CORRELATIONS OF COUPLED LOGISTIC MAPS

by

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ABSTRACT

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The behavior of coupled chaotic systems is not well known. We study the behaviors of two coupled logistic maps. We use three couplings to study the behavior, a master-slave coupling, a symmetric coupling and a variable coupling. We develop methods to study the correlations by looking at the bifurcation diagrams, scatter plots and cobweb plots. With weak couplings correlations are seen. We determine that with strong couplings the two maps completely synchronize.

ACKNOWLEDGMENTS

I would like to thank Dr. Hart for his help in this project. I would also like to thank Ben Wilson for introducing me to the subject, and Bronson Argyle and Richard Taylor for putting up with me.

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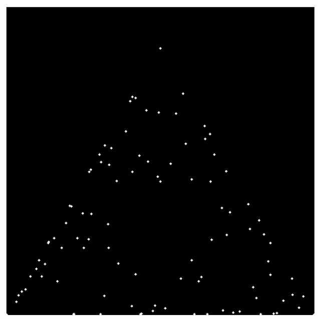
Chapter 1

Introduction

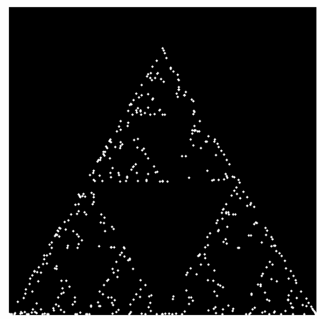
1.1 Chaos

When most people think of chaos they think of disorder and confusion. However, chaotic systems follow rules, and even simple rules can lead to chaos, but still have an underlying structure. Following simple rules can show us the characteristics of a chaotic system. An example of order rising from simple rules is the Sierpinski Gasket. First, on a piece of paper, mark three points to make a triangle. Label one point (1,2), another (3,4), and the last (5,6). Next, select any point inside the triangle. Roll a die to select one of the corners of the triangle at random. Measure the distance from the current point to the corner and mark a new point exactly halfway between. Using this new point, roll the die and select a corner at random, marking half way to that corner from the previous point and continuing until a pattern appears. You might be surprised that a pattern forms in Fig. 1.1.

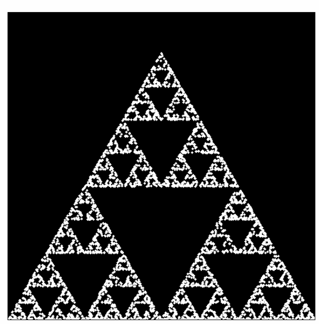
Another feature of a chaotic system is sensitivity to its initial conditions, a trait sometimes called the "butterfly effect," drastic changes to the systems behavior by small changes in the initial conditions. Unlike random systems, chaotic systems pos-



(a) 500 points



(b) 1000 points



(c) 5000 points

Figure 1.1 The Sierpinski Gasket.

sess an underlying structure, as we saw with the Sierpinski Gasket.

1.2 The Logistic Map

The logistic map is given by Eq. (1.1).

$$x_{n+1} = rx_n(1 - x_n) \tag{1.1}$$

A map is similar to a function. For a given number both a map and a function return a specific value. With the logistic map we give it an initial value, we take the value it returns and put that back into the map. We iterate many times and observe the behavior of the values returned. It was originally used as a model for population dynamics, with x, the population, varying between 0 and 1, 0 being extinction and 1 being the maximum population. It exhibits different behaviors depending on the parameter r, sometimes called the biotic constant. The parameter r would be like the reproduction rate or some other factor that could be varied. If r is between 0 and 1, Fig. 1.2(a) shows that after a few iterations the population will settle on zero. The population has completely died off. If r is between 1 and 3, the map will settle on one value. See Fig. 1.2(b). The population has stabilized; it is the same every iteration. That value depends on what r is. For r between 3 and about 3.5, the map will settle on two values as seen in Fig. 1.2(c). It is going back and forth every year between the two values. Again those two values depend on r. Between $r = 3.5$ and 3.6, the map continues to bifurcate, or split, and at about $r = 3.6$ the map has reached chaos. [1] See Fig. 1.2(d). This can be easily seen on the bifurcation diagram. See Fig. 1.3.

