf{R}}(t),
\end{aligned} \tag{3.38}$$

where we have added and subtracted 1 in the last line. The term in brackets of the last line, excluding the -1, sums to $e^{-(2L/c)(d/dt)}$. So we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\frac{2(2L)^{n-1}}{(n+1)}\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t) = \frac{-c}{2L^2}\left[e^{-(2L/c)(d/dt)} - 1\right]\dot{\mathbf{R}}(t). \tag{3.39}$$

The equation of motion (3.17) can therefore be written as

$$m_o\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + \left(\frac{c}{2L}\right)\left(\frac{2e^2}{3c^2L}\right)\left[e^{-(2L/c)(d/dt)} - 1\right]\dot{\mathbf{R}}(t). \tag{3.40}$$

The term $e^{-(2L/c)(d/dt)}$ is a time-delay operator, which we will write in the form

$$e^{-(2L/c)(d/dt)}\dot{\mathbf{R}}(t) = \dot{\mathbf{R}}(t - 2L/c). \tag{3.41}$$

The validity of equation (3.41) can be seen by taking the Taylor series expansion of both sides and comparing term by term. Using $\tau = 2e^2/3mc^3$ and equation (3.41) in (3.40), and dividing both sides by the experimental mass m, we have

$$\frac{m_o}{m}\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}(t)}{m} + \left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)[\dot{\mathbf{R}}(t - 2L/c) - \dot{\mathbf{R}}(t)].$$

Using the result for δm found in expression (3.25), and the expression for τ above, we can write

$$\delta m = \frac{2e^2}{3Lc^2} = \frac{mc\tau}{L},$$

from which we obtain

$$\frac{m_o}{m} = \frac{m - \delta m}{m} = 1 - \frac{c\tau}{L}.$$

Therefore, the final form for the equation of motion becomes

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}(t)}{m(1 - c\tau/L)} + \xi[\dot{\mathbf{R}}(t - 2L/c) - \dot{\mathbf{R}}(t)], \tag{3.42}$$

where we have used the parameter

$$\xi = \frac{(c/2L)(c\tau/L)}{(1 - c\tau/L)} = \frac{1}{2\tau}\frac{(c\tau/L)^2}{(1 - c\tau/L)}.$$

The parameter ξ has dimensions of frequency. Notice that for $0 < L < c\tau$, ξ is negative, whereas for $L > c\tau$, ξ is positive. Also, for $L = c\tau$, ξ is infinite. ξ also goes to infinity if we take the point limit ($L \to 0$). But if we let the radius L go to infinity, ξ goes to zero. We will see that it is the sign of ξ that is important in the construction of a well behaved theory for the extended electron.

The equation (3.42) is called a differential-difference equation. Such an equation involves the difference between two differentials of the same function, but one is evaluated at a different time than the other. With the equation of motion in this form, we are ready to determine the behavior of its solutions. In order to simplify the calculations, we want to examine solutions when there are no external forces present. But first we show the point limit.

In section 3.1 we showed the point limit of equation (3.17) to be equation (3.19). We can check (3.42) by taking the point limit. To be consistent, it should reduce to equation (2.7). In order to take the point limit, we can write (3.42) as

$$m(1-c\tau/L)\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + m(c/2L)(c\tau/L)[\dot{\mathbf{R}}(t-\tfrac{2L}{c}) - \dot{\mathbf{R}}(t)].$$

Expanding the term $\dot{\mathbf{R}}(t-\frac{2L}{c})$ in a Taylor series, we have

$$m\ddot{\mathbf{R}} - \frac{mc\tau}{L}\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + \frac{mc^2\tau}{2L^2}[\dot{\mathbf{R}} - \frac{2L}{c}\ddot{\mathbf{R}} + \frac{4L^2}{2!c^2}\frac{d}{dt}\ddot{\mathbf{R}} + \cdots - \dot{\mathbf{R}}],$$

which reduces to

$$m\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} + \frac{2e^2}{3c^3}\frac{d}{dt}\ddot{\mathbf{R}}.$$

And since $m = m_o + \delta m$, this is indeed equation (2.7) derived for the point electron.

The electron is in its own field. It feels a force back on itself due to its own radiation. In section 2.2 we found that from this self-force for the point electron came the problems of runaway solutions, and preacceleration. Now we want to see if the solutions are well behaved in the case of an extended model, and if so, under what conditions.

When there are no external forces involved, the equation of motion (3.42) becomes

$$\ddot{\mathbf{R}}(t) = \xi[\dot{\mathbf{R}}(t-2L/c) - \dot{\mathbf{R}}(t)]. \tag{3.43}$$

Consider solutions of the form

$$\dot{\mathbf{R}}(t) = \mathbf{A}e^{\beta t/\tau}$$

where $\mathbf{A}$ is a constant, nonzero vector, and β is an arbitrary constant. We can now write the acceleration as

$$\ddot{\mathbf{R}}(t) = \frac{d}{dt}\dot{\mathbf{R}} = \frac{d}{dt}\mathbf{A}e^{\beta t/\tau} = \mathbf{A}\left(\frac{\beta}{\tau}\right)e^{\beta t/\tau}.$$

Inserting these expressions for velocity and acceleration into equation (3.43), and canceling the term $\mathbf{A}e^{\beta t/\tau}$, we obtain a transcendental equation for β as

$$\frac{\beta}{\tau} = \xi\left(e^{-(2L/c)(\beta/\tau)} - 1\right). \tag{3.44}$$

For every solution β of (3.44), there is a corresponding solution to the homogeneous equation (3.43). In order to obtain a general solution, we must take a linear superposition of these solutions with arbitrary coefficients. Since the velocity $\dot{\mathbf{R}}$ is proportional to $\exp(\beta t/\tau)$, if any of the roots β have a positive real part, we will obtain undesirable runaway solutions rather than oscillatory solutions coming from imaginary parts.

Consider the three cases: $L > c\tau$, $L = c\tau$, and $L < c\tau$.

1. $L > c\tau$

In order to discover the behavior of the solutions, we use the dimensionless variables $\eta = 2L\beta/c\tau$ and $g = 1/(L/c\tau - 1)$. Since L, c and τ are real, η and β will have real and imaginary parts simultaneously. Therefore, $\eta \equiv \mu + i\nu$ where μ and ν are real. Now equation (3.44) can be written as

$$\frac{\beta}{\tau} = \frac{c\eta}{2L} = -\xi(1 - e^{-\eta}) = -\left(\frac{c^2\tau}{2L^2}\right)\frac{1}{(1 - c\tau/L)}(1 - e^{-\eta}).$$

Therefore

$$\begin{aligned}
\eta &= \frac{-c\tau}{L(1 - c\tau/L)}(1 - e^{-\eta}) \\
&= \frac{-1}{(L/c\tau - 1)}(1 - e^{-\eta}) \\
&= -g(1 - e^{-\eta}) \\
&= -g + ge^{-(\mu+i\nu)} \\
&= -g + ge^{-\mu}(\cos\nu - i\sin\nu) \\
&= -g(1 - e^{-\mu}\cos\nu) - ige^{-\mu}\sin\nu.
\end{aligned} \tag{3.45}$$

Using the definition for η above, we have the results

$$\mu = -g(1 - e^{-\mu}\cos\nu), \tag{3.46a}$$

and

$$\nu = -ge^{-\mu}\sin\nu. \tag{3.46b}$$

$L > c\tau$ implies that g will always be positive. If we look at (3.46a), which is the real part of η, we see that it will always be negative for $L > c\tau$. This is because equation (3.46a) can only be satisfied when μ is negative. Therefore, if the real part of η is always negative, β will only have a negative real part. But if the real part of β is always negative, the velocity

$$\dot{\mathbf{R}}(t) = \mathbf{A}e^{\beta t/\tau}$$

will go to zero as $t \to \infty$, which implies that there will be no runaway solutions. In other words, for $L > c\tau$, only well-behaved solutions exist. Exact solutions could be found by solving (3.46) graphically (see reference [4]). But what happens when $L = c\tau$, or when L is less than $c\tau$? How small can the model become, and still make sense?

2. $L = c\tau$

Again, we start with equation (3.43), but this time we divide both sides by ξ:

$$\frac{\ddot{\mathbf{R}}(t)}{\xi} = \ddot{\mathbf{R}}\frac{(1 - c\tau/L)}{(c/2L)(c\tau/L)} = [\dot{\mathbf{R}}(t - 2L/c) - \dot{\mathbf{R}}(t)].$$

Since $L = c\tau$ we have

$$\dot{\mathbf{R}}(t - 2L/c) - \dot{\mathbf{R}}(t) = 0. \tag{3.47}$$

Consider solutions of the form $\dot{\mathbf{R}}(t) = \mathbf{A}e^{\beta t/\tau}$ as before, and insert them into equation (3.47), which yields

$$e^{-2L\beta/c\tau} - 1 = 0. \tag{3.48}$$

Equation (3.48) is satisfied for

$$\beta = 2\pi i n, \qquad n = 0, 1, 2, ... \tag{3.49}$$

This can easily be shown by inserting (3.49) into equation (3.48):

$$e^{-2L\beta/c\tau} = e^{-i4\pi Ln/c\tau} = \cos\frac{4\pi Ln}{c\tau} - i\sin\frac{4\pi Ln}{c\tau} = 1.$$

Since $L = c\tau$, we see that the sine term is always zero, whereas the cosine term is always one. Therefore, we find (3.49) to be the correct solution of equation (3.48). In this case β is purely imaginary, which implies no runaway solutions. we have found that as long as L is greater than, or equal to $c\tau$, a well behaved theory exists.

3. $L < c\tau$

In this case g is negative. Using the same definition of μ and ν, (3.45) becomes

$$\begin{aligned} \eta &= |g|(1 - e^{-\eta}) \\ &= |g| - |g|e^{-(\mu+i\nu)} \\ &= |g|\left(1 - e^{-\mu}\cos\nu\right) + i|g|e^{-\mu}\sin\nu. \end{aligned} \tag{3.50}$$

This time, the equation

$$\mu = |g|\left(1 - e^{-\mu}\cos\nu\right).$$

is satisfied for μ positive. Therefore, β always has a positive real part. Thus, for $0 \le L < c\tau$, equation (3.43) has runaway solutions.

In summary, one finds that for $L \ge c\tau$ ($\xi > 0$), equation (3.29) has no runaway, and therefore no preaccelerating solutions. However, for $L < c\tau$ ($\xi < 0$) runaway and acausal solutions return. Notice what these results imply. If we begin with the equation $m = m_o + 2e^2/3Lc^2 = m_o + mc\tau/L$, we see that $m(1 - c\tau/L) = m_o$. For $L < c\tau$, this equation implies that m_o is negative. In other words, the aforementioned problems creep into the theory only when one assumes a negative mechanical mass. Taking the mechanical mass to be positive, one obtains a consistent (well-behaved) theory.

The fact that L must be greater than $c\tau$ is interesting because it shows that the problems associated with the classical point electron (except for infinite self-energy) also apply to the extended charge model. Unless the model's extension is greater than $c\tau = (2/3)r_o$, where $r_o = e^2/mc^2$ is the classical electron radius, the inherent problems remain. So the problems are not confined to the point electron theory.

Before quantum theory, Lorentz and others were looking for an actual model of the electron. Since the birth of quantum mechanics, such efforts have almost been relinquished [22]. However, if we were searching for such a "real" model for the electron, the spherical shell model would fall short. The classical electron radius is of the order of 10^{-15}m. Therefore, L must be greater than $(2/3)r_o \sim 10^{-15}$m. But the electron, by scattering experiments, seems to be smaller than 10^{-17}m [23]. Thus, this particular model requires a size beyond the experimental upperbound of the electron.

3.4 The Equation of Motion and Solutions for a Spherical Charge Distribution in General

The spherical shell turns out to be a simple model to consider. For the other two charge distributions we have considered, the approach followed in the last section is much more difficult. We have not been able to derive the differential-difference equation for the other charge distributions using this approach. However, by using a coordinate transformation, the equation of motion for a general spherically symmetric charge distribution can be found. We will derive this general equation, and then look again at the spherical shell charge distribution in 3.4.1 to show the validity and simplifications obtained by using it. Then, in 3.4.2, these general equations will be used for the uniform sphere in order to obtain the accompanying equation of motion, and discuss the behavior of its solutions. We will not consider the Yukawa distribution, however, since the results in this section are for finite charge distributions only.

The differential-difference equations can be derived in general for any spherically symmetric charge distribution. We use equation (3.23), and assume that the charge density is zero beyond some radius L. In this case (3.23) becomes

$$\gamma_n = \frac{8\pi^2}{n+1}\int_0^L rdr \int_0^L r'dr' \left[(r+r')^{n+1} - |r-r'|^{n+1}\right]\rho(r)\rho(r'). \tag{3.51}$$

We first want to simplify this equation by using the transformation

$$r = (y+y')L, \qquad \text{and} \qquad r' = |y-y'|L,$$

The charge distribution in r and r', and the transformation, are shown in *Figure 3.4.*

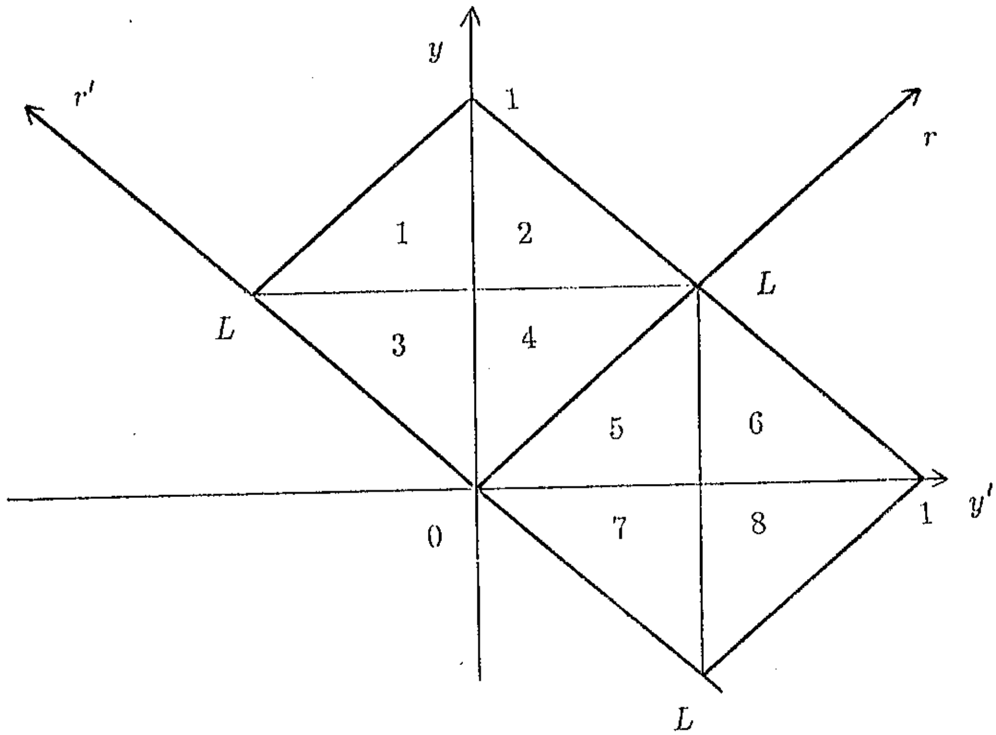

Figure 3.4: *Transformation of r and r' to y and y'.*

In order to rewrite (3.51) in the (y, y') basis, we will begin by transforming the volume element $drdr'$ by using the Jacobian [24]

$$drdr' = \left\| \begin{array}{cc} \frac{\partial r}{\partial y} & \frac{\partial r}{\partial y'} \\ \frac{\partial r'}{\partial y} & \frac{\partial r'}{\partial y'} \end{array} \right\| dydy'.$$

Because of the absolute value signs in the transformation equation for r', we have the two possibilities

$$r' = |y - y'|L = \begin{cases} (y - y')L, & \text{for } y > y'; \\ \\ (y' - y)L, & \text{for } y < y'. \end{cases}$$

Therefore, we have

$$
drdr' = \left\| \begin{matrix} L & L \\ L & -L \end{matrix} \right\| dydy' = 2L^2 dydy',
$$

or

$$
drdr' = \left\| \begin{matrix} L & L \\ -L & L \end{matrix} \right\| dydy' = 2L^2 dydy'.
$$

Performing the transformation, equation (3.51) becomes

$$
\begin{aligned}
\gamma_n = \frac{16\pi^2 L^2}{n+1} \int\int dydy'(y+y')L|y-y'|L\rho[(y+y')L]\rho[|y-y'|L] \\
\times \left\{ [(y+y')L + |y-y'|L]^{n+1} - |(y+y')L - |y-y'|L|^{n+1} \right\}. \qquad (3.52)
\end{aligned}
$$

We will first consider the case $y \geq y'$. Notice that for this case, y is always greater than, or equal to y', and the charge distribution corresponds to triangles 1 through 4 in Figure 3.4. Therefore,

$$
r' = |y-y'|L = (y-y')L,
$$

Introducing the shorthand notation $\rho_+ = \rho[(y+y')L]$, and $\rho_- = [|y-y'|L]$, equation (3.52) becomes

$$
\begin{aligned}
\gamma_n &= \frac{\pi^2(2L)^4}{n+1} \int\int dydy'(y^2 - y'^2) \left[(2Ly)^{n+1} - |2Ly'|^{n+1}\right] \rho_+\rho_- \\
&= \frac{\pi^2(2L)^{n+5}}{n+1} \int\int dydy'(y^2 - y'^2) \left(y^{n+1} - |y'|^{n+1}\right) \rho_+\rho_-,
\end{aligned}
$$

The integration over the charge distribution consists of integrating over the four triangles in Figure 3.4. Therefore

$$\gamma_n = \frac{\pi^2(2L)^{n+5}}{n+1}\left[\int_{\frac{1}{2}}^{1}\int_{y-1}^{0} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_- \right.$$
$$+ \int_{\frac{1}{2}}^{1}\int_{0}^{1-y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-$$
$$+ \int_{0}^{\frac{1}{2}}\int_{-y}^{0} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-$$
$$\left. + \int_{0}^{\frac{1}{2}}\int_{0}^{y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-\right].$$

If we let $y' \to -y'$ in the first and third integrals, we obtain

$$\gamma_n = \frac{\pi^2(2L)^{n+5}}{n+1}\left[\int_{\frac{1}{2}}^{1}\int_{0}^{1-y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_- \right.$$
$$+ \int_{\frac{1}{2}}^{1}\int_{0}^{1-y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-$$
$$+ \int_{0}^{\frac{1}{2}}\int_{0}^{y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-$$
$$\left. + \int_{0}^{\frac{1}{2}}\int_{0}^{y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-\right]. \tag{3.53}$$

The first two integrals are equivalent, as are the second two. We are now integrating twice over triangles 2 and 4. In other words, the charge distribution is symmetric about the y-axis, which allows the simplification,

$$\gamma_n = \frac{2\pi^2(2L)^{n+5}}{n+1}\left[\int_{\frac{1}{2}}^{1}\int_{0}^{1-y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_- \right.$$
$$\left. + \int_{0}^{\frac{1}{2}}\int_{0}^{y} dydy'(y^2 - y'^2)\left(y^{n+1} - |y'|^{n+1}\right)\rho_+\rho_-\right]. \tag{3.54}$$

The limits for the y' integration are only over positive y', so we will drop the absolute value signs (except in ρ_-). We can simplify equation (3.54) further by noticing the symmetry of the integrals with respect to the line $y = y'$. Notice that y and y' can be switched without changing the value of the integrals. This means that integrating over triangles 4 and 2 in Figure 3.4 is equivalent to integrating over triangles 5 and 6 respectively. To simplify (3.54), we will add the integration over triangles 5 and 6:

$$\int_0^{\frac{1}{2}} \int_y^{1-y} dy dy' (y^2 - y'^2)(y^{n+1} - y'^{n+1}) \rho_+ \rho_-, \tag{3.55}$$

to the integrations in brackets of equation (3.54) and divide by 2. Notice that (3.55) can be combined with the second integral expression in (3.54) so that the limits over y' go from 0 to $1 - y$:

$$\gamma_n = \frac{\pi^2 (2L)^{n+5}}{n+1} \left[\int_{\frac{1}{2}}^1 \int_0^{1-y} dy dy' (y^2 - y'^2)(y^{n+1} - y'^{n+1}) \rho_+ \rho_- \right.$$
$$\left. + \int_0^{\frac{1}{2}} \int_0^{1-y} dy dy' (y^2 - y'^2)(y^{n+1} - y'^{n+1}) \rho_+ \rho_- \right].$$

Since the integrands of these two integrals are the same, they can be combined with the limits over y going from 0 to 1:

$$\gamma_n = \frac{\pi^2 (2L)^{n+5}}{n+1} \int_0^1 \int_0^{1-y} dy dy' (y^2 - y'^2)(y^{n+1} - y'^{n+1}) \rho_+ \rho_- . \tag{3.56}$$

We again call upon the symmetry of y and y' to make one last simplification. First, we expand the integrand in (3.56):

$$\begin{aligned}
I &\equiv \int_0^1 \int_0^{1-y} dy dy' (y^2 - y'^2)(y^{n+1} - y'^{n+1}) \rho_+ \rho_- \\
&= \int_0^1 \int_0^{1-y} dy dy' (y^{n+3} - y^2 y'^{n+1} - y^{n+1} y'^2 + y'^{n+3}) \rho_+ \rho_- \\
&= \int_0^1 \int_0^{1-y} dy dy' y^{n+3} \rho_+ \rho_- - \int_0^1 \int_0^{1-y} dy dy' y^2 y'^{n+1} \rho_+ \rho_- \\
&\quad - \int_0^1 \int_0^{1-y} dy dy' y^{n+1} y'^2 \rho_+ \rho_- + \int_0^1 \int_0^{1-y} dy dy' y'^{n+3} \rho_+ \rho_- .
\end{aligned}$$

Interchanging y and y' in the second and fourth integrals of the last line , we obtain an equivalent expression for I as

$$
\begin{aligned}
I &= 2\int_0^1\int_0^{1-y} dydy' y^{n+3}\rho_+\rho_- - 2\int_0^1\int_0^{1-y} dydy' y^{n+1}y'^2\rho_+\rho_- \\
&= 2\int_0^1\int_0^{1-y} dydy'(y^2 - y'^2)y^{n+1}\rho_+\rho_-.
\end{aligned}
$$

Inserting this simplification back into equation (3.56) yields the final result for γ_n as

$$
\gamma_n = \frac{2\pi^2(2L)^{n+5}}{n+1}\int_0^1 dy y^{n+1}\int_0^{1-y} dy'(y^2 - y'^2)\rho[(y+y')L]\rho[|y-y'|L]. \quad (3.57)
$$

We derived this equation for the case $y \geq y'$. What happens for the case $y < y'$? Notice in Figure 3.4 that this case corresponds to integration over the triangles 5 through 8. However, because of the symmetry about the line $y = y'$, integrating over the triangles 7 and 8 yields the same result as integrating over the triangles 3 and 1 respectively. But as was previously shown, integrating over triangles 1 and 3 is equivalent to integrating over triangles 2 and 4 respectively, and we are back to integrating over triangles 2, 4, 5, and 6, which brings us back to equation (3.57). Therefore, (3.57) is the complete equation for the coefficients γ_n in the new basis y and y'.

Now that we have found a general equation for the coefficients γ_n, we can continue further and find a general expression for obtaining the differential-difference equation for any finite spherical charge distribution. To do so, we need the Taylor series expansion

$$
\dot{\mathbf{R}}(t - \tfrac{2L}{c}y) = \sum_{n=0}^{\infty}\frac{\dot{\mathbf{R}}^{(n)}(t)}{n!}\left(-\frac{2Ly}{c}\right)^n. \quad (3.58)
$$

Then, by using the equation of motion (3.17), we can make the association

$$
\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\dot{\mathbf{R}}^{(n+1)}(t) = m\int_0^1 dy f(y)\dot{\mathbf{R}}(t - \tfrac{2L}{c}y), \quad (3.59)
$$

and identify what $f(y)$ needs to be by using equations (3.57) and (3.58) in (3.59):

$$\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\frac{2\pi^2(2L)^{n+5}}{n+1}\int_0^1 dyy^{n+1}\int_0^{1-y}dy'(y^2-y'^2)\rho_+\rho_-\dot{\mathbf{R}}^{(n+1)}(t)=$$

$$m\int_0^1 dyf(y)\sum_{n=o}^{\infty}\frac{\dot{\mathbf{R}}^{(n)}(t)}{n!}\left(-\frac{2Ly}{c}\right)^n.$$

In order to get $\dot{\mathbf{R}}^{(n+1)}(t)$ on the right hand side, as well as to have the powers of L and y to match, we write the right hand side as

$$m\int_0^1 dyf(y)\dot{\mathbf{R}}(t)+m\int_0^1 dyf(y)\sum_{n=0}^{\infty}\frac{\dot{\mathbf{R}}^{(n+1)}(t)}{(n+1)!}\left(-\frac{2Ly}{c}\right)^{n+1}.$$

Equating like powers on right and left sides, we obtain the two equations

$$\frac{2e^2}{3c^2}2\pi^2\sum_{n=o}^{\infty}\frac{(-1)^n}{c^n}\frac{(2L)^{n+5}}{(n+1)!}\int_0^1 dyy^{n+1}\int_0^{1-y}dy'(y^2-y'^2)\rho_+\rho_-\dot{\mathbf{R}}^{n+1}(t)=$$

$$m\int_0^1 dyf(y)\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{c^{n+1}}\frac{(2L)^{n+1}}{(n+1)!}y^{n+1}\dot{\mathbf{R}}^{(n+1)}(t), \tag{3.60}$$

and

$$\int_0^1 dyf(y)=0. \tag{3.61}$$

Now, from equation (3.60), $f(y)$ can be identified as

$$f(y)=\frac{2e^2}{3mc}2\pi^2(2L)^4\int_0^{1-y}dy'(y^2-y'^2)\rho_+\rho_-,$$

or,

$$f(y)=\frac{2c}{L}\left(\frac{c\tau}{L}\right)(4\pi L^3)^2\int_0^{1-y}dy'(y^2-y'^2)\rho[(y+y')L]\rho(|y-y'|L). \tag{3.62}$$

We can check (3.61) and (3.62) by inserting $f(y)$ from (3.62) back into (3.61), and see if we get an equality. The integral becomes

$$\int_0^1 dy \int_0^{1-y} dy'(y^2 - y'^2)\rho[(y+y')L]\rho(|y-y'|L). \tag{3.63}$$

We showed earlier that there is a symmetry about $y = y'$. If we interchange y and y' in the above integral, we get

$$\int_0^1 dy' \int_0^{1-y'} dy(y'^2 - y^2)\rho[(y+y')L]\rho(|y-y'|L) =$$
$$- \int_0^1 dy \int_0^{1-y} dy'(y^2 - y'^2)\rho[(y+y')L]\rho(|y-y'|L).$$

Since the integral becomes negative upon interchanging y and y', the integral must necessarily be equal to zero. This shows us that $f(y)$ acts like a weighting factor, and has just as much negative "weight" as positive.

We can now write the equation of motion (3.17) for a spherically symmetric charge distribution as

$$m_o\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + m \int_0^1 dy f(y)\dot{\mathbf{R}}(t - \tfrac{2L}{c}y), \tag{3.64}$$

where $f(y)$ is defined in equation (3.62). Equation (3.64) is the general form of the equation of motion for a finite, spherically symmetric charge distribution in the form of a differential-difference equation. Notice that (3.64) is a continuous differential-difference equation because we are integrating over y from 0 to 1. Therefore, $\dot{\mathbf{R}}$ is evaluated from $\dot{\mathbf{R}}(t)$ to $\dot{\mathbf{R}}(t - \frac{2L}{c})$. In the case of the spherical shell, there is a delta function which gives the differential-difference between two values only:

$$\dot{\mathbf{R}}(t - \tfrac{2L}{c}) - \dot{\mathbf{R}}(t),$$

as we obtained earlier in (3.42) by summing the series in (3.17). We will find this difference again in section 3.4.1 by using (3.62) and (3.64). But for now, let us follow the same steps we took for the spherical shell to discuss the behavior of the solutions in general. It will turn out that we will still have to turn to specific charge distributions in order to obtain conditions for a well behaved theory.