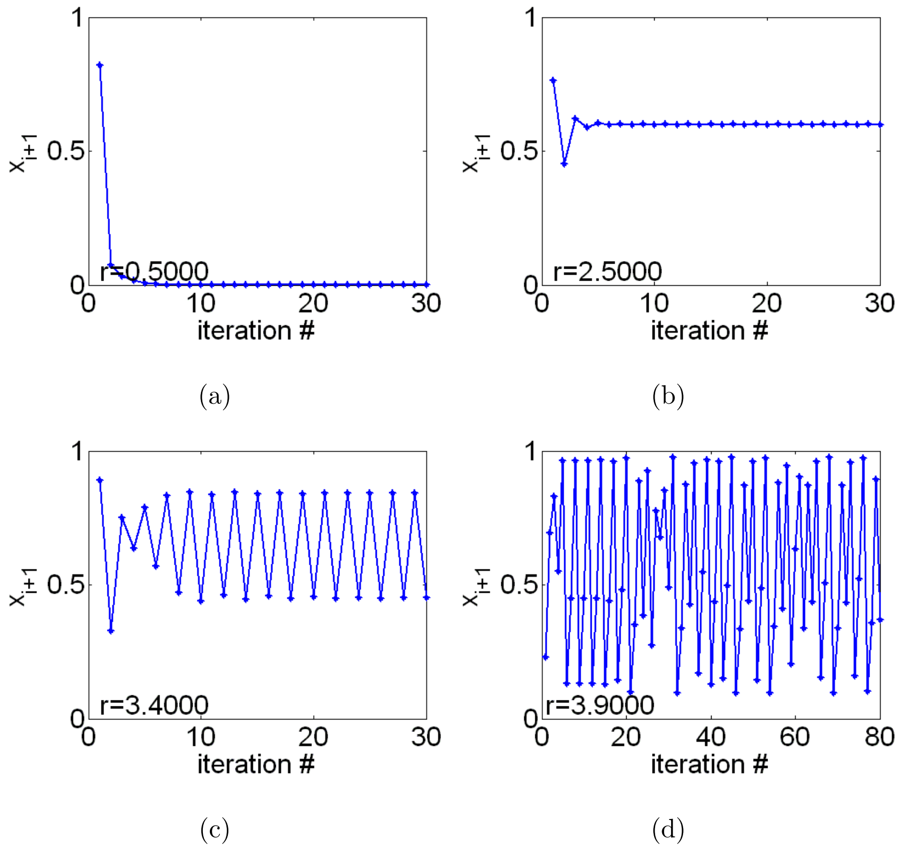


Figure 1.2 The various behaviors of the logistic map

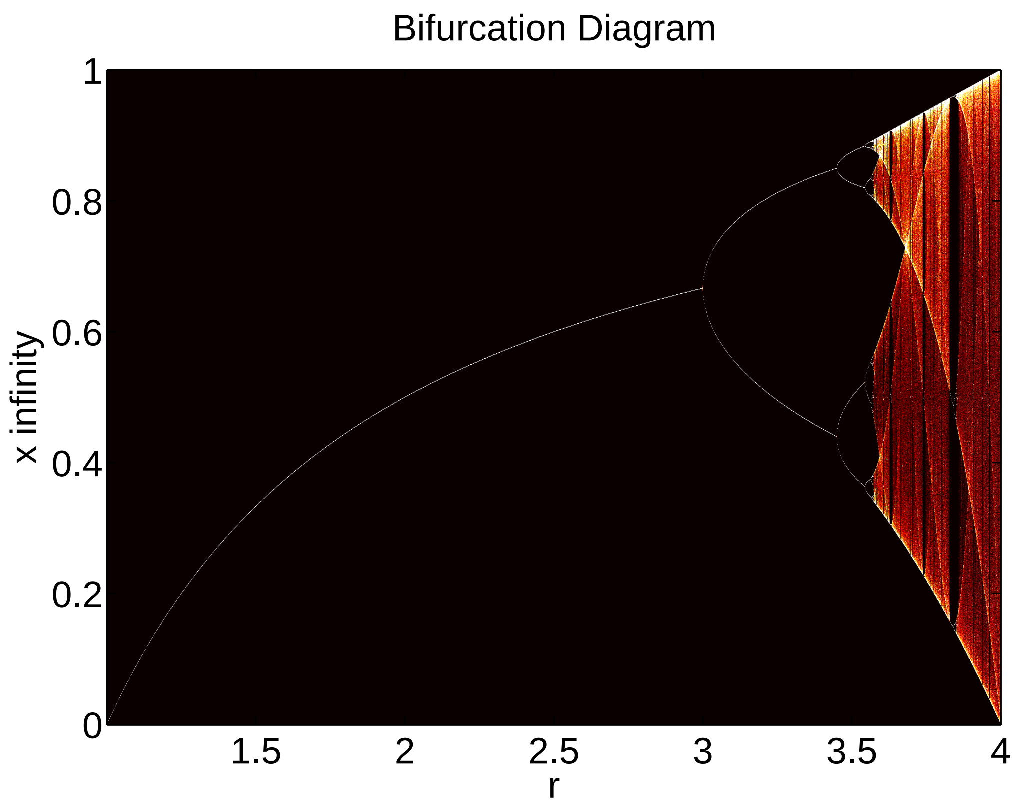


Figure 1.3 The bifurcation diagram of the Logistic Map. For r between 1 and 3, the logistic map settles on one value. For r between 3 and 3.5 it settles on two values. For r greater than 3.6 the logistic map has reached chaos.

1.3 Couplings

As we look at systems in the real world we find that most of them are non-linear and exhibit chaotic behavior given certain conditions. We also find that systems interact. Often a coupling can cause a non-chaotic system to become chaotic. Understanding these systems and their interactions will help us to understand our world. The simplest chaotic system is the logistic map. [1] In order to study interactions of two coupled chaotic systems we used two logistic maps. We use a master-slave weighted average to couple the two maps together. This is a linear coupling so we will not have to be concerned about the coupling itself adding extra non-linearity to our system. In a master-slave coupling the master will affect the slave, but the slave will not affect the master. This is like a system of rabbits. The rabbits like to eat greens, and they can eat to their hearts content. Now if we were to introduce a system of rats that like to eat anything, including greens, they could affect the rabbits. If the rats decide to eat a lot of greens, then the rabbits will not have as much to eat, and depending on how many rabbits there are, some may not survive. Or conversely, the rats might not eat any greens, leaving the rabbits unaffected by the rats. In this system the rats are the masters, unaffected by the rabbits, and the rabbits are the slave, affected by the rats.

We also look at a symmetric coupling. In the symmetric coupling we have the two systems affect each other in an equal manner. This would be like allowing the rabbits to eat anything. With the rabbits eating the rats' food supply the rats would be affected by the rabbits. Another coupling we use is the variable coupling. This coupling can be varied to give us master-slave coupling and symmetric coupling and anything in between. We will look at the master-slave and the symmetric as if they were not part of the variable coupling and look at the variable coupling last.

1.4 Synchronization

Synchronization is one method to classify the behavior of coupled systems. There are many forms of synchronization that can occur. The most known is "complete" synchronization. [2] This is when the two systems have become the same, returning the same value for a given iteration. There is also phase synchronization, where the two systems are both high at the same time and low at the same time, but the magnitudes of their amplitudes are different. Antiphase synchronization is similar to phase synchronization, but when one is high the other is low. The amplitudes in antiphase synchronization can be the same but are not necessarily the same. There are also lag and anticipated synchronization. These are similar forms of synchronization, where one system leads the other system. The difference comes when you distinguish the two systems. With one system as the main system, the other can anticipate [3] [4] the behavior of the main one or it can lag behind.

Chapter 2

Methods

2.1 Purpose

Most systems are non-linear and exhibit chaotic behavior given certain conditions. Systems also interact with other systems. Interactions can cause chaos to occur. Understanding how systems interact is important to understanding the world in which we live. To understand these interactions better, we study two coupled logistic maps. We chose the logistic map because it is the simplest system that exhibits chaos. We look at the interactions between the two maps to see if there are correlations and synchronizations.

2.2 Coupling

We need a linear coupling that has a parameter that we can vary to look at coupling strength. We use a mater-slave coupling, Eq. (2.1). If $\alpha = 0$, the maps are completely uncorrelated (two separate maps) and with $\alpha = 1$ they are fully coupled (the same map). The first system is the master. It is completely unaffected by the second system, and the second system, the slave, is affected by the first system. We observe two different behaviors.

$$\begin{aligned}
x_{n+1} &= r x_n (1 - x_n) \\
y_{n+1} &= r q_n (1 - q_n) \\
q_n &= \alpha x_n + (1 - \alpha) y_n .
\end{aligned} \tag{2.1}$$

The first behavior is seen with a weak coupling. The systems show intricate correlations that are complex and interesting. With higher couplings the two maps completely synchronize.

We also look at a symmetric coupling, Eq. (2.2). In the symmetric case the first map is influenced by the second map the same amount that the second map is influenced by the first.

$$\begin{aligned}
x_{n+1} &= r w_n (1 - w_n) \\
y_{n+1} &= r q_n (1 - q_n) \\
w_n &= (1 - \alpha) x_n + \alpha y_n . \\
q_n &= \alpha x_n + (1 - \alpha) y_n .
\end{aligned} \tag{2.2}$$

This coupling shows similar behaviors to the master-slave coupling. It shows intricate correlations with weak coupling and completely synchronizes with larger coupling strength, (higher α).

The final coupling that we look at is the variable coupling, Eq. (2.3). This is the most general coupling that we used. With $\alpha = 0$ this becomes the master-slave coupling. With $\alpha = \beta$ this becomes the symmetric coupling. Not surprisingly, this gives similar behavior to both the master-slave, and the symmetric couplings.

$$\begin{aligned}
x_{n+1} &= r w_n (1 - w_n) \\
y_{n+1} &= r q_n (1 - q_n) \\
w_n &= (1-\alpha) x_n + \alpha y_n. \\
q_n &= \beta x_n + (1-\beta) y_n.
\end{aligned} \tag{2.3}$$

We will introduce all methods that we use with the master-slave coupling. We will return and look at the other couplings using the methods that we have already developed in the final chapter.