By using (3.59), it is immediate that (3.64) is just the equation of motion (3.17) for an extended charge. But now we are dealing with an integral instead of an infinite sum. With $\mathbf{F}_{\text{ext}} = 0$, equation (3.64) becomes

$$\ddot{\mathbf{R}}(t) = \frac{m}{m_0} \int_0^1 dy f(y) \dot{\mathbf{R}}(t - \tfrac{2L}{c} y). \tag{3.65}$$

As we did in the case of the spherical shell, we again assume solutions of the form

$$\dot{\mathbf{R}}(t) = \mathbf{A} e^{\beta t/\tau}. \tag{3.66}$$

Remember that if β has a positive real part, one obtains the undesirable runaway solutions. Our purpose here is not to obtain the solutions themselves, but to show under what conditions the runaway solutions can be avoided. To begin, we insert (3.66) into (3.65) to obtain

$$\mathbf{A}\frac{\beta}{\tau} e^{\frac{\beta t}{\tau}} = \frac{L}{c\tau} g \int_0^1 dy f(y) \mathbf{A} e^{\beta t/\tau} e^{-\frac{2L}{c} y \frac{\beta}{\tau}}.$$

As in the last section, we define

$$\eta = \frac{2L\beta}{c\tau}.$$

Using this definition, and canceling the term $\mathbf{A}e^{\beta t/\tau}$, we have

$$\eta = \frac{2L^2}{c^2\tau} g \int_0^1 dy f(y) e^{-\eta y}. \tag{3.67}$$

As before, η is a complex constant. Therefore, (3.67) can be written as

$$\begin{aligned} \eta &= \frac{2L^2}{c^2\tau} g \int_0^1 dy f(y) e^{-(\mu + i\nu) y} \\ &= \frac{2L^2}{c^2\tau} g \int_0^1 dy f(y) e^{-\mu y} (\cos \mu y - i \sin \nu y). \end{aligned}$$

Writing the real and imaginary parts separate, we have

$$\mu = \frac{2L^2}{c^2\tau} g \int_0^1 dy f(y) e^{-\mu y} \cos \nu y \qquad (3.68a),$$

$$\nu = -\frac{2L^2}{c^2\tau} g \int_0^1 dy f(y) e^{-\mu y} \sin \nu y. \qquad (3.68b)$$

These equations are the general form of the equations (3.46a) and (3.46b) we found for the spherical shell. In order for β to have a positive real part (which implies runaway solutions), η would need to have a positive real part. In other words, if μ is positive, (3.63) will have runaway solutions. However, we cannot determine the conditions for which μ is positive without having a specific charge distribution in mind. We will verify the results of this section by re-examining the spherical shell. Then, we will use the results of this section to obtain results for the uniform sphere.

3.4.1 *The Spherical Shell Revisited*

In this section, we simply want to try out the results of the last section, and see how it compares with the first method we used in sections 3.2 and 3.3 for the spherical shell. We start by transforming the charge density product from section 3.2.1,

$$\rho(r)\rho(r') = \frac{\delta(r-L)\delta(r'-L)}{16\pi^2 L^4}.$$

The delta functions can be transformed according to the equation [29]

$$\delta(r-L)\delta(r'-L) = \frac{\delta(y-y_0)\delta(y'-y'_0)}{|J|}, \tag{3.69}$$

where J is the Jacobian of the transformation, and is equal to $2L^2$ for the spherical shell as was shown earlier in section 3.4. We need to determine what y_0 and y'_0 are. Notice that for a nonzero value, $r = r' = L$. For this case, the transformation equations can be written

$$y + y' = 1, \qquad \text{and} \qquad |y - y'| = 1,$$

which implies

$$y = 1, \qquad \text{and} \qquad y' = 0,$$

or,

$$y = 0, \qquad \text{and} \qquad y' = 1.$$

Therefore, we must add the two possibilities, and divide by 2 to obtain the desired result under the transformation. The transformation (3.69) becomes

$$\delta(r-L)\delta(r'-L) = \frac{1}{4L^2}[\delta(y-1)\delta(y'-0) + \delta(y-0)\delta(y'-1)].$$

Therefore,

$$\rho[(y+y')L]\rho[|y-y'|L] = \frac{1}{\pi^2(2L)^6}[\delta(y-1)\delta(y'-0)+\delta(y-0)\delta(y'-1)], \quad (3.70)$$

and equation (3.57) for the spherical shell becomes

$$\gamma_n = \frac{2(2L)^{n-1}}{n+1}\left\{\int_0^1 dyy^{n+1}\int_0^{1-y} dy'(y^2-y'^2)\delta(y-1)\delta(y'-0) + \int_0^1 dyy^{n+1}\int_0^{1-y} dy'(y^2-y'^2)\delta(y-0)\delta(y'-1)\right\}.$$

The above integrations are trivial, with the first double integral being equal to 1, and the second being equal to 0. Thus, we have obtained the expression

$$\gamma_n = \frac{2(2L)^{n-1}}{n+1},$$

which is the same result as equation (3.24) previously obtained for the spherical shell. But notice that by using equation (3.57), the calculations necessary to obtain γ_n have been significantly shortened.

Let us use (3.63) to obtain the differential-difference equation (3.43) for the spherical shell. First, we will compute $f(y)$. To do so, we will use the form for the charge densities found in (3.70) for the spherical shell in (3.62), which becomes

$$f(y) = \frac{2c}{L}\left(\frac{c\tau}{L}\right)(4\pi L^3)^2\int_0^{1-y} dy'(y^2-y'^2)\frac{1}{(2L)^2(16\pi^2L^4)} \times[\delta(y-1)\delta(y'-0)+\delta(y-0)\delta(y'-1)].$$

The integrations are once again trivial, in which one obtains

$$f(y) = \frac{2c}{L}\left(\frac{c\tau}{L}\right)\frac{1}{4}[y^2\delta(y-1)+(y^2-1)\delta(y-0)]. \quad (3.71)$$

Let us check this expression of $f(y)$ by seeing if condition (3.61) is satisfied. Inserting (3.71) into (3.61), we have

$$\frac{2c}{L}\left(\frac{c\tau}{L}\right)\frac{1}{4}\int_0^1 dy[y^2\delta(y-1)+(y^2-1)\delta(y-0)] = \frac{2c}{L}\left(\frac{c\tau}{L}\right)\frac{1}{4}[1+(-1)] = 0,$$

and we see that condition (3.61) is indeed satisfied. Now to obtain the equation of motion, we will insert (3.71) for $f(y)$ into equation (3.64), and divide by m_o:

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}}{m_o} + \left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)\frac{m}{m_0}\int_0^1 dy[y^2\delta(y-1)+(y^2-1)\delta(y-0)]\dot{\mathbf{R}}(t-\tfrac{2L}{c}y). \quad (3.72)$$

Notice that

$$m_o = m - \frac{2e^2}{3c^2L} = m - \frac{cm\tau}{L} = m(1 - c\tau/L),$$

and

$$\frac{m}{m_o} = \frac{m}{m - cm\tau/L} = \frac{1}{1 - c\tau/L}.$$

As we used previously in the derivation of (3.43), we define

$$\xi = \frac{(c/2L)(c\tau/L)}{(1 - c\tau/L)}.$$

Using these results and definitions, (3.72) becomes

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}}{m(1 - c\tau/L)} + \xi\int_0^1 dy[y^2\delta(y-1) + (y^2-1)\delta(y-0)]\dot{\mathbf{R}}(t - \tfrac{2L}{c}y).$$

Now performing the integrations over y,

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}}{m(1 - c\tau/L)} + \xi[\dot{\mathbf{R}}(t - \tfrac{2L}{c}) - \dot{\mathbf{R}}(t)].$$

By using the general form for the differential-difference equation (3.63), we have been able to rederive equation (3.43) with much less effort.

Equation (3.68a) can also be verified by inserting $f(y)$ and performing the integration:

$$\begin{aligned}\mu &= g\int_0^1 dy[y^2\delta(y-1) + (y^2-1)\delta(y-0)]e^{-\mu y}\cos\nu y \\ &= -g[1 - e^{-\mu}\cos\nu],\end{aligned}$$

which is the same as equation (3.46a). Now that we have verified the general results of section 3.4, we can use them to obtain the equation of motion for the uniform sphere, and determine the behavior of its solutions.

3.5 The Equation of Motion and Behavior of Solutions for the Uniform Sphere

For the spherical shell, we saw that the general equations derived in this section greatly simplified the derivation of γ_n and the equation of motion. The general equations will also be very useful in our study of the uniform sphere. We start by re-calculating γ_n for the uniform sphere previously obtained in section 3.2.2. We begin with the product of the charge densities

$$\rho(r)\rho(r') = \frac{9}{16\pi^2 L^6}\theta(L-r)\theta(L-r'),$$

where $\theta(L-r)$ is equal to 1 for $r \leq L$, and is equal to 0 otherwise. In order to rewrite the step functions in terms of y and y', notice that the cut-off is at $r = r' = L$. It is useful to refer to Figure 3.5, and notice that we are integrating over the equivalent area of triangles 2,4,5, and 6. Therefore, the cut-off is $y = 1$ and $y' = 1 - y$, and

$$\theta(L-r)\theta(L-r') \to \theta(1-y)\theta[(1-y)-y'],$$

which just gives us back the same limits of integration as we started with. This makes sense because (3.57) was derived with the assumption of the charge density being zero beyond the radius L. Therefore,

$$\rho[(y+y')L]\rho(|y-y'|L) = \frac{9}{16\pi^2 L^6}\theta(1-y)\theta[(1-y)-y'], \tag{3.73}$$

and (3.57) becomes

$$\begin{aligned}\gamma_n &= \frac{2\pi^2(2L)^{n+5}}{n+1}\int_0^1 dy y^{n+1}\int_0^{1-y} dy'(y^2-y'^2)\frac{9}{16\pi^2L^6} \\ &= \frac{9L^{n-1}2^{n+2}}{n+1}\int_0^1 dy y^{n+1}\left[y'y^2 - \frac{y'^3}{3}\right]_0^{1-y}.\end{aligned} \tag{3.74}$$

Inserting the limits of integration, the term in brackets becomes

$$
\begin{aligned}
[\quad] &= y^2 - y^3 - \frac{(1-y)^3}{3} \\
&= y^2 - y^3 - \frac{1}{3} + y - y^2 + \frac{y^3}{3} \\
&= y - \frac{2}{3}y^3 - \frac{1}{3},
\end{aligned}
$$

and equation (3.74) becomes

$$
\begin{aligned}
\gamma_n &= \frac{9L^{n-1}2^{n+2}}{n+1}\int_0^1 dy\left(y^{n+2} - \frac{2}{3}y^{n+4} - \frac{1}{3}y^{n+1}\right) \\
&= \frac{9L^{n-1}2^{n+2}}{n+1}\left(\frac{1}{n+3} - \frac{2}{3(n+5)} - \frac{1}{3(n+2)}\right) \\
&= \frac{9L^{n-1}2^{n+2}}{(n+2)(n+3)(n+5)}.
\end{aligned}
$$

This is the same result obtained in section 3.2.2 for the uniform sphere. We have seen how equation (3.57) has simplified the calculations in order to find the coefficients γ_n. In order to find the differential-difference equation for the uniform sphere, we must first obtain $f(y)$ by using (3.73) in (3.62):

$$
\begin{aligned}
f(y) &= \frac{2c}{L}\left(\frac{c\tau}{L}\right)(4\pi L^3)^2\int_0^{1-y} dy'(y^2 - y'^2)\frac{9}{16\pi^2 L^6}\theta(1-y) \\
&= \frac{2c}{L}\left(\frac{c\tau}{L}\right)9\left(y - \frac{2}{3}y^3 - \frac{1}{3}\right)\theta(1-y). \qquad (3.75)
\end{aligned}
$$

In passing, we can check this $f(y)$ to be sure that the weighting condition (3.61) is satisfied. Ignoring the constant factors, we have

$$
\begin{aligned}
\int_0^1 dy\left(y - \frac{2}{3}y^3 - \frac{1}{3}\right) &= \left[\frac{y^2}{2} - \frac{2}{3}\frac{y^4}{4} - \frac{1}{3}y\right]_0^1 \\
&= \frac{1}{2} - \frac{1}{6} - \frac{1}{3} \\
&= 0,
\end{aligned}
$$

which checks out. Substituting (3.75) in the equation of motion (3.64), we get

$$
m_o\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + m\int_0^1 dy\frac{2c}{L}\left(\frac{c\tau}{L}\right)9\left(y - \frac{2}{3}y^3 - \frac{1}{3}\right)\dot{\mathbf{R}}\left(t - \frac{2L}{c}y\right).
$$

The differential difference equation of motion for the uniform sphere can therefore be written as

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}_{\text{ext}}(t)}{m(1 - c\tau/L)} + 36\xi \int_0^1 dy \left(y - \frac{2}{3}y^3 - \frac{1}{3} \right) \dot{\mathbf{R}}(t - \tfrac{2L}{c}y). \tag{3.76}$$

That this equation reduces to (2.7) in the point charge limit can be seen by using (3.59). In other words, all we have done is changed the variables, and the behavior of the equation in the point limit will not be affected.

This equation represents the motion that an electron would undergo if it were a solid, uniform sphere of charge. It is similar to the spherical shell equation (3.42), but the differential-difference is a continuous one for the uniform sphere.

In order to determine the behavior of the solutions to the equation of motion (3.76), we turn to the general equation (3.68a). If μ is positive, runaway solutions appear. For negative μ, well behaved solutions can be obtained. Inserting expression (3.75) for $f(y)$ into (3.68a), we have

$$\mu = 36g \int_0^1 dy \left(y - \frac{2}{3}y^3 - \frac{1}{3} \right) e^{-\mu y} \cos \nu y. \tag{3.77}$$

By plotting the integrand in (3.77), we can discuss the behavior of the solutions. First, we will consider the integrand without the $\cos \nu y$ term, and for μ positive. The integrand for this case is graphed in *Figure 3.5.*

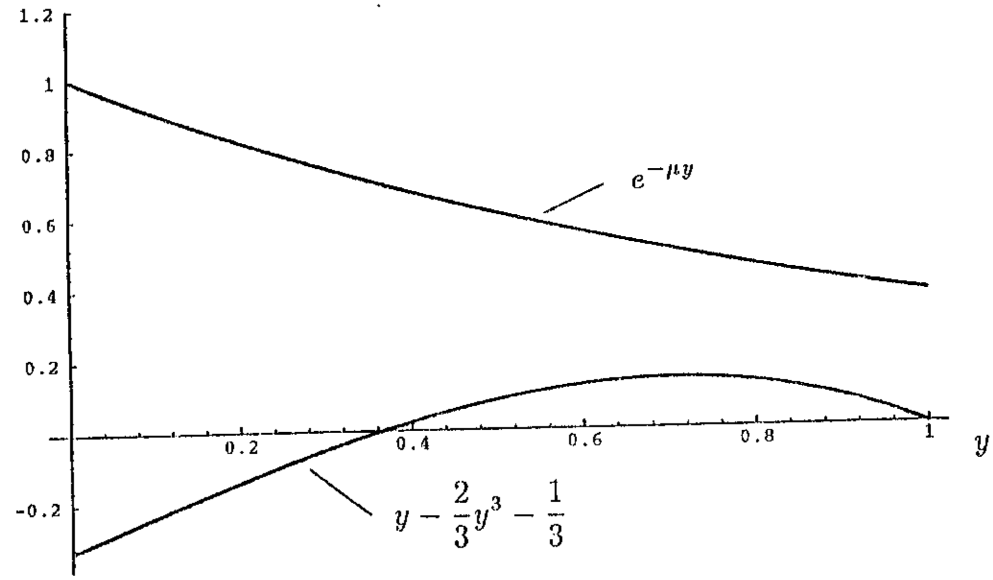

Figure 3.5: *Checking Existence of Runaways for Uniform Sphere* ($\mu > 0$).

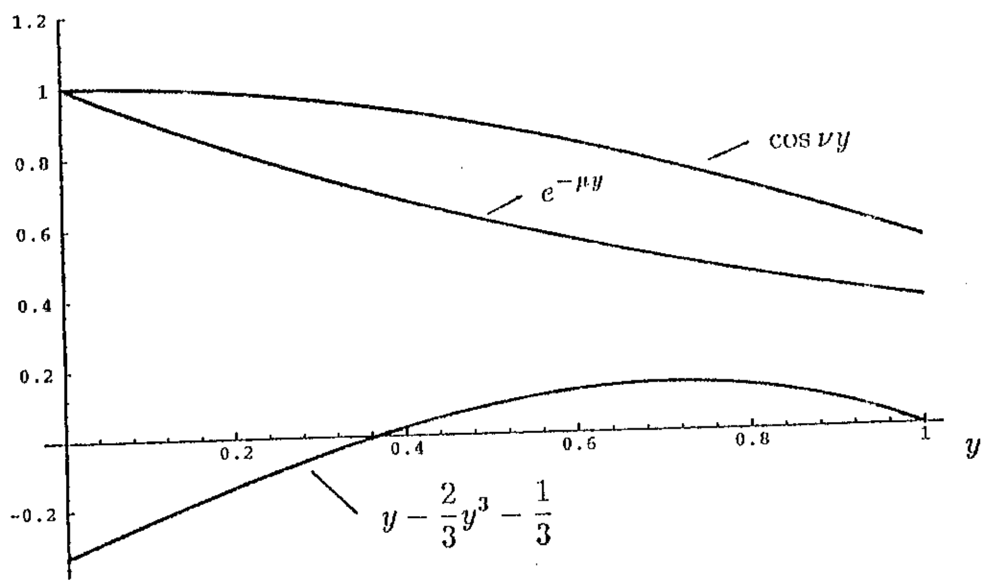

Figure 3.6: *Checking Esistence of Runaways for Uniform Sphere* ($\mu > 0$ and $\nu \approx 1$).

μ positive corresponds to runaway solutions as before. Remember that $f(y)$ has as much negative area as positive. The function $e^{-\mu y}$ is biased towards values close to zero. In other words, the negative area of $f(y)$ is weighted heavier than the positive, and the integral in (3.77) would be negative overall.

But how about the $\cos \nu y$ term? Let us first consider the two extreme values of $\nu = 0$ and $\nu \to \infty$. For $\nu = 0$, $\cos \nu y = 1$. This leaves the integrand unchanged, and the integral would remain negative. For $\nu \to \infty$, $\cos \nu y \to 0$ since the function oscillates rapidly between positive and negative values.

Can the sign of the integral be changed by a value of ν in between zero and infinity? Let us try a value such that $\cos \nu y$ is as graphed in *Figure 3.6.*

Notice that $\cos \nu y$ just contributes to the negative weight of the integral. If a small value of ν is taken, such as in *Figure 3.7*, then even more negative weight is added.

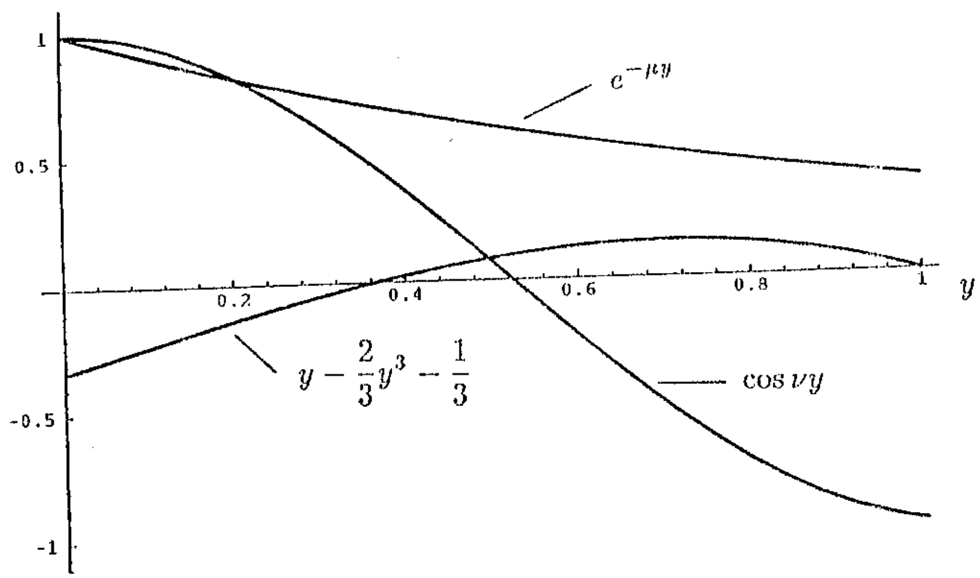

Figure 3.7: *Checking Existence of Runaways for Uniform Sphere* ($\mu > 0$ and $\nu \approx 3$).

Therefore we conclude that the integral in (3.77) will always be negative for $\mu > 0$. Now notice what this implies. For $L > c\tau$, g is positive. But this means that the right hand side of (3.77) is negative, in contradiction with the assumption $\mu > 0$. Therefore, there are no runaway or preaccelerating solutions for $L > c\tau$. This is the same result obtained for the spherical shell.

What if $L < c\tau$? Now g is negative, and we see that there is no contradiction in

(3.77). Therefore, runaway solutions return.

Now let us check to see what happens for $\mu < 0$, which corresponds to well behaved solutions. For this case, the integrand, excluding the $\cos \nu y$ term, is graphed in *Figure 3.8.*

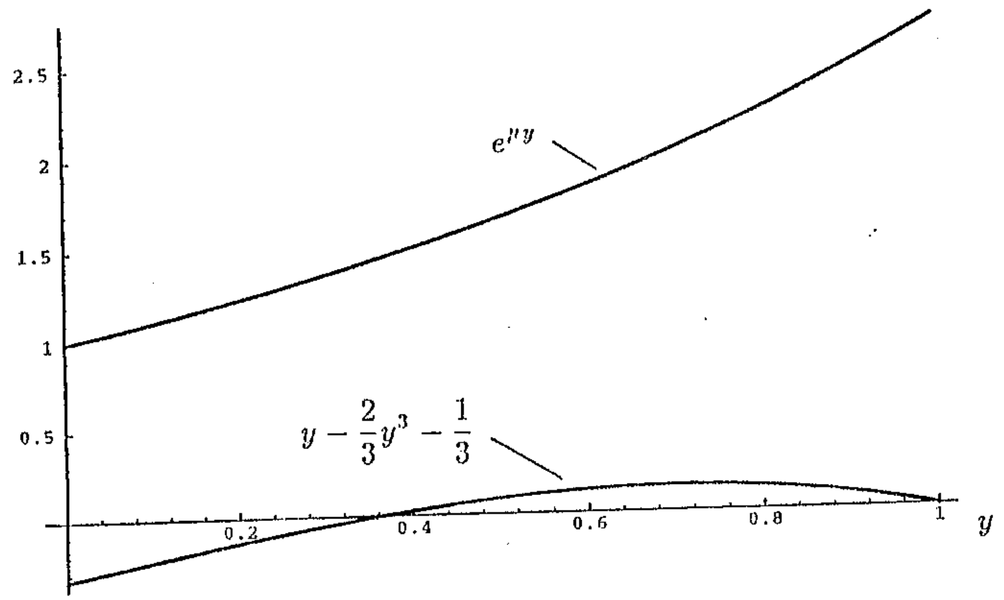

Figure 3.8: *Checking Existence of Runaways for Uniform Sphere* ($\mu < 0$).

Now the $e^{-\mu y}$ term gives preference to the positive part of $f(y)$. The two extreme values $\nu = 0$, and $\nu \to \infty$ yields the same results previously mentioned. But if the integral is positive, $L > c\tau$ leads to a contradiction in (3.77) for $\mu < 0$. However, notice that for middle range values of ν as before (see *Figure 3.9*), the sign of the integral can be changed to be negative overall.

Therefore, there are possible values of ν for which solutions can be found. That means for $L > c\tau$, well behaved solutions do indeed exist as in the case of the spherical shell. These solutions could be found by solving the integrals in (3.68), and simultaneously solving for μ and ν.

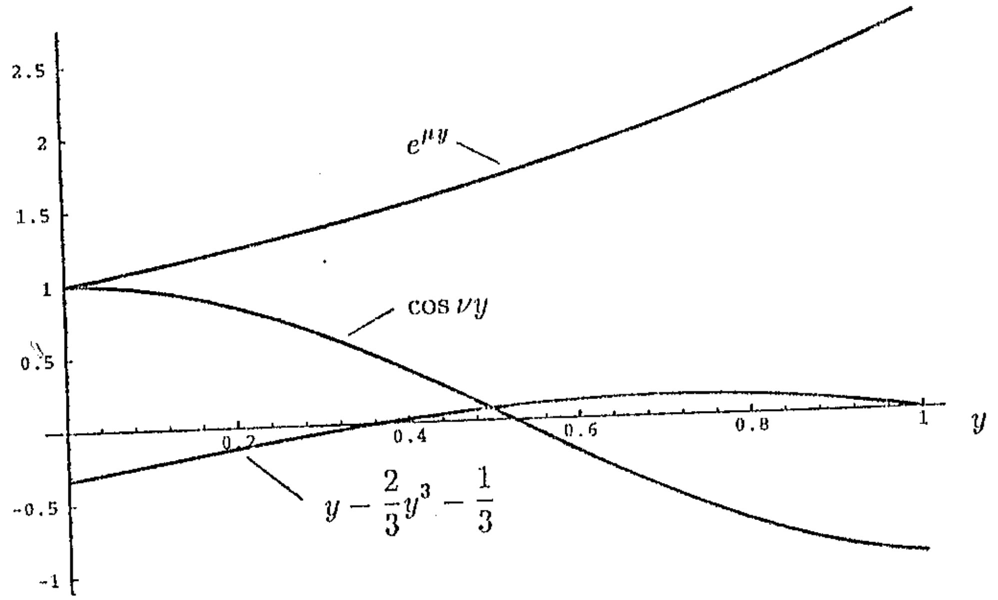

Figure 3.9: *Checking Existence of Runaways for Uniform Sphere* ($\mu < 0$ and $\nu \approx 3$).

3.6 Summary

In this chapter we derived the equation of motion for the extended electron. We saw that well-behaved solutions could be found for $L > c\tau$. But for $L < c\tau$, runaway or preaccelerating solutions return. In other words, these problems are not limited to point charges.

We have also seen how the theory based on an extended charge distribution reduces to the point charge case for $L \to 0$. In doing so, the problems of the point charge theory return. We conclude that the classical extended model theory is not capable of resolving the problems inherent to the point charge theory. In obtaining these results, we used different charge distributions and different approaches.

Looking at the quantum theory will help us answer the following question: is the runaway solution a true solution, or is it a "ghost" solution due to the particlar form of the classical equations. The derivations and results of this chapter will be helpful in deriving a quantum theory. We will be able to follow similar steps, and compare with the classical theory as we go.

CHAPTER 4

THE QUANTUM THEORY OF THE ELECTRON

The classical theory for a point electron has problems which are not resolvable within the domain of classical mechanics. Only when we considered an extended electron with charge radius $L > c\tau$ was classical mechanics able to describe the motion without complications. Since the electron is really a quantum particle, it makes sense that we would need to turn to quantum mechanics in order to obtain an appropriate theory. On the other hand, classical mechanics is sufficient to describe a macroscopic charge.

In this chapter we develop the quantum mechanical theory of radiation reaction for an extended charge and for a point charge. In doing so, we will compare equations derived in the quantum theory with their classical counterparts in order to discover some of the similarities and differences of the two theories. In order to remain as close as possible to our classical treatment, we will make the same assumptions of rigidity and low velocity with respect to the speed of light.

We first derive the operator form of the equation of motion for an extended charge in section 4.1. Then, in section 4.2, we discuss the behavior of the self-mass in the point limit, and investigate whether it diverges as in the classical case. In section 4.3 we derive the point charge equation of motion, and we discuss the behavior of its solutions with no external forces in section 4.4. Finally, in section 4.5, we discuss the behavior of the solutions when external forces are present.