2.3 Bifurcation Diagrams

To observe the correlations of the two systems, we compare the bifurcation diagram of the master to the bifurcation diagram of the slave. To do this for every α value, however, requires a large number of plots. The best way to look at all of the plots was to make an animation that shows the master's bifurcation diagram and the slave's bifurcation diagram as α is increased. We include 16 frames of the slave's bifurcation diagram from the animation in Fig. 2.1. As you can see, the first frame is the same as the master's bifurcation diagram. This is so because with $\alpha = 0$ the two maps are completely uncoupled and uncorrelated. The bifurcation diagram is independent of initial conditions. As α increases we see some changes to the bifurcation. Most notable is the change that occurs around $r = 3.83$ as seen in Fig. 2.2 As α increases we see that this region near $r = 3.83$ gets filled in, becoming chaotic. This region that was periodic is being forced into chaos by the master, which is periodic.

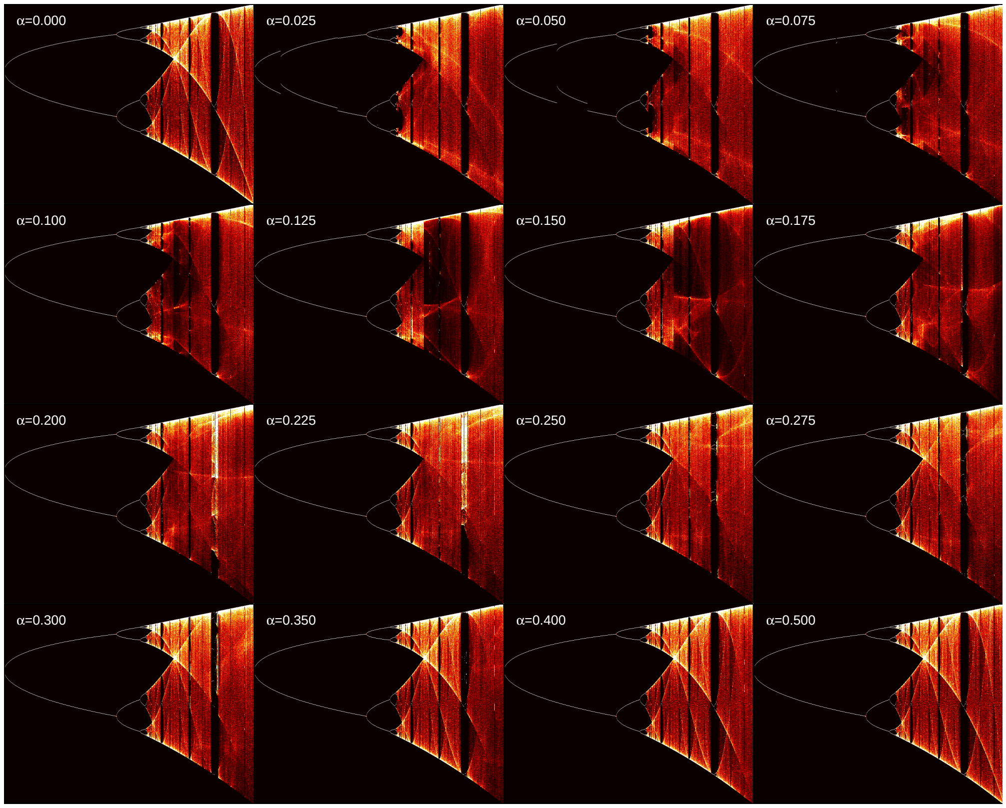


Figure 2.1 The bifurcation diagram of the slave as α is increased.

2.3.1 Subtracted Bifurcation Diagrams

When we look at the bifurcation diagram of the slave we see almost the same bifurcation as the master with a few subtle changes. To see the changes in the bifurcation diagram, we subtracted the master's bifurcation diagram from the slave's bifurcation diagram. This gives us a look at how the slave's bifurcation diagram changes as α changes. It also shows us that for $\alpha = 0$ the master's and the slave's bifurcation diagrams are the same. We can see that with small values of α the slave's bifurcation diagram is affected for most values of r. As α increases the effects from the coupling are restricted to larger values of r until the two maps become completely coupled

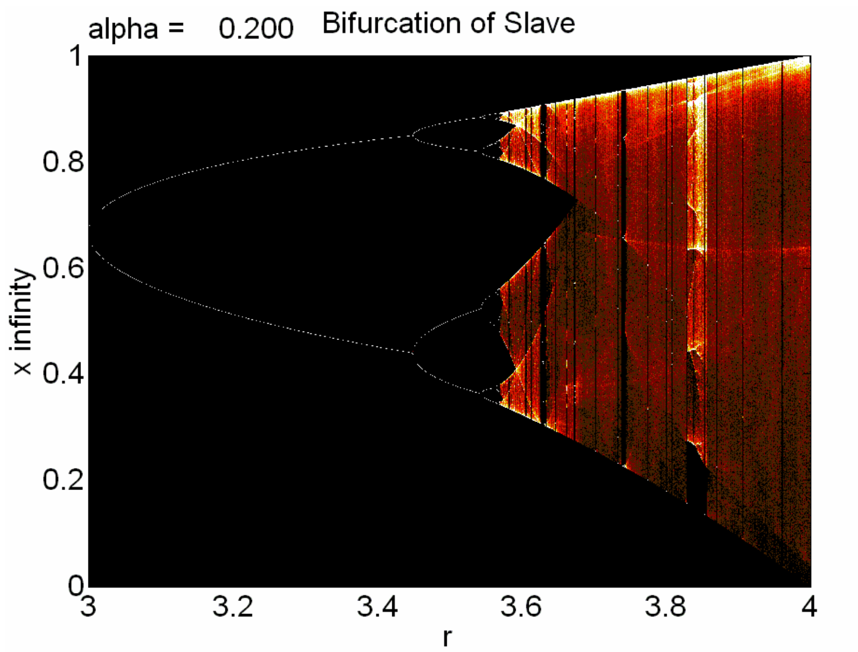


Figure 2.2 The Bifurcation diagram of the Slave with $\alpha = 0.2$. In the region around $r = 3.83$ we can see that the system is chaotic. It is being driven to chaos by a master that is periodic.

at $\alpha = 0.5$. This also was animated through α, 16 frames of which are included in Fig. 2.3.

2.4 Scatter Plots

Scatter plots are another way to look at the correlations between the master and the slave. In these plots we take the value of the master and the corresponding value of the slave. The master becomes the x coordinate and the slave becomes the y coordinate. When the master and the slave are uncoupled, the plot will have points that fill the square from 0 to 1 on the x and y axes as seen in Fig. 2.4(a). As the coupling is increased, the plot will change. As we can see from Fig. 2.4(b), as α increases some lines begin to form in the figure. Also we can see that the bottom right corner doesn't have any points. This means that if the master is large, the slave

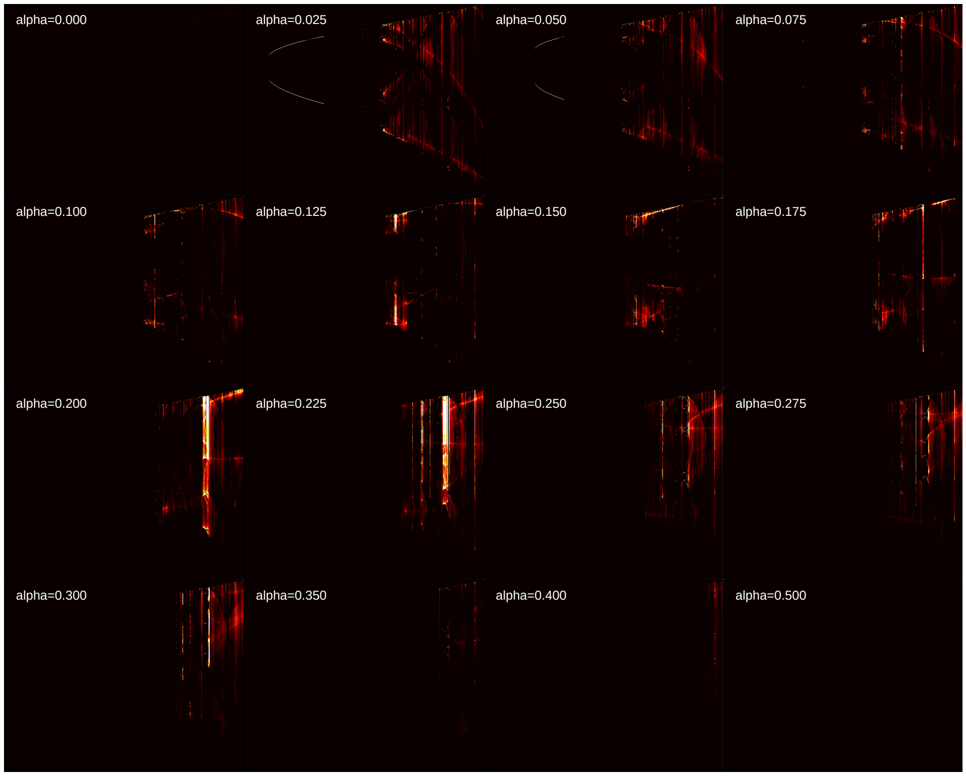
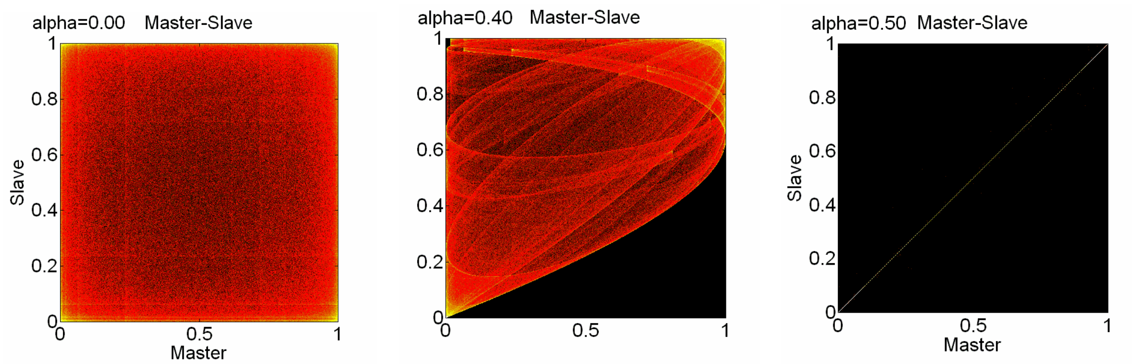