Both in the classical and quantum cases we use models of charge distributions. We consider the use of models simply as a mathematical tool, and not as a means to find an actual charge distribution for the electron. The discussion is based on the development given in [4].

4.1 Equation of Motion for an Extended Model

In order to derive a quantum mechanical equation of motion we follow the procedure used for the classical electron. But in doing so, we take into account that we are now dealing with operators which may not commute. It is to our advantage to use the Heisenberg picture of quantum mechanics because it resembles the classical equations of motion more closely than the Schroedinger picture. This is because in the Heisenberg picture the time dependence is in the operators representing the physical quantities in analogy with the time-dependent classical variables. The equations of motion take the form [25],

$$\frac{d\mathbf{R}}{dt} = \frac{1}{i\hbar}[\mathbf{R}, H], \tag{4.1}$$

where H is the systems Hamiltonian and $\mathbf{R}$ could be any operator. We will primarily be interested in the position operator R representing first the position of the center of charge, and later the position of the point charge. Notice the similarity with the Poisson brackets from classical Hamiltonian mechanics. For the electron, the Hamiltonian is [26]

$$H = \frac{1}{2m_o}\left[\mathbf{P} - \frac{e}{c}\mathbf{A}(\mathbf{R})\right]^2 + e\phi(\mathbf{R}) + \frac{1}{8\pi}\int d\mathbf{r}\left[\mathbf{E}^2(\mathbf{r},t) + [\nabla \times \mathbf{A}(\mathbf{r},t)]^2\right], \tag{4.2}$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{R}) &= \int d\mathbf{r}\frac{1}{2}\left\{\rho(\mathbf{r} - \mathbf{R}(t))\mathbf{A}(\mathbf{r},t) + \mathbf{A}(\mathbf{r},t)\rho(\mathbf{r} - \mathbf{R}(t))\right\} \\ \phi(\mathbf{R}) &= \int d^3r\rho(\mathbf{r} - \mathbf{R}(t))\phi(\mathbf{r},t). \end{aligned} \tag{4.3}$$

Notice that equation (4.2) resembles the classical Hamiltonian, except now we are dealing with operators. The first two terms represent the contribution from the particle, and the last term represents the contribution from the fields. The equation for $\mathbf{A}(\mathbf{R})$ has been accordingly symmetrized. At first glance, it may appear that $\rho(\mathbf{r} - \mathbf{R}(t))$ and $\mathbf{A}(\mathbf{r}, t)$ commute. However, a closer look shows otherwise. We will return to this point later. Another interesting point is that in equation (4.3) we are smearing out at the level of the potential, whereas classically we smeared out at the level of the field.

4.1.1 *The Operator Form of the Lorentz Force Equation*

Our starting point is the same as it was in the classical case; namely, the Lorentz force equation. We can derive the operator form for the Lorentz force equation by using the Hamiltonian (4.2), and equation (4.1) in component form:

$$\frac{dR_i}{dt} = \frac{1}{i\hbar}[R_i, H].$$

Since we are considering a free electron, there are no external fields. We will neglect quadratic and higher terms of the self-fields. We will also neglect the magnetic interactions because of the assumption of low velocities. Now substituting H into the equation of motion, we get

$$\frac{dR_i}{dt} = \frac{1}{i\hbar}\left[R_i, \frac{1}{2m_o}(P - \frac{e}{c}A(\mathbf{R}))^2\right] + \frac{e}{i\hbar}[R_i, \phi(\mathbf{R})].$$

The second commutator goes to zero since a component of R commutes with other components of $\mathbf{R}$ and any function of it. Expanding the first commutator and writing A for $A(\mathbf{R})$, we obtain

$$\frac{dR_i}{dt} = \frac{1}{i\hbar 2m_o}\left[[R_i, P^2] + \frac{e^2}{c^2}[R, A^2] - \frac{e}{c}[R_i, P_jA_j] - \frac{e}{c}[R_i, A_jP_j]\right].$$

The second commutator in this expression is zero (in addition to being of second order), and

$$\frac{dR_i}{dt} = \frac{1}{i\hbar 2m_o}\left[[R_i, P_j]P_j + P_j[R_i, P_j] - \frac{e}{c}[R_i, P_j]A_j - \frac{e}{c}P_j[R_i, A_j]\right.$$
$$\left. - \frac{e}{c}[R_i, A_j]P_j - \frac{e}{c}A_j[R_i, P_j]\right].$$

Notice that the fourth and fifth commutators in this expression are zero, leaving us with

$$\frac{dR_i}{dt} = \frac{1}{i\hbar 2m_o}\left[[R_i, P_j]P_j + P_j[R_i, P_j] - \frac{e}{c}[R_i, P_j]A_j - \frac{e}{c}A_j[R_i, P_j]\right].$$

This equation can be simplified by realizing that $\mathbf{R}$ and $\mathbf{P}$ are conjugate variables and satisfy the relation $[R_i, P_j] = i\hbar\delta_{ij}$, yielding

$$m_o\dot{R}_i = P_i - \frac{e}{c}A_i. \tag{4.4}$$

By taking the time derivative, (4.4) becomes

$$m_o\ddot{R}_i = \dot{P}_i - \frac{e}{c}\frac{dA_i}{dt}. \tag{4.5}$$

Notice that we now have a force on the left hand side. In order to obtain the form for the Lorentz force equation, we need the right side in terms of the operators $\dot{\mathbf{R}}$, $\mathbf{E}$, and $\mathbf{B}$. In order to evaluate the time dependence of the momentum operator, we again turn to the Heisenberg equation of motion,

$$\frac{dP_i}{dt} = \frac{1}{i\hbar}[P_i, H].$$

Substituting H into this equation we have

$$\frac{dP_i}{dt} = \frac{1}{i\hbar 2m_o}\left[[P_i, P_jP_j] + \frac{e^2}{c^2}[P_i, A_jA_j] - \frac{e}{c}[P_i, P_jA_j] - \frac{e}{c}[P_i, A_jP_j]\right] + \frac{e}{i\hbar}[P_i, \phi],$$

where the first commutator is zero. Expanding and keeping only the nonzero commutators, we obtain

$$\begin{aligned} \frac{dP_i}{dt} = \frac{1}{i\hbar 2m_o}\Bigg[\frac{e^2}{c^2}[P_i, A_j]A_j + \frac{e^2}{c^2}A_j[P_i, A_j] - \frac{e}{c}P_j[P_i, A_j] \\ -\frac{e}{c}[P_i, A_j]P_j\Bigg] - \frac{e}{i\hbar}[P_i, \phi]. \end{aligned} \tag{4.6}$$

There are two different commutators in equation (4.6). But we notice that both A and ϕ are functions of $\mathbf{R}$. Therefore, once we find an expression for $[P_i, A_j(\mathbf{R})]$, we can apply the result to $[P_i, \phi(\mathbf{R})]$ as well. We begin by assuming that $A_j(\mathbf{R})$ is analytic in $\mathbf{R}$. Therefore, $A_i(\mathbf{R})$ can be expanded in a power series, and we have

$$\begin{aligned} [P_i, A_j(\mathbf{R})] &= \left[P_i, \sum_n a_n R_j^n\right] \\ &= \sum_n a_n [P_i, R_j^n], \end{aligned} \tag{4.7}$$

where the a_n are the coeffecients of the series. We also used the fact that the commutator of sums is the sum of the commutators. In order to find an expression for $[P_i, R_j^n]$, we begin with $[P_i, R_j] = -i\hbar\delta_{ij}$. Then

$$\begin{aligned} [P_i, R_j^2] &= [P_i, R_j]R_j + R_j[P_i, R_j] \\ &= -2i\hbar\delta_{ij}R_j. \end{aligned}$$

Going one step further, we have

$$\begin{aligned} [P_i, R_j^3] &= [P_i, R_j^2]R_j + R_j^2[P_i, R_j] \\ &= -2i\hbar\delta ij R_j^2 - i\hbar\delta_{ij}R_j^2 \\ &= -3i\hbar\delta_{ij}R_j^2. \end{aligned}$$

From here we can guess the form of the commutator in general to be

$$[P_i, R_j^n] = -ni\hbar\delta_{ij}R_j^{n-1}. \tag{4.8}$$

Our guess can be checked by induction. If we increase n by 1 on the left hand side of the equation, we should obtain the same expression for the right hand side with n increased by 1. Trying this technique we have

$$\begin{aligned}
[P_i, R_j^{n+1}] &= [P_i, R_j^n R_j] \\
&= [P_i, R_j^n]R_j + R_j^n[P_i, R_j] \\
&= -ni\hbar\delta_{ij}R_j^n - i\hbar\delta_{ij}R_j^n \\
&= -(n+1)i\hbar\delta_{ij}R_j^n,
\end{aligned}$$

and we have proved our guess to be correct. Using (4.8) in equation (4.7), we get

$$[P_i, A_j] = -\sum_n a_n ni\hbar\delta_{ij}R_j^{n-1}.$$

Now $nR_j^{n-1} = \partial R_j^n/\partial R_j$. Therefore,

$$\begin{aligned}
[P_i, A_j] &= -\sum_n a_n i\hbar\delta_{ij}\frac{\partial R_j^n}{\partial R_j} \\
&= -\sum_n a_n i\hbar\frac{\partial R_j}{\partial R_i}\frac{\partial R_j^n}{\partial R_j},
\end{aligned}$$

where δ_{ij} has been written as $\partial R_j/\partial R_i$. Now using the inverse chain rule, we have

$$[P_i, A_j] = -\sum a_n i\hbar\frac{\partial R_j^n}{\partial R_i}.$$

Or, by using the power series for A_j as in equation (4.7), our final expression for the commutator is

$$[P_i, A_j] = -i\hbar\frac{\partial A_j}{\partial R_i}. \tag{4.9}$$

Notice that we have not used the fact that $A(\mathbf{R})$ is a vector potential. Therefore our result can be applied to the scalar potential, in which case we obtain

$$[P_i, \phi] = -i\hbar \frac{\partial \phi}{\partial R_i}. \tag{4.10}$$

Our goal is to eliminate $\dot{P}_i$ from equation (4.5). Using (4.9) and (4.10) in equation (4.6) yields

$$\frac{dP_i}{dt} = -\frac{1}{2m_o}\left[\frac{e^2}{c^2}\left(\frac{\partial A_j}{\partial R_i}A_j + A_j\frac{\partial A_j}{\partial R_i}\right) - \frac{e}{c}\left(P_j\frac{\partial A_j}{\partial R_i} + \frac{\partial A_j}{\partial R_i}P_j\right)\right] - e\frac{\partial \phi}{\partial R_i}.$$

But by using equation (4.4), we see that P_j in this equation can be written as

$$P_j = m_o\dot{R}_j + \frac{e}{c}A_j.$$

Therefore one obtains

$$\dot{P}_i = \frac{e}{2c}\left(\dot{R}_j\frac{\partial A_j}{\partial R_i} + \frac{\partial A_j}{\partial R_i}\dot{R}_j\right) - e\frac{\partial \phi}{\partial R_i}.$$

This is the form of the expression for $\dot{P}_i$ we want in equation (4.5). With this result, (4.5) becomes

$$m_o\ddot{R}_i = \frac{e}{2c}\left(\dot{R}_j\frac{\partial A_j}{\partial R_i} + \frac{\partial A_j}{\partial R_i}\dot{R}_j\right) - \frac{e}{c}\frac{dA_i}{dt} - e\frac{\partial \phi}{\partial R_i}. \tag{4.11}$$

To obtain the equation of motion in its desired form, we need to find the time derivative of A_i, which can be found using the chain rule in symmetrized form as

$$\frac{dA_i}{dt} = \frac{1}{2}\left(\dot{R}_k\frac{\partial A_i}{\partial R_k} + \frac{\partial A_i}{\partial R_k}\dot{R}_k\right) + \frac{\partial A_i}{\partial t}.$$

Then equation (4.11) becomes

$$m_o\ddot{R}_i = \frac{e}{2c}\left(\dot{R}_j\frac{\partial A_j}{\partial R_i} + \frac{\partial A_j}{\partial R_i}\dot{R}_j - \dot{R}_j\frac{\partial A_i}{\partial R_j} - \frac{\partial A_i}{\partial R_j}\dot{R}_j\right) - c\frac{\partial \phi}{\partial R_i} - \frac{e}{c}\frac{\partial A_i}{\partial t}. \tag{4.12}$$

Using the electric field operator equation obtained by replacing observables by operators in the classical equation

$$\mathbf{E}(\mathbf{R}) = -\nabla\phi(\mathbf{R}) - \frac{1}{c}\frac{\partial A(\mathbf{R})}{\partial t},$$

or in index notation,

$$eE_i = -e\frac{\partial\phi}{\partial R_i} - \frac{e}{c}\frac{\partial A_i}{\partial t}, \tag{4.13}$$

and rearranging, the equation of motion (4.12) has the form

$$m_o\ddot{R}_i = eE_i + \frac{e}{2c}\left(\dot{R}_j\frac{\partial A_j}{\partial R_i} - \dot{R}_j\frac{\partial A_i}{\partial R_j} + \frac{\partial A_k}{\partial R_i}\dot{R}_k - \frac{\partial A_i}{\partial R_k}\dot{R}_k\right). \tag{4.14}$$

By using the relation

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

the expression inside the brackets of (4.14) becomes

$$\epsilon_{ijk}\epsilon_{klm}\dot{R}_j\frac{\partial A_m}{\partial R_l} - \epsilon_{ijk}\epsilon_{jlm}\frac{\partial A_m}{\partial R_l}\dot{R}_k.$$

Notice that $\epsilon_{klm}(\partial A_m/\partial R_l)$ can be written as $(\nabla\times\mathbf{A})_k$. Therefore,

$$\begin{aligned} I &= \epsilon_{ijk}\dot{R}_j(\nabla\times\mathbf{A})_k - \epsilon_{ijk}(\nabla\times\mathbf{A})_j\dot{R}_k \\ &= [\dot{\mathbf{R}}\times\nabla\times\mathbf{A} - \nabla\times\mathbf{A}\times\dot{\mathbf{R}}]_i. \end{aligned}$$

Substituting this result into equation (4.14) and using the definition

$$\mathbf{B}(\mathbf{R}) = \nabla\times\mathbf{A}(\mathbf{R}),$$

yields the final result

$$m_o\ddot{\mathbf{R}} = e\mathbf{E}(\mathbf{R}) + \frac{e}{2c}\left[\dot{\mathbf{R}}\times\mathbf{B}(\mathbf{R}) - \mathbf{B}(\mathbf{R})\times\dot{\mathbf{R}}\right]. \tag{4.15}$$

This is the operator form of the Lorentz force equation. Notice the similarity of this equation with its classical counterpart

$$m_o\ddot{\mathbf{R}} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{R}} \times \mathbf{B}. \tag{4.16}$$

Notice that we could have obtained (4.15) directly by replacing the classical observables in (4.16) by their corresponding operators. In doing so, we would have to take into account that the operators $\dot{\mathbf{R}}$ and $\mathbf{B}$ do not commute. Therefore we would need to use symmetrization of operators. The minus sign in equation (4.15) comes from the symmetrization of a cross product.

4.1.2 *Equation of Motion in Commutator Form*

Now that we have the operator form of the Lorentz force equation, we can begin the derivation of the quantum equation of motion. In doing so, we closely follow the steps in deriving the classical equation of motion. But now we must be careful to keep the order of non-commuting operators.

In order to obtain the quantum mechanical equation of motion, we need to eliminate the self-fields in the operator form of the Lorentz force equation as we did in the classical case. To do so we can use the operator field equations in the Lorentz gauge

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r}, t),$$

and

$$\nabla^2 \phi(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 \phi(\mathbf{r}, t)}{\partial t^2} = -4\pi\rho.$$

These operator field equations have the solutions

$$\begin{aligned}\mathbf{A}(\mathbf{r}, t) &= \mathbf{A}_{\text{in}}(\mathbf{r}, t) + \frac{e}{c}\int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}', t'_{ret})}{|\mathbf{r} - \mathbf{r}'|} \\ &= \mathbf{A}_{\text{in}} + \mathbf{A}_{\text{self}},\end{aligned} \tag{4.17a}$$

and

$$\begin{aligned}\phi(\mathbf{r}, t) &= \phi_{\text{in}}(\mathbf{r}, t) + e\int d\mathbf{r}' \frac{\rho(\mathbf{r}', t'_{\text{ret}})}{|\mathbf{r} - \mathbf{r}'|} \\ &= \phi_{\text{in}} + \phi_{\text{self}},\end{aligned} \tag{4.17b}$$

where

$$t'_{\text{ret}} = t - (|\mathbf{r} - \mathbf{r}'|/c).$$

The integral terms are called self-potentials since no other charges or currents are present. The retarded time is due to the fact that the field travels at the finite speed c. The term $|\mathbf{r}-\mathbf{r}'|/c$ is the time it takes for the field to travel from a point at $\mathbf{r}$ to a point at $\mathbf{r}'$. Therefore the retarded time is the difference between the time the field was actually emitted from $\mathbf{r}$ and the time it arrived at $\mathbf{r}'$.

Let us now return to equation (4.3), and investigate further why we needed to use symmetrization, or in other words, why $\mathbf{A}$ and $\mathbf{R}$ do not commute. The classical equation of motion for the current density is

$$\mathbf{J}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}(t).$$

Here $\mathbf{v}$ or $\dot{\mathbf{R}}$ represents the velocity of the center of charge, or the velocity of the (rigid) charge distribution. In order to write down the operator form of this equation, we must use symmetrization since the operators $\rho(\mathbf{r},t)$ and $\dot{\mathbf{R}}$ do not commute. Therefore, the operator form of the current density must be written as

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{2}[\rho(\mathbf{r},t),\dot{\mathbf{R}}(t)]_{+}.$$

Now we see that the current density depends on $\dot{\mathbf{R}}$. And by equation (4.17), we see that $\mathbf{A}(\mathbf{r},t)$ depends on the current density. Therefore, $\mathbf{A}(\mathbf{r},t)$ depends on $\dot{\mathbf{R}}$. And since $\rho(\mathbf{r}-\mathbf{R}(t))$ depends on $\mathbf{R}$, it does not commute with $\mathbf{A}(\mathbf{r},t)$. Therefore, we had to use symmetrization in equation (4.3).

The operators $\mathbf{j}$ and ρ in (4.17) are evaluated at the retarded time t_{ret}. We can derive a formula that relates an operator evaluated at the retarded time to that same operator evaluated at time t. To do so we start with the Heisenberg equation [25]

$$O(t'_{ret}) = \exp[+iH(t'_{\text{ret}}-t)]O(t)\exp[-iH(t'_{\text{ret}}-t)]. \tag{4.18}$$

But by using $t'_{\text{ret}} = t - |\mathbf{r}-\mathbf{r}'|/c$ and expanding we can write

$$\begin{aligned}\exp[+iH(t'_{\text{ret}}-t)] &= \exp\left[-iH\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\\ &= 1 + (-i)H\frac{|\mathbf{r}-\mathbf{r}'|}{c} + (-i)^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2} + \cdots,\end{aligned}$$

and

$$\exp[-iH(t'_{\text{ret}} - t)] = \exp\left[iH\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]$$
$$= 1 + iH\frac{|\mathbf{r}-\mathbf{r}'|}{c} + i^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2} + \cdots.$$

By incorperating these expansions into equation (4.18) one obtains

$$O(t'_{\text{ret}}) = \left\{1 + (-i)H\frac{|\mathbf{r}-\mathbf{r}'|}{c} + (-i)^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2} + \ldots\right\}O(t)$$
$$\times\left\{1 + iH\frac{|\mathbf{r}-\mathbf{r}'|}{c} + i^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2} + \cdots\right\}.$$

If we multiply out the first few terms, we begin to see a pattern develop:

$$O(t'_{\text{ret}}) = O(t) + O(t)iH\frac{|\mathbf{r}-\mathbf{r}'|}{c} + (-i)H\frac{|\mathbf{r}-\mathbf{r}'|}{c}O(t) + (-i)^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2}O(t)$$
$$+O(t)(i)^2H^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2} + (-i)H\frac{|\mathbf{r}-\mathbf{r}'|}{c}O(t)(i)H\frac{|\mathbf{r}-\mathbf{r}'|}{c} + \cdots.$$

Combining like terms in the above equation yields

$$O(t'_{\text{ret}}) = O(t) + (-i)\frac{|\mathbf{r}-\mathbf{r}'|}{1!c}[HO - OH] + (-i)^2\frac{|\mathbf{r}-\mathbf{r}'|^2}{2!c^2}[H^2O + OH^2 - 2HOH] + \cdots.$$

The quantity in brackets of the second term on the right hand side is obviously the commutation relation $[H, O]$. Let us take a closer look at the brackets in the third term to see if they too can be simplified.

$$H^2O - 2HOH + OH^2 = H^2 - HOH - HOH + OH^2$$
$$= H(HO - OH) - (HO - OH)H$$
$$= H[H, O] - [H, O]H.$$

Therefore we see that the brackets of the third term can be written as the nested commutator $[H, [H, O]]$. If we continued to the fourth term, we would find the nested commutator $[H, [H, [H, O]]]$, and so on. By using the definition

$$(\mathrm{ad}\ H)O = [H, O]_- \quad \text{and} \quad (\mathrm{ad}^2 H)O = [H, [H, O]_-]_-.$$

we can write equation (4.18) in the form

$$O(t'_{\mathrm{ret}}) = \sum_{n=0} \frac{(-i)^n}{n!} \frac{|\mathbf{r} - \mathbf{r}'|^n}{c^n} (\mathrm{ad}^n H) O(t). \tag{4.19}$$

Now by use of the operator form of the Lorentz force equation, we derive the equation of motion for an extended electron in quantum mechanics. In doing so we follow very closely the classical derivation, being careful to keep the proper ordering of noncommuting operators. We first consider the non-symmetrized form of $\mathbf{A}(\mathbf{R})$ for simplicity. The derivation for the second half of $\mathbf{A}(\mathbf{R})$ will be analogous. We begin by inserting (4.17) into (4.3):

$$\begin{aligned} \mathbf{A}(\mathbf{R}) &= \int d\mathbf{r} \rho[\mathbf{r} - \mathbf{R}(t)] \mathbf{A}(\mathbf{r}, t) \\ &= \int d\mathbf{r} \rho[\mathbf{r} - \mathbf{r}(t)] \left[\mathbf{A}_{\mathrm{in}}(\mathbf{r}, t) + \frac{e}{c} \int d^3 r' \frac{\mathbf{j}(\mathbf{r}, t'_{\mathrm{ret}})}{|\mathbf{r} - \mathbf{r}'|} \right], \end{aligned} \tag{4.20a}$$

and

$$\begin{aligned} \phi(\mathbf{R}) &= \int d\mathbf{r} \rho[\mathbf{r} - \mathbf{R}(t)] \phi(\mathbf{r}, t) \\ &= \int d\mathbf{r} \rho[\mathbf{r} - \mathbf{R}(t)] \left[\phi_{\mathrm{in}}(\mathbf{r}, t) + e \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t'_{\mathrm{ret}})}{|\mathbf{r} - \mathbf{r}'|} \right]. \end{aligned} \tag{4.20b}$$

The magnetic and nonlinear terms should be dropped as in the classical case to remain consistent with the assumption of low velocities. Therefore, the vector form of the equation of motion (4.15) is simply

$$\begin{aligned} m_o \ddot{\mathbf{R}} &= e\mathbf{E}(\mathbf{R}) \\ &= -e\nabla_{\mathbf{R}} \phi(\mathbf{R}) - \frac{e}{c} \frac{\partial}{\partial t} \mathbf{A}(\mathbf{R}). \end{aligned}$$

By using (4.20), we have

$$\begin{aligned} m_o \ddot{\mathbf{R}} = &-e\nabla_{\mathbf{R}} \left\{ \int d\mathbf{r} \rho(\mathbf{r} - \mathbf{R}(t)) \left[\phi_{\mathrm{in}}(\mathbf{r}, t) + e \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t'_{\mathrm{ret}})}{|\mathbf{r} - \mathbf{r}'|} \right] \right\} \\ &- \frac{e}{c} \frac{\partial}{\partial t} \left\{ \int d\mathbf{r} \rho(\mathbf{r} - \mathbf{R}(t)) \left[\mathbf{A}_{\mathrm{in}}(\mathbf{r}, t) + \frac{e}{c} \int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}', t'_{\mathrm{ret}})}{|\mathbf{r} - \mathbf{r}'|} \right] \right\}. \end{aligned}$$

Rearranging, and using the fact that

$$\mathbf{E}_{\text{in}} = -\nabla_{\mathbf{R}}\phi_{\text{in}}(\mathbf{R}) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_{\text{in}}(\mathbf{R}),$$

we have

$$\begin{aligned} m_o\ddot{\mathbf{R}} = e\mathbf{E}_{\text{in}} - e^2\nabla_{\mathbf{R}}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))\frac{\rho(\mathbf{r}',t'_{\text{ret}})}{|\mathbf{r}-\mathbf{r}'|} \\ -\frac{e^2}{c^2}\frac{\partial}{\partial t}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))\frac{\mathbf{j}(\mathbf{r}',t'_{\text{ret}})}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned} \tag{4.21}$$

In order to simplify (4.21) further, we will look at the integral terms separately. Using (4.19) to rewrite ρ as a function of t rather than t_{ret}, the first integral expression in (4.21) becomes (call it I for conveniance)

$$I = -e^2\sum_{n=0}^{\infty}\frac{(-i)^n}{n!c^n}\int\int d\mathbf{r}d\mathbf{r}'\nabla_{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}(ad^nH)\rho(\mathbf{r}',t). \tag{4.22}$$

Notice that $\nabla_{\mathbf{R}}$ acts only on $\rho(\mathbf{r}-\mathbf{R}(t))$, which can be written as

$$\nabla_{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}(t)) = -\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R}(t)).$$

We now use this result in (4.22), and perform partial integration over $\mathbf{r}$ (with vanishing surface densities) in the form of

$$-\int d\mathbf{r}[\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R}(t))]|\mathbf{r}-\mathbf{r}'|^{n-1} = \int d\mathbf{r}\rho(\mathbf{r}-\mathbf{R}(t))\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{r}'|^{n-1}$$

to obtain

$$I = -e^2\sum_{n=0}^{\infty}\frac{(-i)^n}{n!c^n}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{r}'|^{n-1}(\text{ad}^nH)\rho(\mathbf{r}',t). \tag{4.23}$$

The first term ($n=0$) in (4.23) is

$$-e^2\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\rho(\mathbf{r}',t).$$

This is the electrostatic self-force which vanishes for spherically symmetric charge distributions as it did in the classical case.