Figure 2.3 The Difference of the master and slave bifurcation diagrams. As you can see with $\alpha = 0$ and $\alpha = 0.5$ there is nothing in the plot. This means that the bifurcation diagrams of the master and the slave are the same.

is pulled up by that large value and cannot return a low value. As we continue to increase α we eventually reach complete synchronization. This occurs at $\alpha = 0.5$. When this happens both the master and the slave return the same values for each iteration. This produces the $45°$ line seen in Fig. 3.1(b).

One of the methods we use to see how increasing α affects the correlation of the two maps was to animate the scatter plot. To see how the animation looks we have plotted 16 frames of the animation in Fig. 2.5. From these we can see that as α increases there are distinct lines of higher density that form and move in intricate patterns. As α reaches its critical value the systems completely synchronize.



(a) (b) (c)

Figure 2.4 In scatter plots the master is plotted on the x axis and the slave is plotted on the y axis. (a)With $\alpha = 0$, the plot is completely filled. This means that both the master and the slave range from 0 to 1. (b)With α a little larger, patterns appear. We also see that the lower right corner is empty. This means that when the master is large the slave cannot have a small value. (c)With $\alpha = 0.5$, The master and slave have become completely synchronized. Meaning the master and slave return the same value, which give the 45° line.

2.5 Cobweb Plots

Cobweb plots are another possible way to look at the correlations. The cobweb plot is similar to the scatter plot because we make points out of the values that we get from iterating the logistic map. However, the cobweb plot uses only values from one of the maps. What we are plotting is the current value on the x axis and the next on the y axis. If we look at the cobweb plot of the master at $r = 4$, we see a parabola, Fig. 2.6. This comes from the x^2 term we get when we multiply Eq. 2.4 out.

$$x_{n+1} = rx_n - rx_n^2 \tag{2.4}$$

We compare this to the plot we get when we do the same thing for the slave. With $\alpha = 0$ and $r = 4$ the plot is the same parabola that we see when we plot the master, Fig. 2.7(a). As α increases we see that there are some changes to the parabola. See