For the second term ($n = 1$), we get $\nabla|\mathbf{r} - \mathbf{r}'|^0 = \nabla 1 = 0$. Therefore, the series in equation (4.5) can be rewritten by letting $n \to n + 2$ as

$$-e^2 \sum_{n=0}^{\infty} \frac{(-i)^{n+2}}{(n+2)!c^{n+2}} \int\int d\mathbf{r}d\mathbf{r}' \rho(\mathbf{r} - \mathbf{R}(t))\nabla_{\mathbf{r}}|\mathbf{r} - \mathbf{r}'|^{n+1}(\mathrm{ad}^{n+2}H)\rho(\mathbf{r}', t). \quad (4.24)$$

But $(-i)^{n+2} = (-i)^n(-i)^2 = -(-i)^n$ and $(n+2)! = n!(n+1)(n+2)$. Therefore the equation of motion, equation (4.21), can be written as

$$m_o\ddot{\mathbf{R}} = e\mathbf{E}_{\text{in}} - \frac{e^2}{c^2}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!c^n}\int\int d\mathbf{r}d\mathbf{r}'\left\{-\frac{1}{(n+1)(n+2)}\rho(\mathbf{r} - \mathbf{R}(t))\nabla_{\mathbf{r}}|\mathbf{r} - \mathbf{r}'|^{n+1}\right.$$

$$\left.\times(\mathrm{ad}^{n+2}H)\rho(\mathbf{r}', t) - \frac{\partial}{\partial t}[\rho(\mathbf{r} - \mathbf{R}(t))|\mathbf{r} - \mathbf{r}'|^{n-1}(\mathrm{ad}^n H)\mathbf{j}(\mathbf{r}', t)]\right\}. \quad (4.25)$$

where we have again used (4.19) to write $\mathbf{j}$ as a function of t. Now notice that we have

$$(\mathrm{ad}^{n+2}H)\rho(\mathbf{r}', t) = (\mathrm{ad}^{n+1}\mathrm{ad}H)\rho(\mathbf{r}', t),$$

where by definition

$$(\mathrm{ad}H)\rho(\mathbf{r}', t) = [H, \rho(\mathbf{r}', t)] = -i\frac{\partial}{\partial t}\rho(\mathbf{r}', t).$$

Also by definition, we can write

$$\frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}', t) = i[H, \mathbf{j}(\mathbf{r}', t)] = i(\mathrm{ad}H)\mathbf{j}(\mathbf{r}', t),$$

where we have dropped the factors of $\hbar$. The partial derivative in the second term in curly brackets of (4.25) is

$$\left[\frac{\partial}{\partial t}\rho(\mathbf{r} - \mathbf{R}(t))\right]|\mathbf{r} - \mathbf{r}'|^{n-1}(\mathrm{ad}^n H)\mathbf{j}(\mathbf{r}', t) + \rho(\mathbf{r} - \mathbf{R}(t))|\mathbf{r} - \mathbf{r}'|^{n-1}(\mathrm{ad}^n H)\frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}', t).$$

But $\rho(\mathbf{r}-\mathbf{R}(t))$ does not depend explicity on time because of the assumption that the charge distribution is rigid. Putting the above results into (4.25), and factoring out $-i\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}$, we have

$$m_o\ddot{\mathbf{R}} = e\mathbf{E}_{\text{in}} + \frac{e^2}{c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!c^n}\int\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}$$

$$\left\{(\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t) + \frac{1}{(n+1)(n+2)}\frac{\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{r}'|^{n+1}}{|\mathbf{r}-\mathbf{r}'|^{n-1}}(\text{ad}^{n+1}H)\frac{\partial}{\partial t}\rho(\mathbf{r}',t)\right\}. \quad (4.26)$$

We can compare (4.26) to the classical equation (3.7) where H corresponds to taking the time derivative in classical theory. Using the continuity equation

$$\frac{\partial}{\partial t}\rho(\mathbf{r}',t) = -\nabla_{\mathbf{r}'}\cdot\mathbf{j}(\mathbf{r}',t),$$

the expression in brackets of equation (4.26) becomes

$$\{\quad\} = (\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t) - \frac{1}{(n+1)(n+2)}\frac{\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{r}'|^{n+1}}{|\mathbf{r}-\mathbf{r}'|^{n-1}}(\text{ad}^{n+1}H)\nabla_{\mathbf{r}'}\cdot\mathbf{j}(\mathbf{r}',t). \quad (4.27)$$

But,

$$\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{r}'|^{n+1} = (n+1)|\mathbf{r}-\mathbf{r}'|^{n}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.$$

Therefore, equation (4.27) can be written as

$$\{\quad\} = (\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t) - \frac{1}{(n+2)}(\mathbf{r}-\mathbf{r}')(\text{ad}^{n+1}H)\nabla_{\mathbf{r}'}\cdot\mathbf{j}(\mathbf{r}',t). \quad (4.28)$$

We can simplify the second term in (4.28). To do so, we insert (4.28) back into equation (4.26) for the curly brackets, and consider the integration over $\mathbf{r}'$ on the second term in (4.28). Using integration by parts in the form

$$\int_{\text{V}}\mathbf{u}(\nabla\cdot\mathbf{v})dv = \int_{\text{S}}\mathbf{u}(\mathbf{v}\cdot\hat{\mathbf{n}})da - \int_{\text{V}}(\mathbf{v}\cdot\nabla)\mathbf{u}dv,$$

we obtain

$$-\frac{\rho(\mathbf{r}-\mathbf{R}(t))}{(n+2)}\int d\mathbf{r}'|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')(\mathrm{ad}^{n+1}H)\nabla_{\mathbf{r}'}\cdot\mathbf{j}(\mathbf{r}',t)=$$

$$-\frac{\rho(\mathbf{r}-\mathbf{R}(t))}{(n+2)}\left\{\int_S|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')(\mathrm{ad}^{n+1}H)\left[\mathbf{j}(\mathbf{r}',t)\cdot\hat{\mathbf{n}}\right]da\right.$$

$$\left.+\int d\mathbf{r}'(\mathrm{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t)\cdot\nabla_{\mathbf{r}'}\left[|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')\right]\right\}. \tag{4.29}$$

As in the classical case, the surface term vanishes. By following the classical derivation, we see that the integrand of the remaining term on the right hand side of (4.29) can be written as

$$(\mathrm{ad}^{n+1}H)\mathbf{j}\cdot\nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathbf{r}-\mathbf{r}')=$$

$$-|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathrm{ad}^{n+1}H)\left[\mathbf{j}+(n-1)\frac{\mathbf{j}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\right]. \tag{4.30}$$

By inserting (4.30) into (4.29), and then (4.29) back into (4.26), the equation of motion becomes

$$m_o\ddot{\mathbf{R}}=e\mathbf{E}_{\mathrm{in}}+\frac{e^2}{c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!c^n}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}(\mathrm{ad}^{n+1}H)$$

$$\left\{\mathbf{j}(\mathbf{r}',t)-\frac{1}{(n+2)}\mathbf{j}(\mathbf{r}',t)-\frac{(n-1)}{(n+2)}\frac{\mathbf{j}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}')\right\}. \tag{4.31}$$

The first two terms in the curly brackets of equation (4.31) can be added together, and the curly brackets can be written as

$$\{\quad\}=\frac{(n+1)}{(n+2)}\mathbf{j}(\mathbf{r}',t)-\frac{(n-1)}{(n+2)}\frac{\mathbf{j}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}-\mathbf{r}'). \tag{4.32}$$

Expression (4.32) can be compared with the classical expression (3.11). For a spherically symmetric charge distribution, the only relevant direction is that of the current density $\mathbf{j}(\mathbf{r}',t)$. In comparison with the steps taken to obtain (3.13) from (3.12), we see that (4.32) can be replaced by

$$
\begin{aligned}
\{\quad\} \rightarrow \mathbf{j}(\mathbf{r}',t)&\left[\left(\frac{n+1}{n+2}\right)-\left(\frac{n-1}{n+2}\right)\frac{\frac{\mathbf{j}(\mathbf{r}',t)}{|\mathbf{j}(\mathbf{r}',t)|}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{j}(\mathbf{r}',t)}{|\mathbf{j}(\mathbf{r}',t)|}\cdot(\mathbf{r}-\mathbf{r}')\right] \\
= \mathbf{j}(\mathbf{r}',t)&\left[\left(\frac{n+1}{n+2}\right)-\left(\frac{n-1}{n+2}\right)\frac{[\mathbf{j}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')]^2}{|\mathbf{j}(\mathbf{r}',t)|^2|\mathbf{r}-\mathbf{r}'|^2}\right].
\end{aligned}
\tag{4.33}
$$

In order to simplify, we define the angle θ between $\mathbf{j}$ and $(\mathbf{r}-\mathbf{r}')$:

$$
\frac{[\mathbf{j}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')]^2}{|\mathbf{j}(\mathbf{r}',t)|^2|\mathbf{r}-\mathbf{r}'|^2} = \cos^2\theta.
$$

And, when we insert this into equation (4.33), and then into (4.31), the second term in curly brackets of equation (4.33) is an integral of the form

$$
\int f(r,t)\cos^2\theta d\mathbf{r} = \frac{1}{3}\int f(r,t)d\mathbf{r}.
$$

As in the classical case, the curly brackets can therefore be written as

$$
\begin{aligned}
\{\quad\} &= \mathbf{j}(\mathbf{r}',t)\left[\frac{n+1}{n+2}-\frac{n-1}{n+2}\left(\frac{1}{3}\right)\right] \\
&= \frac{2}{3}\mathbf{j}(\mathbf{r}',t).
\end{aligned}
$$

And the final expression for equation (4.31) becomes

$$
m_o\ddot{\mathbf{R}} = e\mathbf{E}_{\text{in}} + \frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!c^n}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}(\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t).
\tag{4.34}
$$

But remember that we need the symmetrized form of equation (4.3), and all we have considered up to this point is the product of $\rho(\mathbf{r}-\mathbf{R}(t))\mathbf{A}(\mathbf{r},t)$. We can follow through with an analogous derivation in which we let the order of the charge density and vector potential operators be reversed. In comparing line by line we see that the final result is the same as (4.34), but with the order of $\rho(\mathbf{r}-\mathbf{R}(t))$ and $(\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t)$ reversed. Remembering the factor of 1/2, we see that the form for the equation of motion is

$$m_o\ddot{\mathbf{R}} = e\mathbf{E}_{\text{in}} + \frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!c^n}$$

$$\int\int d\mathbf{r}d\mathbf{r}'\frac{1}{2}[\rho(\mathbf{r}-\mathbf{R}(t))|\mathbf{r}-\mathbf{r}'|^{n-1}, (\text{ad}^{n+1}H)\mathbf{j}(\mathbf{r}',t)]_+. \qquad (4.35)$$

Equation (4.35) is the quantum mechanical equation of motion for an extended charge as we have derived it thus far. In doing so, we have neglected nonlinear terms, as well as magnetic terms because of the nonrelativistic assumption. Notice that equation (4.35) contains nested commutators which must be evaluated in order to obtain a more useful form for the equation of motion. After the commutators have been evaluated, we will compare the quantum mechanical equation of motion with the classical ones derived in *Chapter 3*. The final form of equation (4.35) will be derived in the next section.

4.1.3 *Evaluation of the Commutators*

We want to write (4.35) in the form of equation (3.17). To do so, we need to evaluate the nested commutators. Once again, we will retain only those terms which are a product of the charge density and some time derivative of the position operator. Since $d^n\mathbf{R}/dt^n$ does not commute with ρ, we define "right-ordering" as the case when all $\mathbf{R}$ operators are on the right of ρ, and "left-ordering" when they are on the left. We will derive only the right–ordered expressions since the left-ordered can be obtained by reversing the order of the operators, and letting $\hbar \to -\hbar$. That $\hbar$ should be replaced by $-\hbar$ can be seen in the following way. Starting with the symmetrized current density operator we have

$$\begin{aligned}
\mathbf{j}(\mathbf{r},t) &= \frac{1}{2}[\rho(\mathbf{r}-\mathbf{R}), \dot{\mathbf{R}}]_+ \\
&= \frac{1}{2}\rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} + \frac{1}{2}\dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) \\
&= \rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} - \frac{1}{2}\rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} + \frac{1}{2}\dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) \\
&= \rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} - \frac{1}{2}[\rho(\mathbf{r}-\mathbf{R}), \dot{\mathbf{R}}]_-.
\end{aligned}$$

The commutator in the second term can be written as

$$
\begin{aligned}
[\rho(\mathbf{r}-\mathbf{R}),\dot{\mathbf{R}}] &= \frac{1}{m_o}[\rho(\mathbf{r}-\mathbf{R}),m_o\dot{\mathbf{R}}] \\
&= \frac{1}{m_o}[\rho(\mathbf{r}-\mathbf{R}),\mathbf{P}] \\
&= \frac{i\hbar}{m_o}\nabla_{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}),
\end{aligned}
$$

where we have used the fact that $[f(-\mathbf{x}),\mathbf{P}] = i\hbar\nabla_{\mathbf{x}}f(-x)$. We can now use $\nabla_{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) = -\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R})$ to obtain the final form

$$
\mathbf{j}(\mathbf{r},t) = \rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} + \frac{i\hbar}{2m_o}\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R}). \tag{4.36}
$$

Equation (4.36) is the operator form for the current density in right-ordered form ($\dot{\mathbf{R}}$ is on the right side of ρ). Following similar steps, the left-ordered form can be obtained as follows:

$$
\begin{aligned}
\mathbf{j}(\mathbf{r},t) &= \frac{1}{2}[\rho(\mathbf{r}-\mathbf{R}),\dot{\mathbf{R}}]_+ \\
&= \dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) - \frac{1}{2}\dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) + \frac{1}{2}\rho(\mathbf{r}-\mathbf{R})\dot{\mathbf{R}} \\
&= \dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) + \frac{1}{2}[\rho(\mathbf{r}-\mathbf{R}),\dot{\mathbf{R}}]_- \\
&= \dot{\mathbf{R}}\rho(\mathbf{r}-\mathbf{R}) - \frac{i\hbar}{2m_o}\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R}).
\end{aligned}
$$

In comparison with (4.36), we see that left-ordering is indeed accomplished by reversing the order of ρ and $\dot{\mathbf{R}}$ and replacing $\hbar$ by $-\hbar$. Returning to the right-ordered case, we evaluate the nested commutators $(\mathrm{ad}^{n+1}H)\mathbf{j}(\mathbf{r},t)$ with the use of (4.35). We begin by looking at the $n=0$ term. By definition and (4.36),

$$
\begin{aligned}
[H,\mathbf{j}(\mathbf{r},t)] &= -i\hbar\frac{\partial}{\partial t}\mathbf{j}(\mathbf{r},t) \\
&= -i\hbar\left[\rho(\mathbf{r}-\mathbf{R})\ddot{\mathbf{R}} + \frac{\partial\rho(\mathbf{r}-\mathbf{R})}{\partial t}\dot{\mathbf{R}} + \frac{i\hbar}{2m_o}\frac{\partial}{\partial t}\nabla_{\mathbf{r}}\rho(\mathbf{r}-\mathbf{R})\right].
\end{aligned}
$$

By switching the order of differentiation in the last term, we have

$$
[H,\mathbf{j}(\mathbf{r},t)] = -i\hbar\left(\rho\ddot{\mathbf{R}} + \frac{\partial\rho}{\partial t}\dot{\mathbf{R}} + \frac{i\hbar}{2m_o}\nabla\frac{\partial\rho}{\partial t}\right).
$$

Using the continuity equation and equation (4.36), we can now write the commutator in the form

$$
\begin{aligned}
[H,\mathbf{j}(\mathbf{r},t)] &= \frac{\hbar}{i}\left[\rho\ddot{\mathbf{R}} - \nabla\cdot\left(\rho\dot{\mathbf{R}} + \frac{i\hbar}{2m_o}\nabla\rho\right)\dot{\mathbf{R}} - \frac{i\hbar}{2m_o}\nabla\nabla\cdot\left(\rho\dot{\mathbf{R}} + \frac{i\hbar}{2m_o}\nabla\rho\right)\right] \\
&= \hbar\left[\frac{\rho\ddot{\mathbf{R}}}{i} - \frac{(\nabla\rho\cdot\dot{\mathbf{R}})}{i}\dot{\mathbf{R}} - \frac{\hbar}{2m_o}\nabla(\nabla\rho)\cdot\dot{\mathbf{R}} - \frac{\hbar}{2m_o}\nabla^2\rho\dot{\mathbf{R}} - \frac{i\hbar^2}{4m_o^2}\nabla\nabla^2\rho\right].
\end{aligned}
$$

Notice that the second term in the square brackets is on the order of $\dot{\mathbf{R}}^2$, and so we neglect it. Thus we have the result

$$
(\mathrm{ad}H)\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{i}\rho\ddot{\mathbf{R}} - \frac{\hbar^2}{2m_o}\nabla(\nabla\rho)\cdot\dot{\mathbf{R}} - \frac{\hbar^2}{2m_o}\nabla^2\rho\dot{\mathbf{R}} - \frac{i\hbar^3}{4m_o^2}\nabla(\nabla^2\rho). \tag{4.37}
$$

Remember that we are after an expression for $(\mathrm{ad}^{n+1}H)\mathbf{j}(\mathbf{r},t)$. By using the Heisenberg equation once again, and pulling out an extra $\hbar$, the nested commutator for $n=1$ becomes

$$
\begin{aligned}
[H,[H,\mathbf{j}]] = \frac{\hbar^2}{i}\frac{\partial}{\partial t}\Bigg[\frac{1}{i}\rho\ddot{\mathbf{R}} - \frac{1}{i}(\nabla\rho\cdot\dot{\mathbf{R}})\dot{\mathbf{R}} - \frac{\hbar}{2m_o}\nabla(\nabla\rho)\cdot\dot{\mathbf{R}} \\
-\frac{\hbar}{2m_o}\nabla^2\rho\dot{\mathbf{R}} - \frac{i\hbar^2}{4m_o^2}\nabla\nabla^2\rho\Bigg].
\end{aligned}
$$

Working out the derivative, we obtain

$$
\begin{aligned}
\frac{[H,[H,\mathbf{j}]]}{\hbar^2} &= -\frac{\partial\rho}{\partial t}\ddot{\mathbf{R}} - \rho\frac{\partial}{\partial t}\ddot{\mathbf{R}} + \left(\nabla\frac{\partial\rho}{\partial t}\cdot\dot{\mathbf{R}}\right)\dot{\mathbf{R}} + (\nabla\rho\cdot\ddot{\mathbf{R}})\dot{\mathbf{R}} + (\nabla\rho\cdot\dot{\mathbf{R}}\ddot{\mathbf{R}} \\
&\quad -\frac{\hbar}{2im_o}\nabla\left(\nabla\frac{\partial\rho}{\partial t}\right)\cdot\dot{\mathbf{R}} - \frac{\hbar}{2im_o}\nabla(\nabla\rho)\cdot\ddot{\mathbf{R}} - \frac{\hbar}{2im_o}\nabla^2\frac{\partial\rho}{\partial t}\dot{\mathbf{R}} - \\
&\quad -\frac{\hbar}{2im_o}\nabla^2\rho\ddot{\mathbf{R}} - \frac{\hbar^2}{4m_o^2}\nabla\nabla^2\frac{\partial\rho}{\partial t}.
\end{aligned} \tag{4.38}
$$

We can replace the $\partial\rho/\partial t$ terms by use of the continuity equation where $\mathbf{j}$ is given in equation (4.36). Doing so, and simplifying, allows us to write (4.38) in the final form

$$\frac{[H,[H,\mathbf{j}]]}{\hbar^2} = -\rho\frac{\partial}{\partial t}\ddot{\mathbf{R}} + 2(\nabla\rho\cdot\dot{\mathbf{R}})\ddot{\mathbf{R}} + (\nabla\rho\cdot\ddot{\mathbf{R}})\dot{\mathbf{R}} - (\nabla(\nabla\rho\cdot\dot{\mathbf{R}})\cdot\dot{\mathbf{R}})\dot{\mathbf{R}} +$$

$$+\frac{i\hbar}{m_o}\nabla^2\rho\ddot{\mathbf{R}} + \frac{i\hbar}{2m_o}\nabla(\nabla\rho\cdot\ddot{\mathbf{R}}) - \frac{i\hbar}{2m_o}\nabla(\nabla(\nabla\rho\cdot\dot{\mathbf{R}}))\cdot\dot{\mathbf{R}} - \frac{i\hbar}{2m_o}\nabla(\nabla(\nabla\rho\cdot\dot{\mathbf{R}})\cdot\dot{\mathbf{R}})$$

$$+\frac{\hbar^2}{4m_o^2}\nabla^2\nabla^2\rho\dot{\mathbf{R}} + \frac{\hbar^2}{2m_o^2}\nabla(\nabla(\nabla^2\rho)\cdot\dot{\mathbf{R}}) + \frac{i\hbar^3}{8m_o^3}\nabla\nabla^2\nabla^2\rho, \tag{4.39}$$

where the ∇-operations act on ρ only. From here it is obvious that the higher order commutators get increasingly complicated. The general form for the nested commutators $(\mathrm{ad}^{n+1}H)\mathbf{j}(\mathbf{r},t)$ turns out to be [27]

$$\frac{[(\mathrm{ad}^{n+1}H)\mathbf{j}(\mathbf{r},t)]}{\hbar^n} = (-i)^n\sum_{m=0}^{n}\left(\frac{-i\hbar}{2m_o}\right)^m\binom{n}{m}(\nabla^2)^m\rho\frac{d^{n+1-m}\mathbf{R}}{dt^{n+1-m}}$$

$$+(-i)^n\sum_{m=1}^{n}\left(\frac{-i\hbar}{2m_o}\right)^m\binom{n}{m-1}\nabla(\nabla^2)^{m-1}\left(\nabla\rho\cdot\frac{d^{n+1-m}\mathbf{R}}{dt^{n+1-m}}\right)$$

$$-i\left(\frac{-\hbar}{2m_o}\right)^{n+1}\nabla(\nabla^2)^n\rho. \tag{4.40}$$

In order to obtain (4.40), all of the non-linear terms have been neglected. We can check this equation for $n=1$, in which case

$$[H,\mathbf{j}] = -i\sum_{m=0}^{1}\left(\frac{-i\hbar}{2m_o}\right)^m\binom{1}{m}(\nabla^2)^m\rho\frac{d^{2-m}}{dt^{2-m}}\mathbf{R}$$

$$-i\sum_{m=1}^{1}\left(\frac{-i\hbar}{2m_o}\right)^m\binom{1}{m-1}\nabla(\nabla^2)^{m-1}\left(\nabla\rho\cdot\frac{d^{2-m}}{dt^{2-m}}\mathbf{R}\right) - i\left(\frac{-\hbar}{2m_o}\right)^2\nabla(\nabla^2)\rho.$$

But, since

$$\binom{1}{0}\equiv 1, \qquad \text{and} \qquad \binom{1}{1}=1,$$

equation (4.40) yields

$$[H,\mathbf{j}] = -i\rho\ddot{\mathbf{R}} - \frac{\hbar}{2m_o}\nabla^2\rho\dot{\mathbf{R}} - \frac{\hbar}{2m_o}\nabla(\nabla\rho\cdot\dot{\mathbf{R}}) - \frac{i\hbar^2}{4m_o^2}\nabla(\nabla^2\rho).$$

By comparing with the linear terms of equation (4.37), equation (4.40) checks out.

When equation (4.40) and the corresponding left-ordered expression are inserted into the equation of motion (4.35), the last term in (4.40) vanishes upon angular integration because of the assumed spherical symmetry of the charge distribution. Also, in symmetrizing, the order of the operators is reversed, $\hbar \to -\hbar$. Therefore, all terms containing odd powers of $\hbar$ drop out of the equation of motion. The equation (4.35) therefore becomes [28]

$$m_o\ddot{\mathbf{R}}(t) = e\mathbf{E}_{\text{in}} + \frac{2e^2}{3c^3}\sum_{n=0}^{\infty}\sum_{k=0}^{(n+1)/2}\frac{(-1)^{n+1}}{n!c^{n-2k}}\left(\frac{-\lambda^2}{4}\right)\binom{n+1}{2k}$$

$$\left[\int\int d\mathbf{x}d\mathbf{x}'\rho(x)|\mathbf{x}-\mathbf{x}'|^{n-1}(\nabla^2_{\mathbf{x}'})^2k\rho(x')\right]\frac{d^{n+2-2k}\mathbf{R}}{dt^{n+2-2k}}$$

$$+\frac{2e^2}{3c^2}\sum_{n=1}^{\infty}\sum_{k=1}^{(n+1)/2}\frac{(-1)^{n+1}}{n!c^{n-2k}}\left(\frac{-\lambda^2}{4}\right)^k\binom{n+1}{2k-1}$$

$$\left[\int\int d\mathbf{x}d\mathbf{x}'\rho(x)|\mathbf{x}-\mathbf{x}'|^{n-1}(\nabla^2_{\mathbf{x}'})^{2k-1}\nabla_{\mathbf{x}'}(\nabla_{\mathbf{x}'})\rho(x')\right]\frac{d^{n+2-2k}}{dt^{n+2-2k}}\mathbf{R},$$

where $\lambda = \hbar/m_o c$ is the Compton wavelength. The equation of motion can be simplified by using the spherical symmetry property to replace $(\nabla_{\mathbf{x}'})_j(\nabla_{\mathbf{x}'})_i$ by $\frac{1}{3}\delta_{ij}\nabla^2_{\mathbf{x}'}$ under the integral sign. Also, a change of indices $\sum_n\sum_k \to \sum_l\sum_k$, where $l \equiv n-2k$ is made. The equation of motion then takes the form [29]

$$m_o\ddot{\mathbf{R}}(t) = e\mathbf{E}_{\text{in}}(\mathbf{R}(t),t) - \frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}A_n\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t), \tag{4.41}$$

where

$$A_n = \left(1+\frac{\lambda}{3(n+2)}\frac{\delta}{\delta\lambda}\right)\left(1+\frac{\lambda}{n+1}\frac{\delta}{\delta\lambda}\right)\Omega_n, \tag{4.42}$$

and

$$\Omega_n = \sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{-\lambda^2}{4}\right)^k\int\int d\mathbf{x}d\mathbf{x}'\rho(x)|\mathbf{x}-\mathbf{x}'|^{n-1+2k}(\nabla^2_{\mathbf{x}'})^{2k}\rho(x'). \tag{4.43}$$

Equation (4.41) is the quantum equation of motion for an extended electron. It is the quantum equivalent of the classical equation of an extended electron model, equation (3.17). The only difference in form of the two equations lies in the coefficients γ_n and A_n. In the proper correspondence limit, (4.41) reduces to (3.17). The classical limit is when $\hbar \to 0$. But this implies $\lambda \to 0$, in which case only the leading term in (4.43) survives. Therefore, equation (4.43) contains an infinite number of quantum "correction" terms which have no classical counterpart.

The infinite series in Ω_n is an important difference between the quantum and classical theories, and allows quantum mechanics to give a more complete desciption of the electron. A useful study would be to investigate the mathematical origin of these extra terms, and see why classical mechanics was unable to generate them. Rather than considering this idea however, we turn our attention to seeing if quantum mechanics can resolve the infinite self-energy problem.

4.2 The Electrostatic Self-Energy for a Point Charge

We now turn our attention to the evaluation of the structure dependent coefficients A_n. The first question we would like to consider is whether or not the quantum mechanical theory suffers from the problem of infinite self-energy in the point limit. Remember that the self-energy term comes from the first term of the series in the equation of motion. In other words, it is the coefficient of the $\ddot{\mathbf{R}}$ term in the series. Therefore, we have

$$\delta m = \frac{2e^2}{3c^2} A_o = \frac{2e^2}{3c^2} \left(1 + \frac{\lambda}{6}\frac{\partial}{\partial\lambda}\right)\left(1 + \lambda\frac{\partial}{\partial\lambda}\right)\Omega_o. \tag{4.44}$$

In order to simplify (4.44), we will first need to calculate Ω_o, and then take the appropriate derivatives. Beginning with (4.43),

$$\Omega_o = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{-\lambda^2}{4}\right)^k \int\int d\mathbf{x}d\mathbf{x}'\rho(x)|\mathbf{x}-\mathbf{x}'|^{2k-1}(\nabla^2_{\mathbf{x}'})^{2k}\rho(x'). \tag{4.45}$$

It will be convenient to evaluate (4.45) in k-space. In order not to confuse the summation index k with k-space, we will change it from k to l. We will also use the Fourier transforms

$$\rho(x) = \frac{1}{(2\pi)^3}\int \tilde{\rho}(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \quad \text{and} \quad \tilde{\rho}(k) = \int \rho(x)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}. \tag{4.46}$$

Also, notice that $r = |\mathbf{x}-\mathbf{x}'|$. Therefore, equation (4.45) can be written as

$$\Omega_o = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\left(\frac{\lambda}{2}\right)^{2l} \int\int d\mathbf{x}d\mathbf{x}'\frac{1}{(2\pi)^3}\int \tilde{\rho}(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\frac{r^{2l}}{r}(\nabla^2_{\mathbf{x}'})^{2l}\rho(x'). \tag{4.47}$$

To simplify, notice that

$$(\nabla^2_{\mathbf{x}'})^{2l}\rho(x') = (\nabla^2_{\mathbf{x}'})^{2l}\frac{1}{(2\pi)^3}\int \tilde{\rho}(k')e^{i\mathbf{k}'\cdot\mathbf{x}'}d\mathbf{k}'$$
$$= \frac{1}{(2\pi)^3}\int \tilde{\rho}(k')(-k'^2)^{2l}e^{i\mathbf{k}'\cdot\mathbf{x}'}d\mathbf{k}'.$$

Also, rewriting $\mathbf{x}$ as $\mathbf{r}+\mathbf{x}'$, we have

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}\cdot\mathbf{x}'}.$$

Also, $d\mathbf{x} = d\mathbf{r}$, and (4.47) becomes

$$\Omega_o = \sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left(\frac{\lambda}{2}\right)^{2l}\int\int\int\int d\mathbf{r}d\mathbf{x}'d\mathbf{k}d\mathbf{k}'$$
$$\frac{1}{(2\pi)^6}\tilde{\rho}(k)\rho(k')e^{i\mathbf{k}\mathbf{r}}e^{i\mathbf{k}\mathbf{x}'}r^{2l-1}(-k'^2)^{2l}e^{i\mathbf{k}'\mathbf{x}'}.$$

Now,

$$\frac{1}{(2\pi)^3}\int d\mathbf{x}'e^{i\mathbf{k}\mathbf{x}'}e^{i\mathbf{k}'\mathbf{x}'} = \delta(\mathbf{k}+\mathbf{k}').$$

Performing the integration over $\mathbf{k}'$ yields

$$\Omega_o = \sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left(\frac{\lambda}{2}\right)^{2l}\int\int d\mathbf{r}d\mathbf{k}\frac{1}{(2\pi)^3}\tilde{\rho}(k)\tilde{\rho}(-k)e^{i\mathbf{k}\cdot\mathbf{r}}r^{2l-1}(-k^2)^{2l}.$$

Bringing the summation inside the integral where it belongs in a careful derivation (see transition from (4.21) to (4.23)), and using the fact that $\tilde{\rho}(-k) = \tilde{\rho}(k)$, Ω_o takes the form

$$\Omega_o = \int\frac{d\mathbf{k}}{(2\pi)^3}\tilde{\rho}(k)^2\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left(\frac{\lambda k^2}{2}\right)^{2l}\int d\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}r^{2l-1}. \qquad (4.48)$$

The last integral can be written as

$$\int \frac{d\mathbf{r}}{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2l} = -\int_0^\infty \int_0^\pi r^2 dr d\cos\theta d\phi e^{ikr\cos\theta} r^{2l-1}$$
$$= 2\pi \int_0^\infty r^{2l+1} dr \int_{-1}^{1} dx e^{ikrx}, \tag{4.49}$$

where we let $x = \cos\theta$. The second integral on the right hand side of (4.49) is easily evaluated, and yields

$$\int_{-1}^{1} dx e^{ikrx} = \frac{2}{kr} \sin kr.$$

Equation (4.48) can therefore be written as

$$\Omega_o = \int \frac{d\mathbf{k}}{2\pi^2} \tilde{\rho}(k)^2 \frac{1}{k} \int_o^\infty \sin kr \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda k^2}{2}\right)^{2l} r^{2l}. \tag{4.50}$$

The series has the sum of $\cos(\lambda k^2 r/2)$. Using the trigonometric identity

$$\sin\alpha cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)],$$

and doing the $\mathbf{k}$ angular integrations, equation (4.50) becomes

$$\Omega_o = \frac{1}{\pi} \int dk \tilde{\rho}(k)^2 k \int_0^\infty dr \left[\sin\left(k + \frac{\lambda k^2}{2}\right) r + \sin\left(k - \frac{\lambda k^2}{2}\right) r\right]. \tag{4.51}$$

We consider the two integrations over r separatly. Using a convergence factor and integral tables [30],

$$\lim_{\mu\to 0} \int_0^\infty dr e^{-\mu r} \sin\left(k + \frac{\lambda k^2}{2}\right) r = \lim_{\mu\to 0} e^{-\mu r} \frac{\left(k + \frac{\lambda k^2}{2}\right)}{\mu^2 + \left(k + \frac{\lambda k^2}{2}\right)}$$
$$= \frac{1}{k + \frac{\lambda k^2}{2}}.$$

Similarly,

$$\lim_{\mu\to 0} \int_0^\infty dr e^{-\mu r} \sin\left(k - \frac{\lambda k^2}{2}\right) r = \frac{1}{k - \frac{\lambda k^2}{2}}.$$

Therefore, the term in square brackets of (4.51) becomes

$$[\ \ ] = \frac{1}{k+\frac{\lambda k^2}{2}} + \frac{1}{k-\frac{\lambda k^2}{2}} = \frac{2}{k\left(1-\frac{\lambda^2 k^2}{4}\right)},$$

and Ω_o can be written in final form as

$$\Omega_o = \frac{2}{\pi} P \int dk \frac{\tilde{\rho}(k)^2}{1-\lambda^2 k^2/4}, \tag{4.52}$$

where we have regularized using the principle value P defined as

$$P\int_a^b dk \equiv \lim_{\epsilon\to 0} \int_a^{\text{singularity}-\epsilon} dk + \int_{\text{singularity}+\epsilon}^b dk.$$

In changing to k-space, we have been able to simplify (4.45). Using equation (4.52) we can study the behavior of δm more easily than we could have in x-space. Once we have δm for an extended charge, we will observe its behavior in the point limit.

In order to evaluate (4.52) further, it will be necessary to choose specific charge distributions. The two charge distributions we will consider are the spherical shell and the Yukawa distributions as used in *Chapter 3*. The most straightforward is the Yukawa distribution:

$$\tilde{\rho}(k) = \frac{1}{1+k^2L^2}.$$

Subsitituting into (4.52), Ω_o becomes

$$\Omega_o = \frac{2}{\pi} P \int_0^\infty dk \frac{1}{(1-\lambda^2k^2/4)(1+k^2L^2)^2}.$$

By using Mathematica to evaluate the integral, Ω_o can be written in the form

$$\Omega_o = 4L\frac{(4L^2+3\lambda^2)}{(4L^2+\lambda^2)^2} = \frac{4}{L}\frac{(4+3\lambda^2/L^2)}{(4+\lambda^2/L^2)^2}. \tag{4.53}$$

Notice that when $\lambda = 0$, Ω_o is equal to $1/L$, which when inserted back into (4.42), yields the classical result for γ_o. Inserting (4.53) into equation (4.44), and allowing Mathematica to perform the differentiations and simplifying algebra, yields

$$\delta m = \frac{2e^2}{3c^2} \frac{8L(32L^6 + 64L^4\lambda^2 - 6L^2\lambda^4 - \lambda^6)}{(4L^2 + \lambda^2)^4},$$

which can be rewritten in the form

$$\delta m = \frac{2e^2}{3c^2} \frac{8L}{\lambda^2} \frac{(32L^6/\lambda^6 + 64L^4/\lambda^4 - 6L^2/\lambda^2 - 1)}{(4L^2/\lambda^2 + 1)^4}.$$

We can now simplify by defining the dimensionless parameter $\xi = 2L/\lambda$, and using the fine structure constant $\alpha = e^2/\lambda mc^2$. The final form of δm is then

$$\delta m = \left(\frac{2}{3}\alpha m\right) \frac{2\xi(\xi^6 + 8\xi^4 - 3\xi^2 - 2)}{(1 + \xi^2)^4}. \tag{4.54}$$

Equation (4.54) is the self-mass for an extended quantum particle using a Yukawa model. To observe the behavior of δm as a function of its size, we can plot (4.54). First, we will divide by $\frac{2}{3}\alpha m$. *Figure 4.1* shows the plot in which it can be seen that $\delta m = 0$ for $\xi = 0$. But $\xi = 0$ corresponds to $L = 0$, or a point charge. Therefore, we see that for a point charge the self-mass is zero. The quantum point theory using a Yukawa model does not seem to stumble on the problem of infinite self-energy as does the classical point theory.

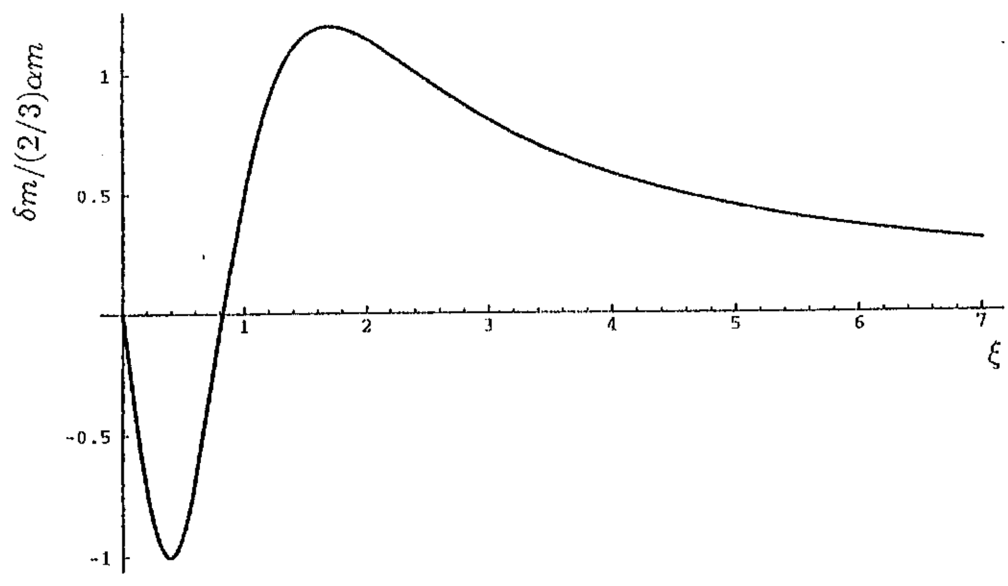

Figure 4.1: *Self-Mass for a Yukawa Distribution.*

Let us check the generality of the above results by deriving δm for the spherical shell, and see if it too goes to zero in the point limit. To do so, we insert the charge density

$$\tilde{\rho}(k) = \frac{1}{kL(2\pi)^{3/2}} \sin kL,$$

into (4.52):

$$\Omega_o = \frac{1}{4\pi^4 L^2} \int_0^\infty dk \frac{\sin^2(kL)}{k^2(1 - \lambda^2 k^2/4)}.$$

Once again, we can use Mathematica to obtain

$$\Omega_o = \frac{1}{4\pi^4 L^2} \left[L\pi - \frac{\pi\lambda \sin(4L/\lambda)}{4} \right].$$

Substituting into (4.51), and using Mathematica, we obtain

$$\delta m = \frac{2e^2}{3c^2} \left[\frac{12L\lambda + 16L\lambda\cos(4L/\lambda) + 8L^2 \sin(4L/\lambda) - 7\lambda^2 \sin(4L/\lambda)}{48L^2\pi^3\lambda} \right].$$

With the use of ξ and α as before, the self-mass can be written in the form

$$\delta m = \left(\frac{2}{3}\alpha m\right) \frac{1}{12\xi^2\pi^3} [6\xi + 8\xi\cos(2\xi) + (2\xi^2 - 7)\sin(2\xi)]. \qquad (4.55)$$

Once again we divide by $\frac{2}{3}\alpha m$, and plot (4.55) in *Figure 4.2.*

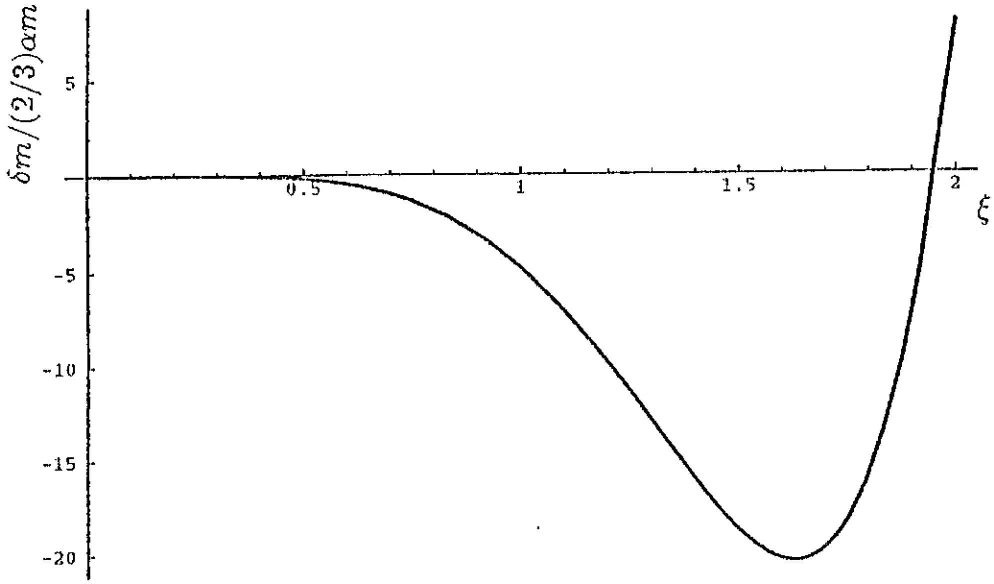

Figure 4.2: *Self-Mass for a Spherical Shell Distribution.*

We see that the self-mass is again gives zero, which turns out to be a general result for any model chosen. This can be seen by taking the point limit of $\rho(x)$ in general:

$$\lim_{L\to 0}\rho(x)=\frac{\delta(x)}{4\pi}.$$

The Fourier transform of $\delta(x)/4\pi$ is $\rho(k)=1$. We can check this transform by inserting $\rho(k)=1$ into

$$\rho(x)=\frac{1}{(2\pi)^3}\int d^3ke^{i\mathbf{k}\cdot\mathbf{x}}\rho(k)=\frac{1}{(2\pi)^3}\int d^3ke^{i\mathbf{k}\cdot\mathbf{x}}=\frac{\delta(x)}{4\pi}.$$

Inserting $\rho(k)$ for a point charge into (4.52) yields

$$\begin{aligned}\Omega_o&=\frac{2}{\pi}P\int_0^\infty dk\frac{1}{1-\lambda^2k^2/4}\\&=\frac{8}{\pi\lambda^2}P\int_0^\infty dk\frac{1}{\frac{4}{\lambda^2}-k^2}.\end{aligned}$$

The integral on the right can be done as follows;

$$
\begin{aligned}
I &= \lim_{\epsilon\to 0}\left[\int_0^{\frac{2}{\lambda}-\epsilon}\frac{dk}{\frac{4}{\lambda^2}-k^2}+\int_{\frac{2}{\lambda}+\epsilon}^{\infty}\frac{dk}{\frac{4}{\lambda^2}-k^2}\right] \\
&= \lim_{\epsilon\to 0}\left[-\frac{\lambda}{4}\ln\left|\frac{k-2/\lambda}{k+2/\lambda}\right|_0^{\frac{2}{\lambda}-\epsilon}-\frac{\lambda}{4}\ln\left|\frac{k-2/\lambda}{k+2/\lambda}\right|_{\frac{2}{\lambda}+\epsilon}^{\infty}\right] \\
&= -\frac{\lambda}{4}\lim_{\epsilon\to 0}\ln\left|\frac{(\epsilon+4/\lambda)}{(\epsilon-4/\lambda)}\right| \\
&= -\frac{\lambda}{4}\ln|-1| \\
&= 0.
\end{aligned}
$$

If $\Omega_o = 0$ in general, then by equation (4.44), the self-mass δm is zero in general. This is a significant difference between this theory and the classical theory. There is no self-energy of a point particle in quantum mechanics, whereas the self-energy of a classical particle is infinite. This has been recognized as a most remarkable result [14, 15]. What are we doing that the classical theory isn't? In order to compare expression (4.52) for Ω_o with the classical expression (3.18) for γ_o, we can transform (4.52) back into coordinate space using a three-dimensional convolution theorem (a straightforward generalization of the one-dimensional convolution theorem [31]). To derive the three-dimensional theory, we use the three-dimensional Fourier transforms as

$$
\begin{aligned}
f(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int \tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \\
\tilde{f}(\mathbf{k}) &= \int f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.
\end{aligned}
\tag{4.56}
$$

Let $f(\mathbf{x})$ and $g(\mathbf{x})$ have Fourier transforms $\tilde{f}(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$ respectively. We define

$$
f * g \equiv \int g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'
$$

to be the convolution of f and g. Now using (4.56), we have

$$
\begin{aligned}
\int g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}' &= \frac{1}{(2\pi)^3}\int g(\mathbf{x}')\int \tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d\mathbf{k}d\mathbf{x}' \\
&= \frac{1}{(2\pi)^3}\int \tilde{f}(\mathbf{k})\left[\int g(\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}'}d\mathbf{x}'\right]e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}.
\end{aligned}
$$

The above integral in brackets is just $\tilde{g}(\mathbf{k})$. Therefore, the three-dimensional convolution theorem is

$$f * g = \int g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}' = \frac{1}{(2\pi)^3}\int \tilde{g}(\mathbf{k})\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}. \tag{4.57}$$

Returning to (4.52), we see that it can be written as an integral over vector $\mathbf{k}$ in the following way:

$$\begin{aligned}\Omega_o &= \frac{2}{\pi}P\int_0^\infty dk\frac{\tilde{\rho}(k)^2}{1-\lambda^2k^2/4} \\ &= \frac{2}{\pi}P\int_0^\infty dk\frac{k^2}{k^2}\frac{1}{4\pi}\int \sin\theta_k d\theta_k d\phi\frac{\tilde{\rho}(k)^2}{1-\lambda^2k^2/4} \\ &= \frac{1}{2\pi^2}P\int\frac{d\mathbf{k}}{k^2}\frac{\tilde{\rho}(k)^2}{1-\lambda^2k^2/4}.\end{aligned}$$

Since $\tilde{\rho}(k) = \tilde{\rho}^*(k)$, we can write

$$\Omega_o = \frac{1}{(2\pi)^3}P\int\frac{4\pi}{k^2}d\mathbf{k}\frac{1}{1-\lambda^2k^2/4}\tilde{\rho}^*(k)\tilde{\rho}(k).$$

The Fourier transform of $\tilde{\rho}^*(k)$ is

$$\tilde{\rho}^*(k) = \int \rho(x)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.$$

Therefore,

$$\Omega_o = \int d\mathbf{x}\frac{1}{(2\pi)^3}P\int\frac{4\pi}{k^2}\frac{1}{1-\lambda^2k^2/4}\tilde{\rho}(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\rho(x). \tag{4.58}$$

To write the second integral in (4.58) as a convolution, remember that $\nabla^2_{\mathbf{x}}$ acting on $\rho(x)$ corresponds to $(-k^2)$ times $e^{i\mathbf{k}\cdot\mathbf{x}}\rho(k)$ in the Fourier integral. Therefore,

$$\frac{1}{1-\lambda^2k^2/4} \to \frac{1}{1+\lambda^2\nabla^2_{\mathbf{x}}/4},$$

and (4.58) becomes

$$\Omega_o = \int d\mathbf{x}\rho(x)\left[\frac{1}{(2\pi)^3}P\int\frac{4\pi}{k^2}\frac{1}{1+\lambda^2\nabla^2_{\mathbf{x}}/4}\tilde{\rho}(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\right]. \tag{4.59}$$

We can now use the convolution theory (4.57) by making the associations

$$\tilde{f}(k) = \frac{4\pi}{k^2}, \quad \text{and} \quad \tilde{g}(k) = \tilde{\rho}(k). \tag{4.60}$$

The operator factor $1/(1+\lambda^2\nabla^2_{\mathbf{x}}/4)$ can be treated as a constant because it is not a function of k, or does not operate on k. However, since it is an operator, we must be careful to keep its relative position to functions of $\mathbf{x}$.

We need the Fourier transform of the functions (4.60). For $\tilde{\rho}(k)$, the Fourier transform is just $\rho(x')$. We can show that the Fourier transform of $4\pi/k^2$ is $1/|\mathbf{x}-\mathbf{x}'|$ by using [32]

$$\nabla^2_r\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(|\mathbf{x}-\mathbf{x}'|),$$

which can be written as

$$\nabla^2_r\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{4\pi}{2\pi}\int e^{ikr}dk.$$

Working backwards, we now ask the question, "What is the Fourier transform of $1/|\mathbf{x}-\mathbf{x}'|$?" To answer this question, we write

$$\nabla^2_r\frac{1}{2\pi}\int \tilde{f}(k)e^{-ikr}dk = -\frac{4\pi}{2\pi}\int e^{ikr}dk.$$

Taking the Laplacian inside the integral, we obtain

$$\frac{1}{2\pi}\int \tilde{f}(k)(-k^2)e^{-ikr}dk = -\frac{4\pi}{2\pi}\int e^{ikr}dk.$$

We equate the integrands to obtain

$$\tilde{f}(k)(-k^2) = -4\pi, \quad \text{or} \quad \tilde{f}(k) = \frac{4\pi}{k^2}.$$

In other words, the Fourier transform of $4\pi/k^2$ is $1/|\mathbf{x}-\mathbf{x}'|$. Going back to (4.57), we see that $f(\mathbf{x}-\mathbf{x}') = 1/|\mathbf{x}-\mathbf{x}'|$. Therefore, the convolution theorem applied to the integral in brackets of (4.59) yields the final result

$$\Omega_o = \int\int \frac{d\mathbf{x}d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\rho(x)\frac{1}{1+\lambda^2\nabla^2_{\mathbf{x}'}/4}\rho(x'). \tag{4.61}$$

Now that Ω_o is written in x-space, we can compare it with the classical coefficients

$$\gamma_o = \int \frac{d\mathbf{x}d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\rho(x)\rho(x').$$

Both expressions have a $1/|\mathbf{x}-\mathbf{x}'|$ term. But notice that in the quantum equation (4.61) there is an operation acting on $\rho(x')$. This operation came about as the sum of the infinite series which was absent in the classical point electron equation. It is the operation which is able to make the integral converge.

The integration in frequency space is definitely supposed to reach beyond the singularities $k = 2/\lambda$ for the principal value regularization to work. In fact the very result $\delta m = 0$ hinges on the fact that one integrates over all frequencies. But high frequencies correspond to high energies. This is a problem since the nonrelativistic assumption is one of the basic ingredients of the theory and allows us *e.g.* to neglect the nonlinear terms in the equation of motion. In conclusion, we believe that this question needs to be looked into further, but this particular point underscores the peculiarity of the Moniz and Sharp approach.

We now want to continue on with the derivation of the quantum point theory. We want to consider whether or not it is free from the problems of runaway or preaccelerating solutions.

4.3 The Point Charge Equation of Motion

4.3.1 *Evaluation of the Even Coefficients* A_{2m}

In order to determine the behavior of the solutions for a quantum point electron, we must first obtain the equation of motion for a point model. The equation of motion for the extended electron (4.41) depends on the structure of the model chosen through the coefficients A_n. It can be argued using dimensional analysis that the ***even*** coefficients A_n must vanish. Consider the Ω_n (4.43). Notice that the first term in the series is just the classical coefficients γ_n (3.18) which have the dimensions of L^{n-1}. So Ω_n equals the classical coefficients multiplied by an infinite series in k:

$$\Omega_n \sim \gamma_n \sum_k \lambda^{2k} |\mathbf{x} - \mathbf{x}'|^{2k} (\nabla^2_{\mathbf{x}'})^{2k}.$$

The $\nabla_{\mathbf{x}'}$ operator has units of $1/L$. Therefore,

$$\Omega_n \sim L^{n-1} \sum_k \frac{\lambda^{2k}}{L^{2k}},$$

and thus $\Omega_n \sim L^{n-1}$. In the point limit $L \to 0$, all of the factors of L must disappear. For n even, $n \to 2m$, and we obtain a factor of L^{2m-1}. But this is an odd power of L, and it is not possible to obtain an odd power from the series in λ^2/L^2. Therefore, there is no possibility of canceling the factors of L unless all even coefficients vanish.

Now let us consider the problem by actually doing the calculations. To obtain a point charge theory, all we need to do is find the point limit of the Ω_n. We can find the even Ω_n by letting $n \to 2m$ in (4.43):

$$\Omega_{2m} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\frac{-\lambda^2}{4}\right)^k \int\int d\mathbf{x} d\mathbf{x}' \rho(x) |\mathbf{x} - \mathbf{x}'|^{2m-1+2k} (\nabla^2_{\mathbf{x}'})^{2k} \rho(x').$$

Following the derivation of (4.52), let $k \to l$, and use $r = |\mathbf{x} - \mathbf{x}'|$, as well as the Fourier transforms (4.46) to write

$$\Omega_{2m} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda}{2}\right)^{2l} \int\int d\mathbf{x} d\mathbf{x}' \frac{1}{(2\pi)^3} \int \tilde{\rho}(k) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \frac{r^{2l+2m}}{r} (\nabla_{\mathbf{x}'})^{2l} \rho(x').$$

In analogy with the simplifying steps for Ω_o from equation (4.47) to (4.48), we have

$$\Omega_{2m} = \int \frac{d\mathbf{k}}{(2\pi)^3}\tilde{\rho}(k)^2 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\left(\frac{\lambda k^2}{2}\right)^{2l} \int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2(l+m)-1}. \tag{4.62}$$

Now, by comparison with (4.49),

$$\begin{aligned}
\int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2(l+m)-1} &= 2\pi \int_0^{\infty} r^{2(l+m)+1} dr \int_{-1}^{1} dx e^{ikrx} \\
&= 2\pi \int_0^{\infty} r^{2(l+m)+1} dr \frac{2}{kr} \sin kr \\
&= \frac{4\pi}{k} \frac{[2(l+m)]!(-1)^{l+m}}{k^{2(l+m)+1}}.
\end{aligned}$$

Therefore, (4.62) can be written as

$$\begin{aligned}
\Omega_{2m} &= \int \frac{d\mathbf{k}}{(2\pi)^3}\tilde{\rho}(k)^2 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\left(\frac{\lambda k^2}{2}\right)^{2l} \frac{4\pi[2(l+m)]!(-1)^{l+m}}{k^{2(l+m+1)}} \\
&= \frac{1}{2\pi^2} \int d\mathbf{k}\tilde{\rho}(k)^2 \frac{(-1)^m}{k^{2m+2}} \sum_{l=0}^{\infty} \frac{[2(l+m)]!}{(2l)!}\left(\frac{\lambda k}{2}\right)^{2l}.
\end{aligned} \tag{4.63}$$

Compare (4.63) for $m = 0$ with equation (4.51). Using the definition of the Fourier transform of $\tilde{\rho}(k)$, equation (4.63) can be written as

$$\Omega_{2m} = (-1)^m \int\int d\mathbf{r} d\mathbf{k} \frac{1}{(2\pi)^3} \frac{4\pi}{k^{2m+2}} \left[\sum_{l=0}^{\infty} \frac{(2l+2m)!}{(2l)!}\left(\frac{\lambda^2 k^2}{4}\right)^l\right] \tilde{\rho}(k) e^{i\mathbf{k}\cdot\mathbf{x}} \rho(x).$$

We now replace k^2 by $-\nabla_{\mathbf{x}}^2$ and write Ω_{2m} in the form

$$\Omega_{2m} = (-1)^m \int\int d\mathbf{r} d\mathbf{k} \rho(x) \left[\sum_{l=0}^{\infty} \frac{(2l+2m)!}{(2l)!}\left(\frac{-\lambda^2 \nabla_{\mathbf{x}}^2}{4}\right)^l\right] \frac{4\pi}{(2\pi)^3} \frac{\tilde{\rho}(k) e^{i\mathbf{k}\cdot\mathbf{x}}}{k^{2m+2}}.$$

The series in brackets has the sum [33]

$$\sum_{l=0}^{\infty} \frac{(2l+2m)!}{(2l)!}\left(\frac{-\lambda^2 \nabla_{\mathbf{x}}^2}{4}\right)^l = \frac{\partial^{2m}}{\partial \lambda^{2m}}\left[\lambda^{2m} \frac{1}{1+\lambda^2 \nabla_{\mathbf{x}}^2/4}\right]. \tag{4.64}$$

We can easily check (4.64) for $m = 0$. In this case the sum is just the geometric sum of operators which can be written formally as

$$\sum_{l=0}^{\infty} \left(\frac{-\lambda^2 \nabla_{\mathbf{x}}^2}{4} \right)^l = \frac{1}{1 + \lambda^2 \nabla_{\mathbf{x}}^2/4},$$