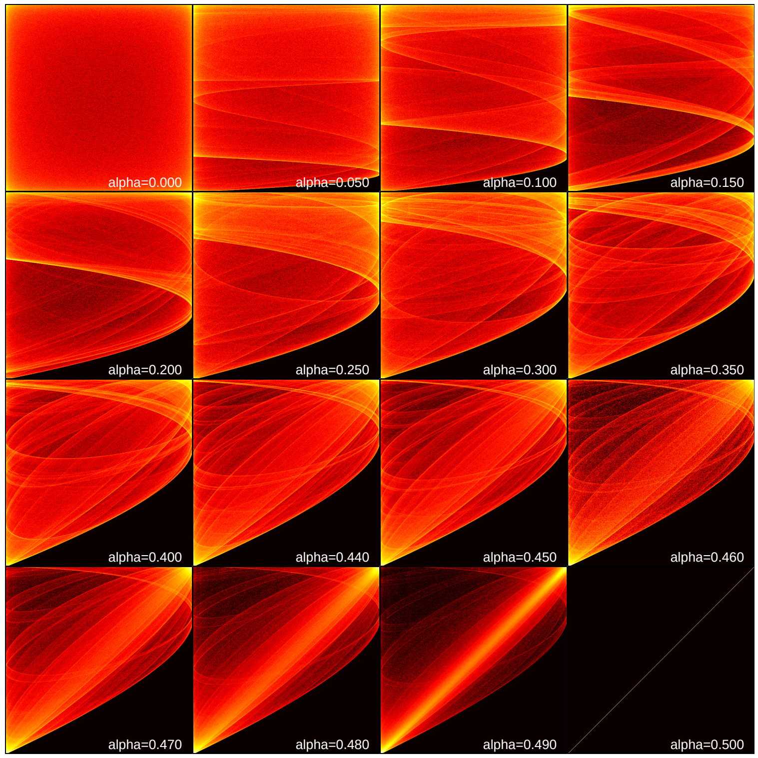


Figure 2.5 It is convenient to look at an animation of the scatter plot as α is increased. Unfortunately in a paper we can't have an animation. We therefore look at 16 plots.

Fig. 2.7(b). Then when α reaches 0.5 the original parabola is returned. See Fig. 2.7(c).

Again, we animate the cobweb plots through α. We can see a little of how the animation looks from Fig. 2.8. The correlations are apparent as we see lines of higher density appear and move around as α is varied. As α reaches a critical value, the two maps become completely synchronized.

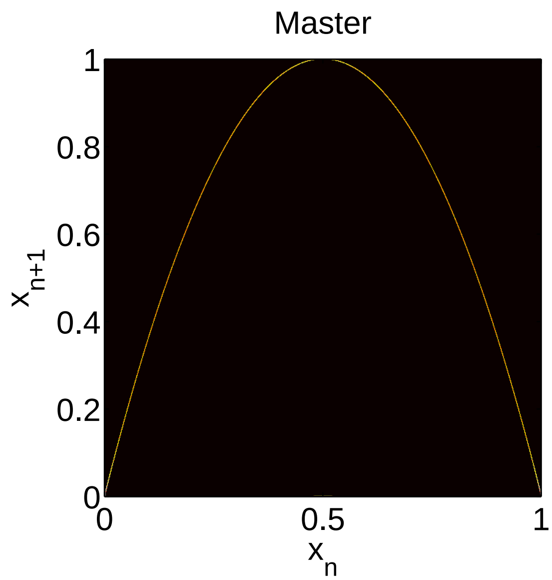
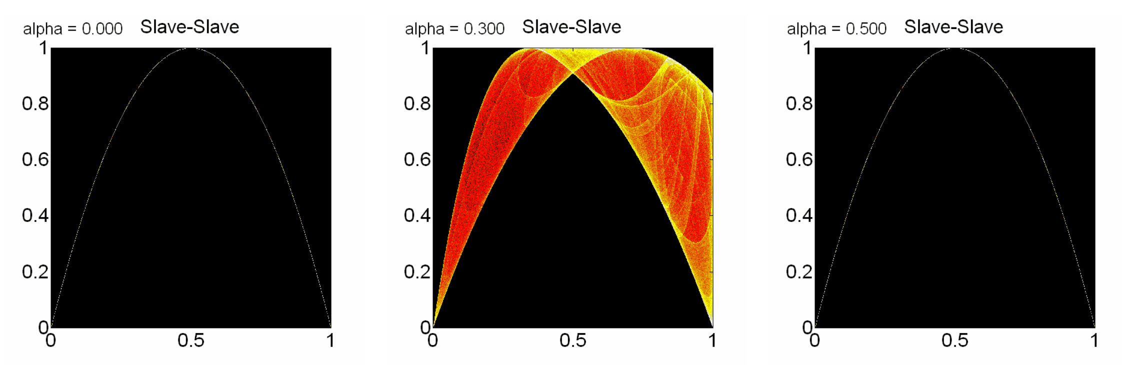


Figure 2.6



(a) (b) (c)

Figure 2.7 Cobweb plots for the slave at $r = 4$. (a)With $\alpha = 0$ and $r = 4$ the plot is the same parabola that we saw when we plotted the master. (b)As α is increased we see that there are some changes to the parabola. (c)When α reaches 0.5 the original parabola is returned.