We can also check for $m = 1$, in which case the left hand side of (4.64) is

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{(2l+2)!}{(2l)!} \left(\frac{\lambda^2 \nabla_{\mathbf{x}}^4}{2} \right)^l &= \sum_{l=0}^{\infty} (2l+1)(2l+2) \left(\frac{\lambda^2 \nabla_{\mathbf{x}}{}^2}{4} \right)^l \\ &= 4 \sum l^2 a^l + 6 \sum l a^l + 2 \sum a^l, \end{aligned} \tag{4.65}$$

where $a \equiv \lambda^2 \nabla_{\mathbf{x}}^2/4$. The last two series in (4.65) have the sums [34]

$$\sum_{l=0}^{\infty} a^l = \frac{1}{1-a}, \tag{4.66}$$

and

$$\sum_{l=0}^{\infty} l a^l = \frac{a}{(1-a)^2}. \tag{4.67}$$

In order to find the first sum in (4.65), notice that

$$\begin{aligned} \sum_{l=0}^{\infty} l(l-1) a^{l-2} &= \frac{d^2}{da^2} \sum_{l=0}^{\infty} a^l = \frac{d^2}{da^2} (1-a)^{-1} \\ &= \frac{d}{da} (1-a)^{-2} = \frac{2}{(1-a)^3}. \end{aligned} \tag{4.68}$$

Then, let $f(x) = \sum l^2 a^l$, so that

$$\frac{f(x)}{a^2} = \sum_{l=0}^{\infty} l^2 a^{l-2}.$$

Now if we subtract $\sum l a^l$ from both sides,

$$\frac{1}{a^2}\left(f(x) - \sum l a^l\right) = \sum l^2 a^{l-2} = \sum l(l-1)a^{l-2} = \frac{2}{(1-a)^3},$$

where the last equality follows by equation (4.68). Solving for $f(x)$ yields

$$f(x) = \sum_{l=0}^{\infty} l^2 a^l = \frac{2a^2}{(1-a)^3} + \frac{a}{(1-a)^2}. \tag{4.69}$$

Inserting the results from (4.66), (4.67), and (4.69) into (4.65) yields

$$\sum_{l=0}^{\infty} \frac{(2l+2)!}{(2l)!} a^l = \frac{8a^2}{(1-a)^3} + \frac{10a}{(1-a)^2} + \frac{2}{1-a}. \tag{4.70}$$

Now for the right hand side of (4.63), we define $b \equiv \nabla_{\mathbf{x}}^2/4$, and obtain

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}[\lambda^2(1+b\lambda^2)^{-1}] &= \frac{\partial}{\partial\lambda}[2\lambda(1+b\lambda^2)^{-1} - 2b\lambda^3(1+b\lambda^2)^{-2} \\ &= \frac{2}{1+b\lambda^2} - \frac{10b\lambda^2}{(1+b\lambda^2)^2} + \frac{8b^2\lambda^4}{(1+b\lambda^2)^3},\end{aligned}$$

or in terms of a,

$$\frac{\partial^2}{\partial\lambda^2}[\lambda^2(1+b\lambda^2)^{-1}] = \frac{2}{1-a} + \frac{10a}{(1-a)^2} + \frac{8a^2}{(1-a)^3}. \tag{4.71}$$

So equation (4.64) checks for $m=1$ also. We will therefore accept (4.63) as being valid. We must use the three-dimensional convolution theorem, which allows us to make the association [35]

$$\frac{1}{(2\pi)^3}\int d\mathbf{k} \frac{4\pi}{k^{2m+2}}\tilde{\rho}(k)e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{(-1)^m}{(2m)!}\int d\mathbf{x}'\rho(x')|\mathbf{x}-\mathbf{x}'|^{2m-1}, \tag{4.72}$$

where $\tilde{g}(k) = \tilde{\rho}(k)$ and $\tilde{f}(k) = 4\pi/k^{2m+2}$ with the Fourier transform of $\tilde{\rho}(k)$ being $\rho(x')$, and for $\tilde{f}(k)$, it is $\frac{(-1)^m}{(2m)!}|\mathbf{x}-\mathbf{x}'|^{2m-1}$. For $m=0$, (4.72) collapses to our previous convolution theorem in obtaining Ω_o. Equation (4.63) can now be written as

$$\Omega_{2m} = \frac{1}{(2m)!}\frac{\partial^{2m}}{\partial\lambda^{2m}}\left[\lambda^{2m}\int\int d\mathbf{x}d\mathbf{x}'\rho(x)|\mathbf{x}-\mathbf{x}'|^{2m-1}\frac{1}{1+\lambda^2\nabla_{\mathbf{x}'}^2/4}\rho(x')\right], \tag{4.73}$$

This is our final expression for the even coefficients of an extended electron. Since we are after an equation of motion for the point electron, we can evaluate (4.73) in the point limit. We already found that $\Omega_o = 0$. Notice that for $2m \geq 2$, the factor $|\mathbf{x} - \mathbf{x}'|^{2m-1}$ goes to zero. Therefore, all of the even coefficients Ω_{2m} vanish.

4.3.2 *Evaluation of the Odd Coefficients* A_{2m+1}

The odd coefficients are found by letting $n \to 2m+1$ in (4.43):

$$\Omega_{2m+1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\frac{-\lambda^2}{4} \right)^k \int \int d\mathbf{x} d\mathbf{x}' \rho(x) |\mathbf{x} - \mathbf{x}'|^{2m+2k} (\nabla^2_{\mathbf{x}'})^{2k} \rho(x').$$

Again following the derivation of (4.52), we let

$$k \to l, \quad \text{and} \quad \rho(x) = \frac{1}{(2\pi)^3} \int \tilde{\rho}(k) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}.$$

Since $r^{2(m+k)} = |\mathbf{x} - \mathbf{x}'|^{2(m+k)}$, we have

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda}{2} \right)^{2l} \int \int d\mathbf{x} d\mathbf{x}' \frac{1}{(2\pi)^3} \int \tilde{\rho}(k) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} r^{2m+2l} (\nabla^2_{\mathbf{x}'})^{2l} \rho(x').$$

We will also use

$$\mathbf{x} = \mathbf{r} + \mathbf{x}' \quad \text{and} \quad \tilde{\rho}(k) = \tilde{\rho}^*(k) = \int d\mathbf{x}' e^{i\mathbf{k}\cdot\mathbf{x}'} \rho(x').$$

Therefore,

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda}{2} \right)^{2l} \int d\mathbf{k} \tilde{\rho}(k)^2 (k^2)^{2l} \frac{1}{(2\pi)^3} \int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2m+2l}.$$

The integrals over θ_k and ϕ_k yield a factor of 4π, which leaves

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda}{2} \right)^{2l} \int_0^{\infty} dk k^2 \tilde{\rho}(k)^2 (k^2)^{2l} \frac{1}{2\pi^2} \int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2m+2l}. \qquad (4.74)$$

The last integral on the right integrates as follows:

$$
\begin{aligned}
I \equiv \frac{1}{2\pi^2}\int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^{2m+2l} &= \frac{1}{\pi}\int_0^\infty r^{2m+2l+2}\int_{-1}^{1} e^{ikr\cos\theta} d\cos\theta \\
&= \frac{2}{\pi k}\int_0^\infty r^{2m+2l+1}\sin kr dr \\
&= \frac{2}{\pi k}\int_0^\infty r^{2m+2l+1}\frac{(e^{ikr}-e^{-ikr})}{2i}dr. \qquad (4.75)
\end{aligned}
$$

Notice that the exponential factors can be written in the form

$$e^{ikr} = \frac{1}{(ir)^{2m+2l+1}}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1} e^{ikr},$$

or

$$e^{-ikr} = \frac{1}{(-ir)^{2m+2l+1}}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1} e^{-ikr}.$$

The factors of r will then cancel, and the factors of i, along with the i in (4.75), become $(-1)^{m+l+1}$. Equation (4.75) is therefore

$$I = \frac{2}{\pi k}(-1)^{m+l+1}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1}\int_0^\infty dr\frac{(e^{ikr}-e^{-ikr})}{2}.$$

where we have reversed the order of integration and differentiation. If we consider the integral over the second exponential term separately, we can simplify I even further. Notice that if we let $r \to -r$ in the second integral, switch the limits of integration, and add the two integrals back together, we obtain

$$I = \frac{2}{k}(-1)^{m+l+1}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1}\frac{1}{2\pi}\int_{-\infty}^{\infty} dr e^{ikr}.$$

But by using the definition of the delta function, we have

$$I = \frac{2}{k}(-1)^{m+l+1}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1}\delta(k).$$

We insert this result back into (4.74), and also change the limits of integration over k to go from $-\infty$ to ∞ and divide by 2. We can make this change since the integrand is *even* in k. Equation (4.74) can therefore be written as

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left(\frac{\lambda}{2}\right)^{2l} \int_{-\infty}^{\infty} dk k \tilde{\rho}(k)^2 (k^2)^{2l} (-1)^{m+l+1} \left(\frac{\partial}{\partial k}\right)^{2m+2l+1} \delta(k). \tag{4.76}$$

Since we want to obtain the equation of motion for a point electron, we take the point limit of (4.76). In the point limit, $\tilde{\rho}(k) = 1$, and we have

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^{m+l+1}}{(2l)!} \left(\frac{\lambda}{2}\right)^{2l} \int_{-\infty}^{\infty} dk k^{4l+1} \left(\frac{\partial}{\partial k}\right)^{2m+2l+1} \delta(k).$$

By partial integration, the partial derivative with respect to k can be switched from operating on the delta function to operating on k^{4l+1}. We partial integrate $2m+2l+1$ times in order to kill all of the partial derivatives. Doing so leaves us with a simple integration over k. Notice that by partial integrating $2m+2l+1$ times brings down the power of k that many times, yielding $2m+2l+1$ terms in the product $(4l+1)(4l)(4l-1)\cdots$. Each time we partial integrate, we also obtain a minus sign. Therefore, we have an additional $(-1)^{2m+2l+1}$. Taking all of this into account,

$$\Omega_{2m+1} = \sum_{l=0}^{\infty} \frac{(-1)^{3m+4l+2}}{(2l)!} \left(\frac{\lambda}{2}\right)^{2l} (4l+1)(4l)(4l-1)\cdots(4l+1-2m-2l-1)$$
$$\int_{-\infty}^{\infty} dk k^{4l+1-2m-2l-1} \delta(k).$$

With the delta function, the integral over k is easily evaluated. But notice what it implies. Since k becomes zero, the only nonzero value is when its exponent is equal to zero. Therefore, $l = m$. Notice that the product $(4l+1)(4l)(4l-1)\cdots$ now has exactly $4m+1$ terms, and can be written as $(4m+1)!$. Therefore,

$$\Omega_{2m+1} = \frac{(-1)^m}{(2m)!} \left(\frac{\lambda}{2}\right)^{2m} (4m+1)!.$$

Now converting $2m+1$ back to n we have $m = (n-1)/2$. Therefore,

$$\Omega_n = (-1)^{(n-1)/2} \frac{(2n-1)!}{(n-1)!} \frac{\lambda^{n-1}}{2^{n-1}}.$$

The factorials and factors of 2 combine into the single term $(2n-1)!!$. Therefore, the final result for Ω_n is

$$\Omega_n = \begin{cases} (-1)^{(n-1)/2}(2n-1)!!\lambda^{n-1}, & \text{for } n \text{ odd,} \\ 0, & \text{for } n \text{ even.} \end{cases}$$

These are the Ω_n for the point charge which will give us the point charge equation of motion. Inserting the Ω_n into (4.42), and performing the first differentiation, we have

$$\begin{aligned} A_{n\ \mathrm{odd}} &= \left(1+\frac{\lambda}{3(n+2)}\frac{\partial}{\partial\lambda}\right)\left(1+\frac{\lambda}{n+1}\frac{\partial}{\partial\lambda}\right)(-1)^{(n-1)/2}(2n-1)!!\lambda^{n-1} \\ &= \left(1+\frac{\lambda}{3(n+2)}\frac{\partial}{\partial\lambda}\right)\left[(-1)^{(n-1)/2}(2n-1)!!\lambda^{n-1}\right. \\ &\qquad\qquad \left. +\frac{(-1)^{(n-1)/2}(2n-1)!!}{n+1}(n-1)\lambda^{n-1}\right], \end{aligned}$$

or,

$$A_{n\ \mathrm{odd}} = (-1)^{(n-1)/2}(2n-1)!!\left(1+\frac{\lambda}{3(n+2)}\frac{\partial}{\partial\lambda}\right)\lambda^{n-1}\left(1+\frac{n-1}{n+1}\right).$$

But

$$\left(1+\frac{n-1}{n+1}\right) = \frac{2n}{n+1},$$

and the second differentiation yields

$$A_{n\ \mathrm{odd}} = (-1)^{(n-1)/2}(2n-1)!!\left(1+\frac{(n-1)}{3(n+2)}\right)\lambda^{n-1}\frac{2n}{n+1}.$$

Since

$$\left(1+\frac{n-1}{3(n+2)}\right) = \frac{4n+5}{3(n+2)},$$

we obtain the final form of $A_{n\ \mathrm{odd}}$ as

$$A_{n\ \mathrm{odd}} = (-1)^{(n-1)/2}\frac{2n(4n+5)}{3(n+1)(n+2)}(2n-1)!!\lambda^{n-1}.$$

These are the odd coefficients for a point charge, and the even coefficients are zero. Therefore, by (4.41), we see that the final expression for the equation of motion of a point charge is

$$m_o\ddot{\mathbf{R}}(t) = e\mathbf{E}_{in}(\mathbf{R}(t), t) + \frac{2e^2}{3c^2}\sum_{n \text{ odd}} \frac{1}{n!c^n} A_n \frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t) \tag{4.77}$$

where

$$A_n = \begin{cases} (-1)^{(n-1)/2}\frac{2n(4n+5)}{3(n+1)(n+2)}(2n-1)!!\lambda^{n-1}, & \text{for } n \text{ odd,} \\ 0, & \text{for } n \text{ even.} \end{cases} \tag{4.78}$$

This is the equation we have been looking for. It is the quantum mechanical equation of motion for the point electron. Notice that this equation of motion for a quantum point charge has maintained an infinite series. The classical point equation (2.6) on the other hand only has one term coming from radiation reaction effects. Equation (4.77) is similar to the classical equation (3.17) for the *extended* electron, where λ plays the role of a size parameter instead of L as in the classical case. We now study the behavior of the solutions of equation (4.77). In what follows, the symbol m_o represents the mechanical rest mass from the Moniz and Sharp theory including the δm term from the A_o coefficient.

4.4 Solutions in the Absence of External Forces

In this section we consider the behavior of the solutions for the quantum mechanical equation of motion (4.77). Is this equation for a point electron well-behaved, or does it display runaway solutions like the classical equation? We showed in section 4.2 that in quantum mechanics, the self-energy for a point electron is zero. If there are no runaway or preaccelerating solutions, then the problems of the classical theory do not exist in quantum mechanics.

4.4.1 *The Constant Velocity Solution*

In order to write the equation of motion (4.77) in a form in which we can search for solutions, we return to the Hamiltonian equation (4.2). Consider taking matrix elements of the equation of motion (4.77) between exact stationary states $|m>$ and $|n>$ of the Hamiltonian in order to determine the behavior of these states. After dividing both sides by m_o, we have

$$<m|\ddot{\mathbf{R}}(t)|n>=\frac{e}{m_o}<m|\mathbf{E}_{\rm in}|n>+\frac{2e^2}{3m_oc^2}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n!c^n}\frac{2n(4n+5)}{3(n+1)(n+2)}$$

$$\times(2n-1)!!\lambda^{n-1}<m|\frac{d^{n+1}}{dt^{n+1}}\dot{\mathbf{R}}(t)|n>. \qquad (4.79)$$

We want to find states for which the matrix elements $<m|\mathbf{E}_{\rm in}|n>$ are negligible. This is possible since we are dealing with a free electron. We want to examine the effects of the self-force, and not the effects due to an external electric field. In the Heisenberg picture, the time development of an operator is given by

$$O(t)=e^{iHt}O(0)e^{-iHt}.$$

We use this equation to find the matrix elements of (4.79), in which we have

$$\begin{aligned}<m|\dot{\mathbf{R}}(t)|n> &=<m|e^{iHt}\dot{\mathbf{R}}(0)e^{iHt}|n>\\ &=e^{iE_mt/\hbar}<m|\dot{\mathbf{R}}(0)|n>e^{-iE_nt/\hbar}\\ &=e^{i(E_m-E_n)t/\hbar}<m|\dot{\mathbf{R}}(0)|n>.\end{aligned}$$

Let $E_{mn}=E_m-E_n$, and $\dot{\mathbf{R}}(0)_{mn}=<m|\dot{\mathbf{R}}(0)|n>$. Then

$$<m|\dot{\mathbf{R}}(t)|n>=e^{iE_{mn}t/\hbar}\dot{\mathbf{R}}(0)_{mn}. \qquad (4.80)$$

We now use the Heisenberg equation (4.1) to obtain the matrix element for the second time derivative of $\mathbf{R}(t)$:

$$\begin{aligned}<m|\frac{d}{dt}\dot{\mathbf{R}}(t)|n> &=\frac{1}{i\hbar}<m|[\dot{\mathbf{R}}(t),H]|n>\\ &=-\frac{i}{\hbar}[<m|\dot{\mathbf{R}}(t)H|n>-<m|H\dot{\mathbf{R}}(t)|n>]\\ &=-\frac{i}{\hbar}[<m|\dot{\mathbf{R}}(t)|n>E_n-E_m<m|\dot{\mathbf{R}}(t)|n>]\\ &=\frac{iE_{mn}}{\hbar}<m|\dot{\mathbf{R}}(t)|n>.\end{aligned}$$

Therefore,

$$<m|\frac{d}{dt}\dot{\mathbf{R}}(t)|n>=\frac{iE_{mn}}{\hbar}e^{iE_{mn}t/\hbar}\dot{\mathbf{R}}(0)_{mn}.$$

These are the needed matrix elements for the left hand side of the equation of motion (4.79). However, notice that on the right hand side, there are matrix elements for higher time derivatives (up to infinity) of $\mathbf{R}(t)$. In order to see what the general formula might be for these matrix elements, consider the matrix element for the triple time derivative of $\mathbf{R}(t)$:

$$\begin{aligned}
<m|\frac{d^2}{dt^2}\dot{\mathbf{R}}(t)|n> &= \frac{1}{i\hbar} <m| \left[\frac{d}{dt}\dot{\mathbf{R}}(t), H\right] |n> \\
&= -\frac{i}{\hbar}\left[<m|\frac{d}{dt}\dot{\mathbf{R}}(t)H|n> - <m|H\frac{d}{dt}\dot{\mathbf{R}}(t)|n>\right] \\
&= -\frac{i}{\hbar}\left[<m|\frac{d}{dt}\dot{\mathbf{R}}(t)|n> E_n - E_m <m|\frac{d}{dt}\dot{\mathbf{R}}(t)|n>\right] \\
&= \frac{iE_{mn}}{\hbar} <m|\frac{d}{dt}\dot{\mathbf{R}}(t)|n> \\
&= \left(\frac{iE_{mn}}{\hbar}\right)^2 e^{iE_{mn}t/\hbar}\dot{\mathbf{R}}(0)_{mn}.
\end{aligned}$$

In general, it is found that

$$<m|\frac{d^{n+1}}{dt^{n+1}}\dot{\mathbf{R}}(t)|n>= \left(\frac{iE_{mn}}{\hbar}\right)^{n+1} e^{iE_{mn}t/\hbar}\dot{\mathbf{R}}(0)_{mn}. \tag{4.81}$$

We define the parameter $\beta \equiv iE_{mn}/\hbar$, so that (4.81) becomes

$$< m|\frac{d^{n+1}}{dt^{n+1}}\dot{\mathbf{R}}(t)|n >= \beta^{n+1}e^{\beta t}\dot{\mathbf{R}}(0)_{mn}. \tag{4.82}$$

With the use of (4.82), the equation of motion (4.79) becomes

$$0 = \beta e^{\beta t}\dot{\mathbf{R}}(0)_{mn}\left[1 - \frac{2e^2}{3m_o c^2}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n!c^n}\frac{2n(4n+5)}{3(n+1)(n+2)}(2n-1)!!\lambda^{n-1}\beta^n\right]. \tag{4.83}$$

Now that we have taken stationary states of the operator form for the equation of motion, we are able to solve for the roots β. Since $\beta = iE_{mn}/\hbar$, by equation (4.80) we see that β positive corresponds to runaway solutions. In other words, by finding the roots β, we are able to determine the behavior of the solutions. Notice that one solution of (4.83) is for $\beta = 0$. But, by equation (4.80), this implies

$$\dot{\mathbf{R}}(t)_{mn} = \dot{\mathbf{R}}(0)_{mn}.$$

In other words, the velocity at any time t just equals the initial velocity. This is of course the desired solution when there are no external forces present. In the correspondence limit, this solution yields the corresponding classical solution of final velocity equaling the initial velocity. Therefore, we have obtained the appropriate solution for the classical case which is well-behaved. We have not completely examined all of the roots of (4.83), however. There could still be "bad" roots lurking. Rather than $\beta = 0$ in (4.83), we could have

$$1 = \frac{2e^2}{3m_oc^2} \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n!c^n} \frac{2n(4n+5)}{3(n+1)(n+2)} (2n-1)!!\lambda^{n-1}\beta^n. \tag{4.84}$$

We investigate possible roots of this equation in the subsections that follow. To begin with, we must consider the possibility of summing the series in (4.84).

4.4.2 *The Radius of Convergence*

Before we can find roots of equation (4.84), we must do some simplification and preliminary work. We simplify by using the fine structure constant $\alpha = e^2/\lambda m_o c^2$ to write

$$\frac{2e^2}{3m_oc^2} = \frac{2}{3}\alpha\lambda.$$

Then,

$$\frac{\lambda^{n-1}}{c^n}\beta^n = \frac{\beta}{c}\left(\frac{\beta\lambda}{c}\right)^{n-1}.$$

To further simplify, we define $\eta \equiv \beta\lambda/c$, which is dimensionless. We combine the λ in (4.83) with the β/c above to obtain another factor of η. With these modifications, equation (4.84) becomes

$$\begin{aligned} 1 &= \frac{2}{3}\alpha\eta \sum_{n \text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\left[\frac{1}{3}\left(\frac{2n}{n+1}\right)\left(\frac{4n+5}{n+2}\right)\right]\eta^{n-1} \\ &= \frac{2}{3}\alpha f(\eta). \end{aligned} \tag{4.85}$$

Equation (4.85) is a power series in η. It is the equation for finding roots obtained from the operator equation (4.77).

Our next concern is whether or not the series in (4.85) converges. And if so, what is its radius of convergence? And within this radius of convergence, what is the sum of the series? The radius of convergence can be found using the general formula

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{4.86}$$

But the series in (4.85) is only for n odd. Therefore, to get a series over all values of n, we will let $n \to 2n+1$. The series then becomes

$$\sum_{n=0}^{\infty} \frac{(2(2n+1)-1)!!}{(2n+1)!}(-1)^{((2n+1)-1)/2} \left[\frac{1}{3}\left(\frac{2(2n+1)}{2n+1+1}\right)\left(\frac{4(2n+1)+5}{(2n+1)+2}\right)\right] \eta^{2n+1-1}$$

$$= \sum_{n=0}^{\infty} \frac{(4n+1)!!}{(2n+1)!}(-1)^n \left[\frac{1}{3}\left(\frac{2n+1}{n+1}\right)\left(\frac{8n+9}{2n+3}\right)\right] \eta^{2n}.$$

We now have an equivalent expression for the series in (4.85). Notice that it is a power series in η^2. We can compute the radius of convergence by using the above formula (4.86):

$$r = \lim_{n\to\infty} \left| \frac{(4n+1)!!}{(2n+1)!} \frac{1}{3} \left(\frac{2n+1}{n+1}\right)\left(\frac{8n+9}{2n+3}\right) \frac{(2n+3)!}{(4n+5)!!} 3 \left(\frac{n+2}{2n+3}\right)\left(\frac{2n+5}{8n+17}\right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{(4n+1)!!}{(4n+5)!!} \frac{(2n+3)!}{(2n+1)!} \frac{(2n+1)}{(2n+3)} \frac{(n+2)}{(n+1)} \frac{(8n+9)}{(8n+17)} \frac{(2n+5)}{(2n+3)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(2n+1)(n+2)(8n+9)(2n+5)}{(4n+5)(4n+3)(8n+17)(2n+3)} \right|.$$

As $n \to \infty$, the highest power of n dominates. The highest power of n for both the numerator and the denominator is n^4. Therefore, as $n \to \infty$, we keep only the coefficients of the n^4 terms, which yields

$$r = \left| \frac{64}{256} \right| = \frac{1}{4}.$$

Therefore, the radius of convergence for η^2 is $1/4$, and the series in (4.85) converges on the interval $\eta^2 \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. Or equivalently, the series converges for $|\eta| < \frac{1}{2}$.

4.4.3 *Summing the Series*

The series in (4.85) can be summed inside its radius of convergence. We need to sum the series in order to discover any other roots of the equation of motion. To do so we need to rewrite it so as to include all values of n, and to put it in the simplest possible form. With a little thought and manipulation, we see that this can be done by noticing that if we add 0 in the form of 8-14+16, and rewrite $5n$ as $12n - 7n$, we obtain

$$\begin{aligned}
\left[\frac{1}{3}\left(\frac{2n}{n+1}\right)\left(\frac{4n+5}{n+2}\right)\right] &= \frac{2}{3}\frac{4n^2+5n}{(n+1)(n+2)} \\
&= \frac{2}{3}\left[\frac{4n^2+12n+8-7n-14+6}{(n+1)(n+2)}\right] \\
&= \frac{2}{3}\left[\frac{4(n+1)(n+2)-7(n+2)+6}{(n+1)(n+2)}\right].
\end{aligned}$$

Therefore, $f(\eta)$ can be written as

$$\begin{aligned}
f(\eta) &= \frac{2}{3}\sum_{n\text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\eta^n \left[4 - \frac{7}{(n+1)} + \frac{6}{(n+1)(n+2)}\right] \\
&= \frac{2}{3}\left[4\sum_{n\text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\eta^n - 7\sum_{n\text{ odd}} \frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^n \right. \\
&\qquad \left. +6\sum_{n\text{ odd}} \frac{(2n-1)!!}{(n+2)!}(-1)^{(n-1)/2}\eta^n\right],
\end{aligned} \tag{4.87}$$

and we have succesfully rewritten $f(\eta)$ as the sum of three simpler series which can be summed. Once these series have been summed, we can search for roots of equation (4.85). Similar to the classical case, any real roots of (4.85) will give runaway solutions. Therefore, once we have found the roots, we will be able to determine the behavior of the solutions for the equation of motion, and see if quantum mechanics is free from the problems inherent to the classical point theory.

Let us consider each of the three series of (4.87) individually. To sum the first series, we begin by writing down the first few terms as

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\eta^n = \eta - \frac{5!!}{3!}\eta^3 + \frac{9!!}{5!}\eta^5 - \frac{13!!}{7!}\eta^7 + \cdots. \tag{4.88}$$

Since n is odd only in this sum, the even terms

$$\frac{3!!}{2!}\eta^2, \quad \frac{7!!}{4!}\eta^4, \quad \frac{11!!}{6!}\eta^6, \quad \ldots$$

are missing. In order to sum this series, we rewrite it to contain all values of n. In doing so, we will be able to rewrite it as the difference of two series which contain all values of n, where the even terms cancel, and the odd terms add together to create the original series. In order to get the correct signs, we need to use powers of i such that:

$$\sum i^n = i - 1 - i + 1 + \cdots \tag{4.89a}$$

$$\sum (-i)^n = -i - 1 + i + 1 + \cdots \tag{4.89b}$$

Using (4.89), consider the series difference

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!} i^n \eta^n - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!}(-i)^n \eta^n = 2i\eta - \frac{2 \cdot 5!!}{3!} i\eta^3 + \frac{2 \cdot 9!!}{5!}\eta^5 + \cdots.$$

This is almost what we need. It is obvious that if we multiply the above difference by $-i/2$, we will obtain an equivalent expression for the series (4.88), but with the advantage of having series which include all values of n. Therefore

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\eta^n = \frac{-i}{2}\left[\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!}(i\eta)^n - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!}(-i\eta)^n\right]. \tag{4.90}$$

To get a feel for what the sum of the first series in the brackets of (4.90) might be, we will write out the first few terms:

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!}(i\eta)^n = i\eta + \frac{3!!}{2!}(i\eta)^2 + \frac{5!!}{3!}(i\eta)^3 + \frac{7!!}{4!}(i\eta)^4 + \cdots.$$

This is actually a binomial expansion, which can be seen by introducing some factors of 2, as well as adding 1 to both sides:

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}(i\eta)^n = 1+\frac{2i\eta}{2}+\frac{3!!}{2!2^2}(2i\eta)^2-\frac{5!!}{3!2^3}(2i\eta)^3+\frac{7!!}{4!2^4}(2i\eta)^4+\cdots.$$

Let us rewrite this series in the form

$$\begin{aligned}1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}(i\eta)^n &= 1+\frac{1}{2}(2i\eta)+\frac{1}{2}\cdot\frac{3}{2}\frac{(2i\eta)^2}{2!}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\frac{(2i\eta)^3}{3!}+\cdots\\ &= 1+\left(-\frac{1}{2}\right)(-2i\eta)+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\frac{(-2i\eta)^2}{2!}\\ &\quad+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\frac{(-2i\eta)^3}{3!}+\cdots\end{aligned}$$

With the series written in this form, we can compare it with the binomial theorem:

$$(1+x)^r = 1+rx+r(r-1)\frac{x^2}{2!}+r(r-1)(r-2)\frac{x^3}{3!}+\cdots,$$

where $x\in(-1,1)$, and r is any real number, from which we can make the identifications

$$x=-2i\eta, \quad \text{and} \quad r=-\frac{1}{2}.$$

We therefore have the sum of the first series in the brackets of (4.90) in the form of the binomial expression

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}(i\eta)^n = (1-2i\eta)^{-1/2}-1.$$

In a similar fashion, one can show that the second series in the brackets of (4.90) is

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}(-i\eta)^n = (1+2i\eta)^{-1/2}-1.$$

When we insert these results into (4.90), we are left with

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\eta^n = -\frac{i}{2}\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{1/2}\right]. \tag{4.91}$$

It turns out that the other two series in brackets of equation (4.87) can also be shown to be the sum or difference of binomial expressions. Both of these series are the same as the one just considered, except for the denominator; instead of $n!$, we have $(n+1)!$ and $(n+2)!$.

In order to sum the second series, and show that it is a binomial expansion, we start with

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^n = \frac{1}{\eta}\sum_{n\ odd} \frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^{n+1}.$$

If we write out the first few terms of this series

$$\frac{1}{\eta}\sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^{n+1} = \frac{1}{\eta}\left[\frac{1}{2!}\eta^2 - \frac{5!!}{4!}\eta^4 + \frac{9!!}{6!}\eta^6 + \cdots\right] \tag{4.92}$$

we see that once again there are only odd terms. To write the sum over all n, and include the even terms, we need to write it as the sum of two series. We construct (4.92) using (4.89) and making the appropriate shift in the powers of i. Consider the sum

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+1)!} i^{n+3}\eta^{n+1} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+1)!}(-i)^{n+3}\eta^{n+1} =$$

$$\frac{2}{2!}\eta^2 - \frac{2\cdot 5!!}{4!}\eta^4 + \frac{2\cdot 9!!}{6!}\eta^6 + \cdots$$

Notice that we have chosen the signs of the series so that we get a cancellation of the unwanted terms. We now have two regular series (summations over all values of n) which can be shown to be binomial expansions. Let us use the above combination of series to rewrite the series in (4.92) as

$$\frac{1}{\eta}\sum_{n\ \mathrm{odd}}\frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^{n+1}=\frac{1}{2\eta}\left[-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}(i\eta)^{n+1}\right.$$
$$\left.-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}(-i\eta)^{n+1}\right]. \qquad (4.93)$$

Consider the first series in brackets of (4.93). Again, it will be to our advantage to write out the first few terms:

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}(i\eta)^{n+1}=\frac{1}{2!}(i\eta)^2+\frac{3!!}{3!}(i\eta)^3+\frac{5!!}{4!}(i\eta)^4+\frac{6!!}{5!}(i\eta)^5+\cdots.$$

Making this series negative, and adding $1-i\eta$ to both sides, we can show that this is indeed a binomial expansion. To do so, we will insert the appropriate factors of 2, and rearrange. Let

$$S=1-i\eta-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}(i\eta)^{n+1}.$$

Therefore,

$$\begin{aligned}
S&=1-\frac{1}{2}(2i\eta)-\frac{1}{2!2^2}(2i\eta)^2-\frac{3!!}{3!2^3}(2i\eta)^3-\frac{5!!}{4!2^4}(2i\eta)^4-\cdots\\
&=1-\frac{1}{2}(2i\eta)-\frac{1}{2}\cdot\frac{1}{2}\frac{(2i\eta)^2}{2!}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\frac{(2i\eta)^3}{3!}-\cdots\\
&=1+\frac{1}{2}(-2i\eta)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(-2i\eta)^2}{2!}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-2i\eta)^3}{3!}+\cdots\\
&=1+\frac{1}{2}(-2i\eta)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(-2i\eta)^2}{2!}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(-2i\eta)^3}{3!}+\cdots,
\end{aligned}$$

from which we can again compare with the binomial theorem to obtain the result

$$-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}(i\eta)^{n+1}=(1-2i\eta)^{1/2}-1+i\eta.$$

And similarly one can show that

$$-\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+1)!}(-i\eta)^{n+1} = (1+2i\eta)^{1/2} - 1 - i\eta.$$

Returning to (4.93), we see that it can be written as

$$\frac{1}{\eta} \sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+1)!}(-1)^{(n-1)/2}\eta^{n+1} = \frac{1}{2\eta}\left[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - 2\right]. \quad (4.94)$$

Now following the same steps for the third series in equation (4.87), we have

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+2)!}(-1)^{(n-1)/2}\eta^{n} = \frac{1}{\eta^2} \sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+2)!}(-1)^{(n-1)/2}\eta^{n+2}.$$

The first few terms of the series are

$$\frac{1}{\eta^2}\left[\frac{1}{3!}\eta^3 - \frac{5!!}{5!}\eta^5 + \frac{9!!}{7!}\eta^7 + \cdots\right].$$

Again, we include all values of n by taking the difference of two series; namely

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+2)!} i^{n+2}\eta^{n+2} - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+2)!}(-i)^{n+2}\eta^{n+2} =$$

$$-\frac{2}{3!}i\eta^3 + \frac{2\cdot 5!!}{5!}i\eta^5 - \frac{2\cdot 9!!}{7!}i\eta^7 + \cdots. \quad (4.95)$$

If we multiply both sides of (4.95) by $i/2\eta^2$, we see that

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+2)!}(-1)^{(n-1)/2}\eta^{n} = \frac{i}{2\eta^2}\left[\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+2)!}(i\eta)^{n+2}\right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+2)!}(-i\eta)^{n+2}\right]. \quad (4.96)$$

In order to show that the series in brackets of (4.96) are binomial expansions, let us look at the first few terms of the first series:

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+2)!}(i\eta)^{n+2} = \frac{1}{3!}(i\eta)^3 + \frac{3!!}{4!}(i\eta)^4 + \frac{5!!}{5!}(i\eta)^5 + \frac{7!!}{6!}(i\eta)^6 + \cdots$$
$$= \frac{1}{3!2^3}(2i\eta)^3 + \frac{3!!}{4!2^4}(2i\eta)^4 + \frac{5!!}{5!2^5}(2i\eta)^5 + \frac{7!!}{6!2^6}(2i\eta)^6 + \cdots. \tag{4.97}$$

If we multiply both sides by 3, we have

$$3\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+2)!}(i\eta)^{n+2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{(-2i\eta)^3}{3!} + \frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-2i\eta)^4}{4!}$$
$$+\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{(-2i\eta)^5}{5!}+\cdots. \tag{4.98}$$

For a binomial expansion, we are missing the first three terms

$$1, \quad \frac{3}{2}(-2i\eta), \quad \frac{3}{2}\cdot\frac{1}{2}\frac{(-2i\eta)^2}{2!}.$$

If we add these three terms to both sides of (4.98), and again compare with the binomial theorem, then

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+2)!}(i\eta)^{n+2} = \frac{1}{3}\left\{(1-2i\eta)^{3/2} - 1 - \frac{3}{2}(-2i\eta) - \frac{3}{2}\cdot\frac{1}{2}\frac{(-2i\eta)^2}{2!}\right\}. \tag{4.99}$$

Following similar steps, the second series in (4.96) can be written as

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+2)!}(-i\eta)^{n+2} = \frac{1}{3}\left\{(1+2i\eta)^{3/2} - 1 - \frac{3}{2}(2i\eta) - \frac{3}{2}\cdot\frac{1}{2}\frac{(2i\eta)^2}{2!}\right\}. \tag{4.100}$$

Inserting (4.99) and (4.100) into (4.96) gives

$$\sum_{n \text{ odd}} \frac{(2n-1)!!}{(n+2)!}(-1)^{(n-1)/2}\eta^n = \frac{i}{6\eta^2}\left[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2} + 6i\eta\right]. \quad (4.101)$$

We have shown that the three series of (4.87) can be written as the sum, or the difference, of binomial expressions. Let us use (4.91), (4.94), and (4.101) in equation (4.87) to write the expression for $f(\eta)$ as

$$\begin{aligned} f(\eta) = \frac{2}{3}\Bigg\{ & 4\left(-\frac{i}{2}\right)\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right] \\ & -7\left(\frac{1}{2\eta}\right)\left[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - 2\right] \\ & +6\left(\frac{i}{6\eta^2}\right)\left[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2} + 6i\eta\right]\Bigg\}, \end{aligned}$$

or,

$$\begin{aligned} f(\eta) = & -\frac{4i}{3}\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right] \\ & -\frac{7}{3\eta}\left[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2}\right] + \frac{14}{3\eta} \\ & +\frac{2i}{3\eta^2}\left[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2}\right] - \frac{12}{3\eta}. \end{aligned}$$

But,

$$\frac{14}{3\eta} - \frac{12}{3\eta} = \frac{1}{\eta}\left(\frac{14}{3} - \frac{12}{3}\right) = \frac{2}{3\eta},$$

and $f(\eta)$ becomes

$$\begin{aligned} f(\eta) = & -\frac{4i}{3}\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right] \\ & -\frac{7}{3\eta}\left[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - \frac{2}{7}\right] \\ & +\frac{2i}{3\eta^2}\left[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2}\right]. \quad (4.102) \end{aligned}$$

We started out with the operator form of the equation of motion (4.77). We then found matrix elements of this equation, and were able to sum the series within its circle of convergence to obtain (4.102). Therefore, we now have an equation for which we can obtain the roots, and determine the behavior of the solutions.

4.4.4 *Correspondence with the Classical Equation of Motion*

Let us first observe what sort of behavior is obtained from the equation in the classical limit $\hbar \to 0$. Notice that

$$\eta = \frac{\beta\lambda}{c} = \frac{\beta\hbar}{m_o c^2}.$$

Therefore, when $\hbar$ goes to zero, η also goes to zero. β, which equals $iE_{mn}/\hbar$, is unaffected since the energy contains an $\hbar$. We want to obtain the classical counterpart of equation (4.102). To do so, we will first expand the binomials in $f(\eta)$:

$$\begin{aligned} f(\eta) = &-\left(\frac{4i}{3}\right)\left[1 + i\eta - \frac{3!!}{2!}\eta^2 - \frac{5!!}{3!}i\eta^3 + \cdots - 1 + i\eta + \frac{3!!}{2!}\eta^2 - \frac{5!!}{3!}i\eta^3 + \cdots\right] \\ &-\frac{7}{3\eta}\left[1 - i\eta + \frac{1}{2!}\eta + \frac{3!!}{3!}i\eta^3 + \cdots + 1 + i\eta + \frac{1}{2!}\eta^2 - \frac{3!!}{3!}i\eta^3 + \cdots - \frac{2}{7}\right] \\ &+\frac{2i}{3\eta^2}\left[1 - 3i\eta - \frac{3}{2!}\eta^2 - \frac{3}{3!}i\eta^3 + \cdots - 1 - 3i\eta + \frac{3}{2!}\eta^2 - \frac{3}{3!}i\eta^3 + \cdots\right]. \end{aligned}$$

Adding like terms, we obtain

$$\begin{aligned} f(\eta) = -\frac{4i}{3}\left[2i\eta - \frac{2\cdot 5!!}{3!}i\eta^3 + \cdots\right] - \frac{7}{3\eta}\left[2 + \frac{2}{2!}\eta^2 + \cdots - \frac{2}{7}\right] \\ + \frac{2i}{3\eta^2}\left[-6i\eta - \frac{6}{3!}i\eta^3 + \cdots\right], \end{aligned}$$

or,

$$f(\eta) = \frac{8}{3}\eta - \frac{8\cdot 5!!}{3\cdot 3!}\eta^3 - \frac{14}{3\eta} - \frac{14}{3\cdot 2!}\eta + \frac{2}{3\eta} + \frac{12}{3\eta} + \frac{12}{3\cdot 3!}\eta + \cdots.$$

As $\eta \to 0$, the dominating terms are those containing $1/\eta$. However, if we add all of the $1/\eta$ terms together,

$$-\frac{14}{3\eta}+\frac{2}{3\eta}+\frac{12}{3\eta}=0,$$

we see that they vanish. The dominating terms are therefore the terms containing η. As $\eta \to 0$, $f(\eta)$ becomes

$$\begin{aligned}\lim_{\eta\to 0} f(\eta) &= \frac{8}{3}\eta - \frac{14}{3\cdot 2!}\eta + \frac{12}{3\cdot 3!}\eta \\ &= \frac{2}{3}\eta\left(4-\frac{7}{2}+1\right) \\ &= \frac{2}{3}\eta\left(\frac{3}{2}\right) \\ &= \eta.\end{aligned}$$

Referring back to equation (4.102), in the limit as $\hbar \to 0$,

$$\lim_{\hbar\to 0}\left\{\frac{2}{3}\alpha f(\eta)\right\} = \frac{2}{3}\alpha\eta = \frac{2}{3}\left(\frac{e^2}{\lambda m_0 c^2}\right)\left(\frac{\beta\lambda}{c}\right) = \frac{2}{3}\frac{e^2}{m_0 c^2}\beta.$$

But $\tau = 2e^2/3em_oc^3$. Therefore we have the result

$$\lim_{\hbar\to 0}\left\{\frac{2}{3}\alpha f(\eta)\right\} = \beta\tau.$$

We have shown that the classical counterpart of (4.85) and (4.102) is

$$1 = \beta\tau. \tag{4.103}$$

But notice this equation implies that β is real and positive since τ is real and positive. From equation (4.80) we see that

$$\dot{\mathbf{R}}(t)_{mn} = e^{\beta t}\dot{\mathbf{R}}(0)_{mn}.$$

In the correspondence limit to the classical theory ($\hbar \to 0$), the matrix elements $\dot{\mathbf{R}}(t)_{mn}$ and $\dot{\mathbf{R}}(0)_{mn}$ go to the corresponding classical observables $\dot{\mathbf{R}}(t)$ and $\dot{\mathbf{R}}(0)$. Therefore, the classical equation (4.103) corresponds to the equation of motion

$$\dot{\mathbf{R}}(t) = e^{\beta t}\dot{\mathbf{R}}(0).$$

This equation of motion is just the classical equation (2.2). With β real, this classical equation exibits runaway acceleration regardless of the initial velocity $\dot{\mathbf{R}}(0)$. And so, in the limit as $\hbar \to 0$, we return to the classical result for the point electron of runaway solutions that we obtained in section 2.2. In other words, we cannot obtain a well behaved equation of motion for the classical point electron. We now turn again to the question of whether or not there are runaway or preaccelerating solutions for the quantum point electron.

4.4.5 *Other Solutions of the Quantum Equation of Motion*

In section 4.4.1 we discovered that the quantum theory contains the expected constant velocity solution when no external forces are present. We want to discover if there are any roots to equation (4.85). But because of the difficulty of finding the roots of this exact equation, it will be to our advantage to consider the simplified form of $f(\eta)$ as

$$\tilde{f}(\eta) = -\left(\frac{4i}{3}\right)\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right].$$

The behavior of the solutions for this truncated form will also be the behavior of the general solutions with minor modifications of the coefficients [36]. Indeed, both $f(y)$ and $\tilde{f}(y)$ are odd functions of y with the same asymptotic behavior and a similar behavior for $y \to 0$. Using $\tilde{f}(\eta)$, equation (4.85) can be written as

$$1 = -\frac{8\alpha}{9}i\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right].$$

To simplify further, we make the definitions

$$g \equiv \frac{8\alpha}{9}, \quad \text{and} \quad \xi \equiv 2\eta.$$

Notice that we are now searching for the roots ξ of the equation

$$1 = -gi\left[(1-i\xi)^{-1/2} - (1+i\xi)^{-1/2}\right], \tag{4.104}$$

where the circle of convergence can be written in the form $|\xi| < 1$. Our task is to solve for ξ. To do so we rewrite equation (4.104) as

$$
\begin{aligned}
1 &= -gi\left[\frac{1}{\sqrt{1-i\xi}} - \frac{1}{\sqrt{1+i\xi}}\right] \\
&= -gi\left[\frac{\sqrt{1+i\xi}-\sqrt{1-i\xi}}{\sqrt{1+\xi^2}}\right],
\end{aligned}
$$

and square both sides to obtain

$$
\begin{aligned}
1 &= g^2\left[\frac{\sqrt{1+\xi^2}+\sqrt{1+\xi^2}-1-i\xi-1+i\xi}{1+\xi^2}\right] \\
&= g^2\left[\frac{-2+2\sqrt{1+\xi^2}}{1+\xi^2}\right] \\
&= 2g^2\left[\frac{-1+\sqrt{1+\xi^2}}{1+\xi^2}\right].
\end{aligned}
$$

We then have

$$
1+\xi^2+2g^2 = 2g^2\sqrt{1+\xi^2}.
$$

In order to isolate ξ, we square both sides again. After collecting like terms, we obtain

$$
\xi^4 + 2(1+2g^2-2g^4)\xi^2 + (1+4g^2) = 0.
$$

By use of the quadratic formula,

$$
\begin{aligned}
\xi^2 &= \frac{2(2g^4-2g^2-1)\pm 2\sqrt{(2g^4-2g^2-1)^2-(1+4g^2)}}{2} \\
&= (2g^4-2g^2-1)\pm\sqrt{4g^8-8g^6}.
\end{aligned}
$$

And finally, we have the result

$$
\xi^2 = (2g^4-2g^2-1)\pm 2g^3\sqrt{g^2-2}. \tag{4.105}
$$

Remember that the roots must lie within the circle of convergence $|\xi|<1$ to be valid. First, observe the case for $g^2<2$. In this case, (4.105) becomes

$$\xi^2 = (2g^4 - 2g^2 - 1) \pm i2g^3(2 - g^2)^{1/2}.$$

Notice that ξ is complex. Therefore we can write

$$\begin{aligned}
|\xi|^2 &= \left[(2g^4 - 2g^2 - 1)^2 + 4g^6(2 - g^2)\right]^{1/2} \\
&= (4g^8 - 4g^6 - 2g^4 - 4g^6 + 4g^4 + 2g^2 - 2g^4 + 2g^2 + 1 + 8g^6 - 4g^8)^{1/2} \\
&= (4g^2 + 1)^{1/2}.
\end{aligned}$$

And for real g,

$$|\xi|^2 = (4g^2 + 1)^{1/2} > 1.$$

These roots lie outside the circle of convergence $|\xi| < 1$, and are therefore not valid. So far we have not found any other roots comming from equation (4.85). Notice what this means as far as the fine structure constant is concerned. Since $g = 8\alpha/9$, for the condition $g^2 < 2$,

$$\alpha = \frac{9}{8}g < \frac{9}{8}\sqrt{2} \approx 1.59.$$

But the physical value of $\alpha = 1/137$. Since there are no roots inside the circle of convergence for $g^2 < 2$, and g^2 has to be less than 2 in order to obtain the physical value of α, there are no roots of the quantum equation of motion other than the constant velocity solution obtained earlier. In other words, when no external forces are present, the quantum theory contains only the expected solution of constant velocity for the physical value of α.

Notice what the radius of convergence $|\eta| < 1/2$ for the series in (4.85) implies. Since $\eta = \beta\lambda/c$ and $\beta = iE_{mn}/\hbar$,

$$\begin{aligned}
|\eta|^2 &< \frac{1}{4} \\
\left|\frac{iE_{mn}\lambda}{\hbar c}\right|^2 &< \frac{1}{4} \\
\frac{E_{mn}\lambda}{\hbar c} &< \frac{1}{2}.
\end{aligned}$$

Since $\lambda = \hbar/mc$, we obtain the condition on the energy as

$$E_{mn} < \frac{1}{2}mc^2.$$

This means that only nonrelativistic energies are compatible with the convergence of the series. But within this condition, there are no runaway solutions for the physical value of α.

4.4.6 *Strong-Coupling or Semiclassical Limit*

When no external forces are present, the only solution for the quantum electron is the expected constant velocity solution. Since we are also interested in the classical theory and the correspondence limit from quantum to classical, we investigate the case $g^2 > 2$. Notice that $\alpha = e^2/\hbar c$ implies that as $\hbar \to 0$, α gets large. For $g^2 > 2$, the value of α is large compared to the physical value $1/137$, and can be interpreted as a semiclassical limit. An alternative interpretation is to let the physical value of e^2 grow. This is the strong-coupling limit. In this section we consider the validity of such a limit. When $g^2 > 2$, from equation (4.105) we have

$$\xi = \pm\sqrt{(2g^4 - 2g^2 - 1) \pm 2g^3(g^2 - 2)^{1/2}}. \tag{4.106}$$

Consider the four possible roots of (4.106) coming from the fact that there are four different combinations of signs possible. We will evaluate these roots in the classical limit $\hbar \to 0$, which implies that $g \to \infty$ since

$$g = \frac{8\alpha}{9} = \frac{8}{9}\left(\frac{e^2}{\lambda m_o c^2}\right) = \frac{8}{9}\left(\frac{e^2}{m_o c^3}\right)\left(\frac{m_o c}{\hbar}\right) = \frac{8}{9}\frac{e^2}{\hbar c}.$$

$\underline{\pm\pm}$

$$\xi_{++} = \left[(2g^4 - 2g^2 - 1) + 2g^3(g^2 - 2)^{1/2}\right]^{1/2},$$

which, in the limit as g goes to infinity, is

$$\lim_{g\to\infty} \xi_{++} = \sqrt{2g^4 + 2g^4} = 2g^2.$$

$\underline{\pm-}$

$$\xi_{+-} = \left[(2g^4 - 2g^2 - 1) - 2g^3(g^2 - 2)^{1/2}\right]^{1/2}.$$

The g^4 factor cancels. Therefore we will need to expand the binomial in the second term in brackets as

$$\begin{aligned}
\xi_{+-} &= \left[2g^4 - 2g^2 - 1 - 2g^4\left(1 - \frac{2}{g^2}\right)^{1/2}\right]^{1/2} \\
&= \left[2g^4 - 2g^2 - 1 - 2g^4 + \frac{2g^4}{2}\frac{2}{g^2} + \frac{2g^4}{2}\frac{1}{2}\frac{1}{2}\frac{4}{g^4}\right. \\
&\qquad \left. - \frac{2g^4}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}\left(-\frac{2}{g^2}\right)^3 + \cdots\right]^{1/2}.
\end{aligned}$$

From this we see that the g^2 factor, as well as the zeroth order factor, also cancel. The first term that does not cancel is the $1/g^2$ term. This term will therefore be the dominating term in the limit as $g \to \infty$, and we obtain

$$\lim_{g\to\infty} \xi_{+-} = \frac{1}{g}.$$

The other two roots are just the negatives of the two roots found above. Therefore, in the limit as $g \to \infty$, the four roots are:

$$\xi_{++} = 2g^2, \qquad \xi_{-+} = -2g^2$$

$$\xi_{+-} = \frac{1}{g}, \qquad \xi_{--} = -\frac{1}{g}.$$

Only two of these roots are solutions of the original equation, however. This can be seen by inserting each root back into the original equation (4.104). Let us begin with ξ_{+-} to see if it corresponds to a solution. Since $\xi_{+-} = 1/g$, in the limit as $g \to \infty$, equation (4.104) becomes

$$\begin{aligned}
1 &= -gi\left[\left(1 - \frac{i}{g}\right)^{-1/2} - \left(1 + \frac{i}{g}\right)^{-1/2}\right] \\
&= -gi\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{i}{g}\right) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}\left(-\frac{i}{g}\right)^2 + \cdots\right. \\
&\qquad \left. - 1 - \left(-\frac{1}{2}\right)\left(\frac{i}{g}\right) - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}\left(\frac{i}{g}\right)^2 + \cdots\right] \\
&= -gi\left(\frac{i}{g}\right),
\end{aligned}$$

where the first terms in the expansions cancel, and all other terms exept for the second term go to zero in the limit as g goes to infinity. Therefore we get an indentity, and ξ_{+-} is a solution of (4.104).

We can show that ξ_{++} is also a solution. We do so by inserting $\xi_{++} = 2g^2$ into (4.104) to obtain

$$1 = -gi\left[(1-2ig^2)^{-1/2} - (1+2ig^2)^{-1/2}\right].$$

In order to expand these binomials, we need to be more careful than before. Notice that i and g can be rewritten as

$$i = e^{i\pi/2}, \quad \text{and} \quad g = \left(g^{-2}\right)^{-1/2}.$$

The g can be taken inside the binomial expressions to obtain

$$1 = -e^{i\pi/2}\left[\left(\frac{1}{g^2} - 2e^{i\pi/2}\right)^{-1/2} - \left(\frac{1}{g^2} + 2e^{i\pi/2}\right)\right].$$

In order to have the binomial expressions in the appropriate form for expansion, we write

$$\begin{aligned} 1 = -e^{i\pi/2}\Bigg[&\left(-2e^{i\pi/2}\right)^{-1/2}\left(1 - \frac{1}{2g^2e^{i\pi/2}}\right)^{-1/2} \\ &- \left(2e^{i\pi/2}\right)^{-1/2}\left(1 + \frac{1}{2g^2e^{i\pi/2}}\right)^{-1/2}\Bigg]. \end{aligned}$$

But notice that in the limit as $g \to \infty$, only the first term of an expansion survives. Therefore,

$$\begin{aligned} 1 &= -e^{i\pi/2}\left(-2e^{i\pi/2}\right)^{-1/2} + e^{i\pi/2}\left(2e^{i\pi/2}\right)^{-1/2} \\ &= -\frac{e^{i\pi/2}}{\sqrt{2}}\left(-e^{i\pi/2}\right)^{-1/2} + \frac{e^{i\pi/2}}{\sqrt{2}}\left(e^{i\pi/2}\right)^{-1/2}. \end{aligned}$$

But $-e^{i\pi/2} = e^{-i\pi/2}$, and we have

$$
\begin{aligned}
1 &= -\frac{e^{i\pi/2}e^{i\pi/4}}{\sqrt{2}} + \frac{e^{i\pi/2}e^{-i\pi/4}}{\sqrt{2}} \\
&= \frac{-e^{i3\pi/4} + e^{i\pi/4}}{\sqrt{2}} \\
&= \frac{-\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}}{\sqrt{2}} \\
&= \frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{\sqrt{2}} \\
&= 1.
\end{aligned}
$$

Therefore we see that ξ_{++} is indeed a solution. However, if we insert ξ_{-+} and ξ_{--} into (4.33), we see that they differ from their positive counterparts by an overall minus sign. Therefore, in both of these cases, we get the contradiction that $1 = -1$. Therefore, we have found that out of the four possible roots, only two are solutions of (4.33).

Notice that $\xi{-+}$ and ξ_{--}, which are not solutions to the original equation, came into play because of first squaring the equation containing ξ, and then taking the square root, which introduced a plus and a minus sign.

In the limit as $g \to \infty$, the root $\xi_{++} \to \infty$, and therefore lies outside the circle of convergence $|\xi| < 1$. But what about ξ_{+-}? For $g \to \infty$,

$$
\xi_{+-} \to \frac{1}{g} = \frac{9}{8\alpha} = \frac{9}{8}\frac{m\lambda c^2}{e^2} = \frac{3}{4}\left(\frac{\lambda}{c}\right)\frac{1}{\tau}.
$$

But this is a *real* root. For this case, β becomes

$$
\beta = \frac{c\eta_{+-}}{\lambda} = \frac{c\xi_{+-}}{2\lambda} = \frac{1}{2}\left(\frac{c}{\lambda}\right)\frac{1}{g} = \frac{1}{2}\left(\frac{c}{\lambda}\right)\frac{3}{4}\left(\frac{\lambda}{c}\right)\frac{1}{\tau} = \frac{3}{8}\left(\frac{1}{\tau}\right),
$$

which corresponds to the classical runaway solution (4.103). The 3/8 comes into play because of the truncated form $\tilde{f}(\eta)$. In other words, for $g^2 > 2$, there is a root inside the circle of convergence which corresponds to the classical runaway solution. However, let us investigate this solution further by considering a semiclassical approach in which we let g get large, but remain finite. In this case we have

$$\begin{aligned}
\beta &= \frac{1}{2}\left(\frac{c}{\lambda}\right)\xi_{+-} \\
&= \frac{1}{2}\left(\frac{c}{\lambda}\right)\left[(2g^4 - 2g^2 - 1) - 2g^3(g^2 - 2)^{1/2}\right]^{1/2} \\
&= \frac{1}{2}\left(\frac{c}{\lambda}\right)\left[(2g^4 - 2g^2 - 1) - 2g^4\left(1 - \frac{2}{g^2}\right)^{1/2}\right]^{1/2}.
\end{aligned}$$

We now expand the binomial to obtain

$$\begin{aligned}
\beta &= \frac{1}{2}\left(\frac{e}{\lambda}\right)\left[2g^4 - 2g^2 - 1 - 2g^4 + 2g^2 + 1 + \frac{1}{g^2} + \frac{5}{4}\frac{1}{g^4} + \cdots\right]^{1/2} \\
&= \frac{1}{2}\left(\frac{c}{\lambda}\right)\frac{1}{g}\left[1 + \frac{5}{4}\frac{1}{g^2} + \cdots\right]^{1/2}.
\end{aligned}$$

Since g is large, we will neglect all values of order $1/g^4$ on, and expand. In doing so we will get the first two terms in a small $\hbar$- expansion about the classical runaway solution $\beta = 3/8\tau$. We first write $1/g$ as

$$\frac{1}{g} = \left(\frac{9}{8}\right)\frac{m_o \lambda c^2}{e^2}.$$

Therefore, β can be written as

$$\begin{aligned}
\beta &= \frac{1}{2}\left(\frac{c}{\lambda}\right)\frac{9}{8}\frac{m_o \lambda c^2}{e^2}\left[1 + \frac{1}{2}\left(\frac{5}{4}\right)\frac{1}{g^2} + \cdots\right] \\
&= \frac{1}{2}\left(\frac{3}{4}\right)\frac{1}{\tau}\left[1 + \frac{5}{8}\frac{1}{g^2} + \cdots\right] \\
&= \frac{3}{8}\frac{1}{\tau}\left[1 + \frac{5}{8}\frac{1}{g^2} + \cdots\right].
\end{aligned}$$

Or, since $1/g$ can be written as

$$\frac{1}{g} = \frac{9}{8}\frac{m_o c^2}{e^2}\left(\frac{\hbar}{m_o c}\right) = \frac{9}{8}\frac{\hbar c}{e^2},$$

we have

$$\beta = \frac{3}{8}\frac{1}{\tau}\left[1 + \frac{5}{8}\frac{c^2}{e^4}\hbar^2 + \cdots\right]. \tag{4.107}$$

Notice that this is an expansion in $\hbar$ about the classical runaway solution $\beta = 3/8\tau$. However, we need to consider the validity of such a semiclassical approach. Since $\alpha = 9/8g$, the possible values of α are separated by the earlier conditions, $g^2 < 2$, and $g^2 > 2$; the critical value of α being

$$\alpha_{\text{crit}} = \frac{9}{8}\sqrt{2} \approx 1.59. \tag{4.108}$$

Remember that

$$\xi^2 = (2g^4 - 2g^2 - 1) \pm 2g^3(g^2 - 2)^{1/2}.$$

If we take the derivative, we will be able to obtain an expression for $d\xi/dg$:

$$2\xi d\xi = 8g^3 dg - 4g dg \pm 6g^2 dg(g^2 - 2)^{1/2} \pm 2g^3 \left(\frac{1}{2}\right) \frac{2g dg}{(g^2 - 2)^{1/2}}.$$

Therefore,

$$\frac{d\xi}{dg} = \frac{1}{\xi}\left[4g^3 - 2g \pm 3g^2(g^2 - 2)^{1/2} \pm \frac{g^4}{(g^2 - 2)^{1/2}}\right].$$

Using the original definitions $g = 8\alpha/9$ and $\xi = 2\eta$, the derivative can be written as

$$\frac{d\eta}{d\alpha} = \frac{4}{9}\frac{1}{\xi}\left[4g^3 - 2g \pm 3g^2(g^2 - 2)^{1/2} \pm \frac{g^4}{(g^2 - 2)^{1/2}}\right].$$

At the critical value of α, $g^2 = 2$, and we have

$$\left.\frac{d\eta}{d\alpha}\right|_{\alpha \to \alpha_{\text{crit}}} \to \infty.$$

This suggests that there is a "first order phase transition" at $\alpha = \alpha_{\text{crit}}$. In other words, there is a boundary or a cusp at α_{crit}. Within the circle of convergence, and for $\alpha < \alpha_{\text{crit}}$, the quantum theory is perfectly well behaved.

In the correspondence limit, there are some runaway solutions which begin to appear, but they appear only after the boundary has been crossed ($\alpha > \alpha_{\text{crit}}$). Because of the cusp, there is no meaningful connection or extrapolation from the domain containing the physical value of α with the semiclassical to classical domain of α (see

Figure 4.3). Therefore, the semiclassical approach is not meaningful for studying the question of runaways in quantum mechanics.

What if we consider at what value of g or α we first get a root which is inside the circle of convergence of the original series in (4.85)? To do so we use the truncated form $\tilde{f}(\eta)$ in equation (4.85) as before,

$$\begin{aligned}
1 &= -\frac{8}{9}\alpha i\left[\frac{1}{\sqrt{1-2i\eta}} - \frac{1}{\sqrt{1+2i\eta}}\right] \\
&= -\frac{8}{9}\alpha i\left[\frac{\sqrt{1+2i\eta}-\sqrt{1-2i\eta}}{\sqrt{1+4\eta^2}}\right].
\end{aligned}$$

Squaring both sides, we obtain

$$\begin{aligned}
1 &= -\left(\frac{8}{9}\right)^2\alpha^2\left[\frac{1+2i\eta+1-2i\eta-2\sqrt{1+4\eta^2}}{1+4\eta^2}\right] \\
&= -\left(\frac{8}{9}\right)^2\alpha^2 2\left[\frac{1-\sqrt{1+4\eta^2}}{1+4\eta^2}\right].
\end{aligned}$$

If we insert $1/2$ in for η and solve for α, we obtain the first value of α which is inside the circle of convergence. When $\eta = 1/2$ we have

$$\begin{aligned}
1 &= -\left(\frac{8}{9}\right)^2\alpha^2 2\left[\frac{1-\sqrt{2}}{2}\right] \\
&= -\left(\frac{8}{9}\right)^2\alpha^2[1-\sqrt{2}].
\end{aligned}$$

Therefore,

$$\alpha^2 = -\left(\frac{9}{8}\right)^2\frac{1}{1-\sqrt{2}} = -\left(\frac{9}{8}\right)^2\frac{1+\sqrt{2}}{(1-\sqrt{2})(1+\sqrt{2})} = \left(\frac{9}{8}\right)^2(1+\sqrt{2}).$$

Or equivalently,

$$\alpha = \frac{9}{8}(1+\sqrt{2})^{1/2} \approx 1.75. \tag{4.109}$$

This is the first value of α we obtain, and corresponds to the first root ξ inside the radius of convergence. But as is shown in *Figure 4.3*, this first value of α is beyond

α_{crit} and cannot be considered as coming from a quantum solution. As $\alpha \to \infty$, the root ξ corresponding to (4.109) goes to the root ξ_{+-} mentioned earlier.

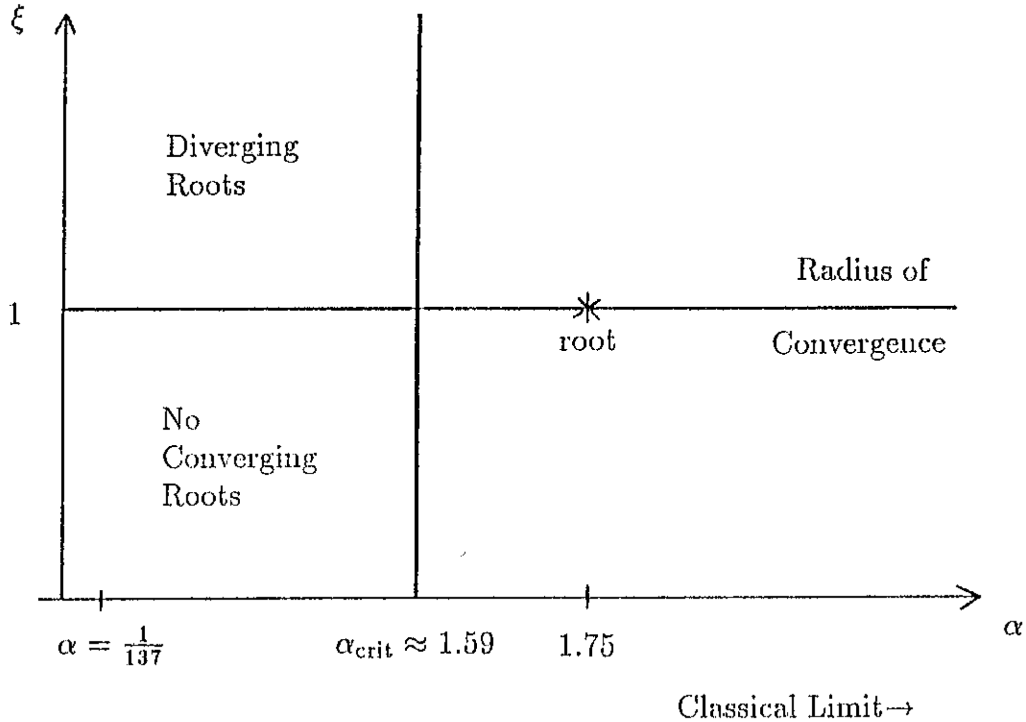

Figure 4.3: *The Limit From Quantum to Classical Solutions.*

Remember that ξ_{+-} was the root corresponding to classical runaway solutions. But since there is no connection across the cusp at α_{crit}, we are not able to obtain the classical runaway solutions from quantum mechanics. Therefore, since there is no connection, the classical runaway solutions should be non-existent and can be neglected in the classical theory. In other words, the runaways are strictly a mathematical byproduct of the classical theory with no physical significance. It arises simply from the inability of the classical theory to handle a quantum particle such as the electron.

4.5 Solutions with External Forces

We have considered the behavior of the solutions when no external forces are present, and found that there are no runaways in the quantum point theory. We also found that the classical runaways do not come from the correspondence limit, and should not exist in classical theory. But what if there are external forces present? Can the problem of runaway solutions also be resolved through a correspondence with the quantum theory?

We assume a classical time-dependent external force, and again assume that the effects of the in-field are negligible. In this case, equation (4.77) becomes

$$m_o \ddot{\mathbf{R}}(t) = \mathbf{F}(t) + \frac{2e^2}{3c^2} \sum_{n \text{ odd}} \frac{A_n}{n! c^n} \frac{d^{n+2}}{dt^{n+2}} \mathbf{R}(t). \tag{4.110}$$

If we multiply equation (4.85) by $m_o \ddot{\mathbf{R}}(t)$, and add in the external force, we see that equation (4.110) can be reduced to

$$m_o \ddot{\mathbf{R}}(t) \left(1 - \frac{2}{3}\alpha f(\eta)\right) = \mathbf{F}(t). \tag{4.111}$$

Using the definitions $\eta = \beta\lambda/c$ and $\beta = iE/\hbar$, as well as the relation $E = \hbar\omega$, equation (4.111) can be written as

$$m_o \ddot{\mathbf{R}}(t) = \frac{\mathbf{F}(t)}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)}, \tag{4.112}$$

where $f(\eta)$ is given by equation (4.102). Now multiplying both sides of (4.112) by $\frac{1}{\sqrt{2\pi}} dt e^{-i\omega t}$, integrating over all times, and using the Fourier transforms

$$\ddot{\mathbf{R}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} \ddot{\mathbf{R}}(t), \quad \text{and} \quad \mathbf{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} \mathbf{F}(t), \tag{4.113}$$

we obtain

$$m_o \ddot{\mathbf{R}}(\omega) = \frac{\mathbf{F}(\omega)}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)}. \tag{4.114}$$

Projecting back using the inverse transforms of (4.113), we obtain

$$m_o\ddot{\mathbf{R}}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\frac{\mathbf{F}(\omega)}{1-\frac{2}{3}\alpha f(i\omega\lambda/c)}.$$

We once again write $F(\omega)$ using (4.113),

$$m_o\ddot{\mathbf{R}}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} dt'\mathbf{F}(t')\frac{e^{i\omega(t-t')}}{1-\frac{2}{3}\alpha f(i\omega\lambda/c)}. \tag{4.115}$$

Equation (4.115) can now be written as

$$m_o\ddot{\mathbf{R}}(t) = \int_{-\infty}^{\infty} dt' G(t-t')\mathbf{F}(t'), \tag{4.116}$$

where $G(t-t')$ is the response function given by

$$\begin{aligned} G(t-t') &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G(\omega)e^{i\omega(t-t')} \\ &= \int_{-c/2\lambda}^{c/2\lambda}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{1-\frac{2}{3}\alpha f(i\omega\lambda/c)}, \end{aligned} \tag{4.117}$$

and the limits of integration come from the condition $|\omega| < c/2\lambda$. This condition comes from the radius of convergence $|\eta| < 1/2$, in which case we have

$$\left|\frac{iE\lambda}{\hbar c}\right| < \frac{1}{2}.$$

Since $E = \hbar\omega$, this condition becomes

$$|\omega| < \frac{c}{2\lambda}. \tag{4.118}$$

Condition (4.118) means that the applied force changes slowly in the time required for light to cross the Compton wavelength of an electron. We must require $\mathbf{F}(\omega)$ to vanish for $|\omega| > c/2\lambda$. In other words, this model is consistent only for frequencies less than $c/2\lambda$, which implies that the response function is spread about the origin $(t = t')$ with a minimum width given by the Heisenberg uncertainty relation as

$$\Delta t \sim \frac{\hbar}{E_{mn}} \sim \frac{\lambda}{c}.$$

By equation (2.10) we see that preacceleration is on the order of τ, and for physical values of α,

$$\tau \sim \alpha(\lambda/c) \ll \Delta t.$$

Therefore, the noncausal behavior acts over too short of a time to be observable. The quantum mechanical equation of motion for the point electron, therefore, exibits no observable violations of causality.

It has been shown [37] that since the cutoff frequency is small compared to c/λ, the denominator of (4.117) can be expanded in order to obtain the classical solution corresponding to the quantum solution (4.116). In doing so, one obtains

$$m_o\ddot{\mathbf{R}}(t) = \int_0^\infty dse^{-s}\mathbf{F}(t+\tau s). \tag{4.119}$$

But equation (4.119) is just the classical solution (2.10). Therefore, when external forces are present, the quantum solution yields the classical solution (2.10). In obtaining this equation the runaway solutions of the classical equation (2.9) were not even introduced.

Even though the classical solution that contains acausal behavior has been recovered from quantum mechanics, the problem of preacceleration has still been resolved. This is because of the condition of the cut off frequency, making any noncausal effects unmeasurable. We conclude therefore that the classical theory is completely consistent within the domains of classical mechanics.

4.6 Summary

In this chapter, we first derived the quantum equation of motion for an extended, nonrelativistic electron. We next discussed the self-energy for a quantum point electron. This is an important difference with the classical theory which claims infinite self-energy. We noticed that the difference between the expressions for the self-energy of the two theories can be related to the fact that the quantum expression contained an infinite series which was not present in the classical expression.

The quantum point theory was next obtained from the quantum extended model theory in the appropriate limit. The quantum point theory was shown to contain no runaway solutions. It has also been shown that the point theory contains no observable acausality. Therefore, we can conclude with Moniz and Sharp that the quantum mechanical theory for the point electron does not contain the problems inherent to the classical point theory.

We showed that the classical limit of the quantum point theory reduced back to the classical equation of motion for the point electron. Therefore, the problems of the classical theory cannot be resolved in such a limit. However, by considering the correspondence limit of the solutions, answers were found. First, the constant velocity solution was found when no external forces were present. This solution corresponds to the classical constant velocity solutions. Second, it was found that the runaway solutions enter in as one is taking the classical limit only after jumping over a cusp. Therefore, the runaway solution in classical theory has no meaningful connection with the quantum theory. Third, it has been shown [37] that when external forces are present, the time over which acausal behavior occurs is over too short of a time scale to be measurable, and therefore has no meaning. In the classical limit, the solution (2.10) was found. However, the problems of runaways, preacceleration, and infinite self-mass have been resolved through the quantum theory. Finally, we learn from the limiting process the importance of taking limits in the right order. This study illustrates how the order in which limits are taken can have a major effect on the final results.

CHAPTER 5

SUMMARY AND CONCLUSIONS

The electron is one of the simplest charged particles we know of. And yet, as we have seen, the equations of motion for the electron are complicated when radiation reaction effects are included. In *Chapter 2* for instance, we found that the equation of motion was a third order differential equation. The solutions to this classical equation exhibit the well-known problems of runaway and preaccelerating behavior, along with infinite self-energy. These problems of the theory have sparked and continue to spark much research in an effort to understand and resolve them.

A more thorough theory was developed by Abraham and Lorentz for the classical electron in which the physical extension in space of the charge was taken into account. This theory does not contain the problems of the classical point electron theory as long as the effective charge radius L is taken to be greater than $c\tau$. This is interesting because it shows that the runaway and acausal behavior are not a characteristic of the point theory only, but occur for an extended model as well if the extension is not sufficient. This condition on L was obtained using the spherical shell model. We tested the generality of this result by using a uniform sphere. Once again, we found that the effective charge radius L had to be greater than $c\tau$ for well behaved solutions. We consider the use of different models simply a mathematical tool for understanding the theory, and not an attempt to find a real model of the electron.

For Lorentz the electron was a real, physical particle. He calculated its radius by assuming that all of the mass was electromagnetic in nature. However, we now know that the electron is three orders of magnitude smaller than that predicted by Lorentz. Many believe it to be a true point particle in modern theory. That the electron has structure beyond or in addition to its spatial extension is illustrated by spin (usually described as a relativistic effect) and its anomalous magnetic moment (quantum field theoretic effect). Any attempt to take the point limit of the classical extended electron theory, however, results in recovering the problems of the point theory. Therefore, the problems of the point theory cannot be resolved within the framework of classical mechanics, and we must turn our attention to a more comprehensive theory.

Using the Heisenberg picture of quantum mechanics, we followed the derivation of the nonrelativistic quantum theory for an extended electron in *Chapter 4*. From this equation we obtained an expression for the self-mass.

From the extended model theory, a quantum point electron theory is obtained. This equation of motion was shown to have no runaway solutions and no observable acausal behavior. In other words, major problems from the classical theory disappear from the framework of quantum mechanics for a nonrelativistic point equation.

Taking the classical limit of the quantum point equation of motion just brings us back to the classical equation for the point electron with most of the associated problems. Therefore, there is not much to be learned from such a limit. However, a more productive approach consists in looking at the correspondence between the solutions of the two theories. When no external forces are present, the constant velocity solution corresponds to the classical constant velocity solution. The runaway solution cannot come from the correspondence of the quantum solutions, however, because of the discontinuity between the two theories as illustrated by α_{crit} in Figure 4.3. Therefore, the runaway solutions of the classical theory have no meaning and are simply due to the inability of classical mechanics to describe a quantum particle.

The acausal behavior was also shown to be nonobservable in quantum mechanics. The preacceleration happens on too short of a time scale to be observable as dictated by the Heisenberg uncertainty principle. Therefore, the acausal behavior in the classical theory is not measurable, and therefore has no meaning. Since the runaway solutions should not exist, and any acausal behavior is not observable, we can use the classical theory in confidence within the appropriate domain. The problems of the classical theory have been resolved by taking an appropriate route through quantum mechanics. These results are summarized in *Figure 5.1*.

Keeping in mind the limitations inherent to our nonrelativistic, idealized free electron and the limitations of the (nonperturbative) series of the Moniz and Sharp theory, the conclusion emerging from this research is threefold: First, it appears that the problems of the classical point electron theory are understood. They cannot be resolved within classical theory. They cannot be resolved by realistically or artificially considering an extended classical charge.

Secondly, the problem is "manageable" in the sense that there exists a more general theory in which the solutions do not exhibit the problematic behavior. The two theories are related in particular through a limiting procedure ($\lambda \to 0$, or $\hbar \to 0$, or $\alpha \to \infty$). Through this relation, it can be seen that the unphysical solutions from the classical theory do not correspond to the limiting values of the physical quantum solutions. This is to be contrasted with the unphysical classical point electron solutions obtained as a zero radius limit of the physical extended electron within classical mechanics.

Thirdly, as pointed out by Rohrlich [15], a great deal of physics can be learned by taking the correspondence limit between theories. By observing what one theory is doing which another theory is not, we learn more about the two theories. This observation could be extended to the next level of sophistication: quantum field

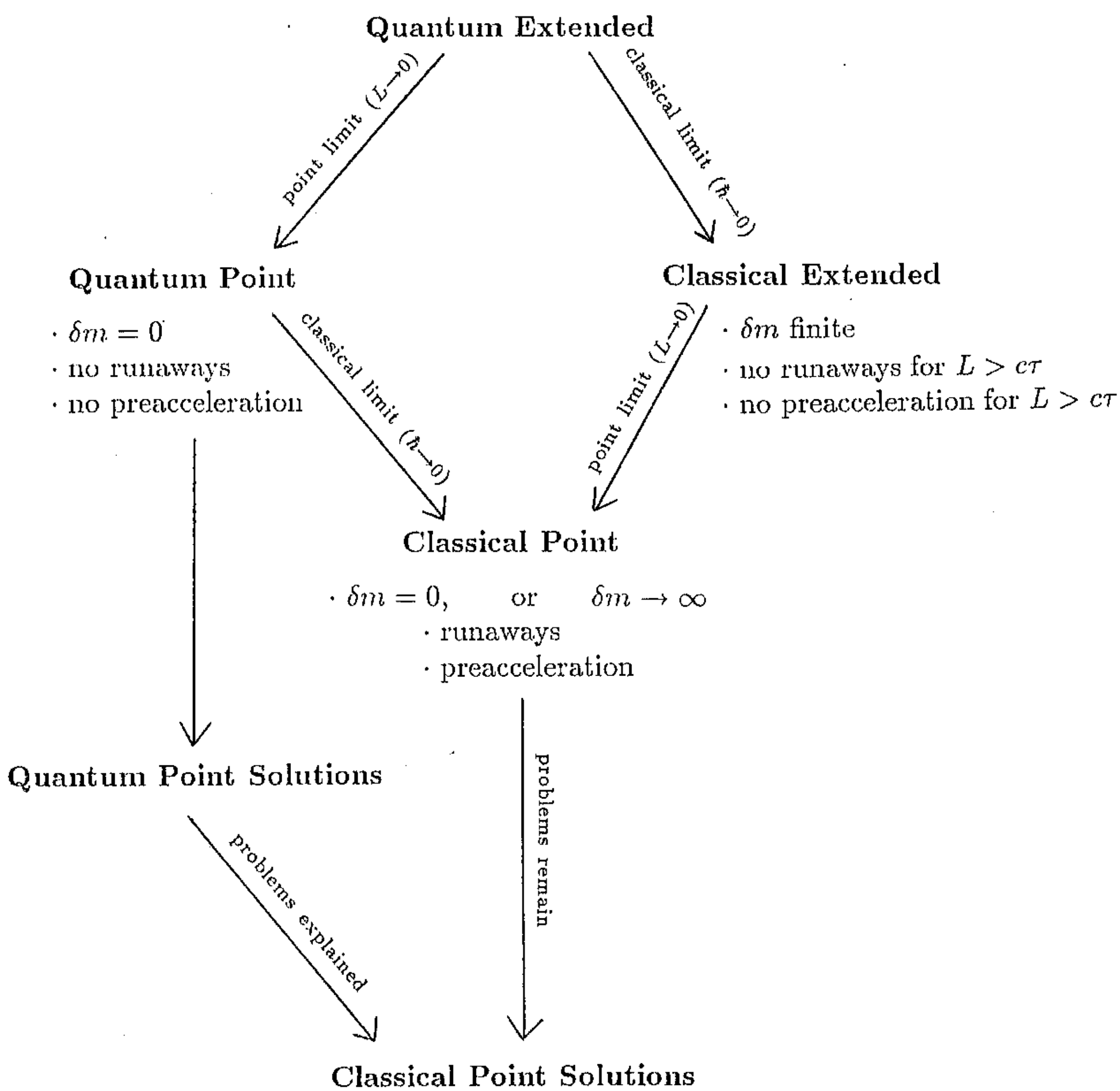

Figure: *Summary of the Correspondence Between Theories.*

theory. We would then ask the question," What is quantum mechanics doing different so as not to contain the problems of the classical theory?" Obviously the relation between classical and quantum theory is very complex and goes beyond the limiting procedure used in this research. This work can only hope to lift one corner of the veil. In the process of comparing the two theories, we observed that the quantum theory maintained an infinite series in the point theory not present in the classical point theory. In fact, the quantum point theory looks much like for an extended model. What the quantum theory seems to be doing is maintaining an extension in space even for a point electron. This "extension" is represented by the Compton wavelength.

Somehow the additional structure present in a quantum particle (whether represented by a spatial extension or not), is probably the same structure which leads to the intriguing quantum behavior present in so many "quantum parodoxes" studied elsewhere, and fulfils here a very worthy purpose: it gives a more definite answer to a problem raised one hundred years ago.

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A STUDY OF THE CLASSICAL AND QUANTUM MECHANICAL EQUATIONS OF MOTION FOR THE RADIATING NONRELATIVISTIC ELECTRON

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ABSTRACT

According to classical electrodynamics, accelerating charged particles radiate. This radiation carries away some of the particles energy and momentum, and consequently affects its motion. This phenomenon is called radiation reaction.

We study the equations of motion for the nonrelativistic electron including radiation reaction. In doing so, we choose to emphasize the classical quantum connection in the approach of Moniz and Sharp [*Phys Rev D*, **15** 2850 (1977)]. We provide detailed calculations and analysis of their results. In the classical case, it is shown that the solutions to the equation of motion exhibit the well-known problems of either runaway behavior or preacceleration for a point electron. Any attempt to resolve these problems within the realms of classical electrodynamics is fruitless. Using the Heisenberg picture of quantum mechanics, we study the quantum equations of motion, and find that the quantum point electron theory does avoid some of the problems of the classical theory. Through an appropriate correspondence limit we study how the quantum theory is able to resolve some of these problems. In so doing, we gain further understanding of quantum mechanics, and its connection with classical mechanics.

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