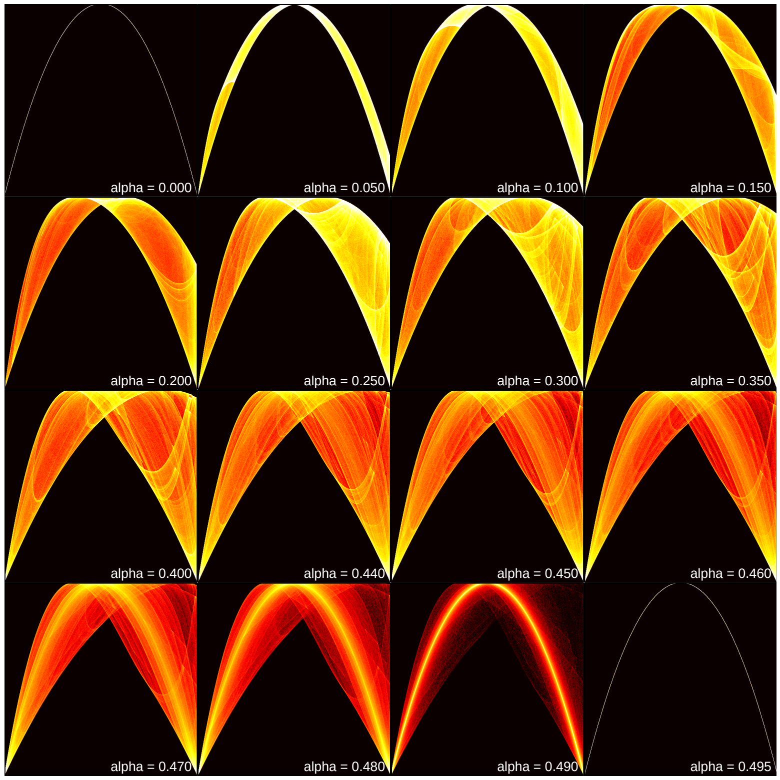


Figure 2.8 16 cobweb plots of the slave at $r = 4$. The first frame and the last frame both show the same parabola that we get from the master with $r = 4$.

2.6 Analysis Methods

Through these methods we are able to look at all of the couplings and analyze their behaviors. The scatter plots are particularly useful at showing correlations and synchronizations in the symmetric and variable coupling cases.

Chapter 3

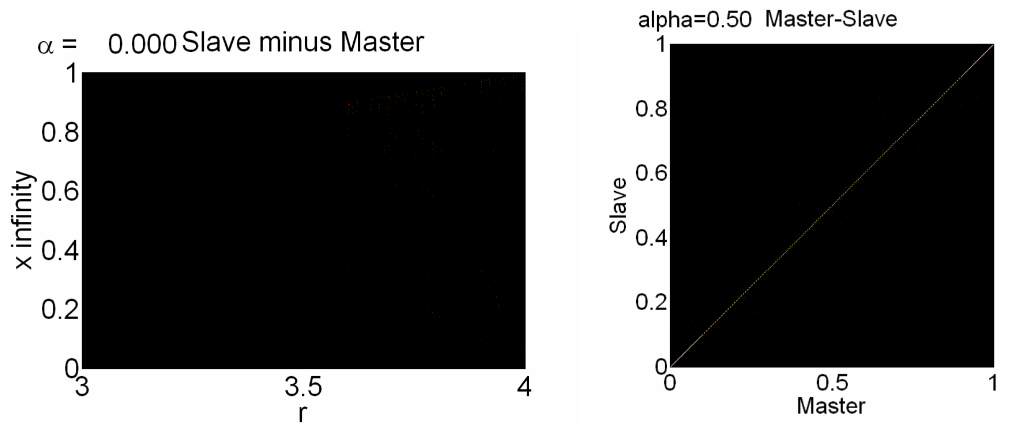
Analysis of Results and Conclusions

3.1 Analysis of Results

Coupled chaotic systems often have correlations. We attempt to classify the correlations of two coupled logistic maps. We find similar behaviors for all three couplings that we use. For low coupling strength we observe intricate correlations between the two systems. At higher coupling strength, the two systems completely synchronize.

3.1.1 Master-Slave Coupling

Given that chaotic systems are difficult to analyze by solving equations, we use the plots that we generate to analyze the synchronization. Synchronization is easiest to see on the subtracted bifurcation diagrams and on the scatter plots. When the two systems reach complete synchronization the subtracted bifurcation diagram is blank, and the scatter plot is a single 45° line as seen in Fig. 3.1. We see on both plots that when $r = 4$ and $\alpha = 0.5$, the two systems have completely synchronized.



(a) (b)

Figure 3.1 (a)The blank plot indicates complete synchronization since all of the points are the some in both the master's and the slave's bifurcation diagram. (b)The 45° line indicates that the two systems are completely synchronized.

3.1.2 Symmetric Coupling

The symmetric coupling is similar to the master-slave coupling in that it is easier to analyze the plots than it is to analyze the equations. The scatter plot of the variable coupling behaves similarly to the master-slave coupling. For higher couplings, the maps synchronize giving the 45° line. We see in Fig. 3.2 that when $r = 4$ and $\alpha = 0.25$ the two systems completely synchronize. For lower values of α, we see that there are correlations. The top left and the bottom right corners both have areas where there are no points. This means that if the first system has a large value, the second system can not have a small value. Also if the second system is large, the first system can not be small. For couplings around $\alpha = 0.11$ we see that there are only two regions on the plot that have points.

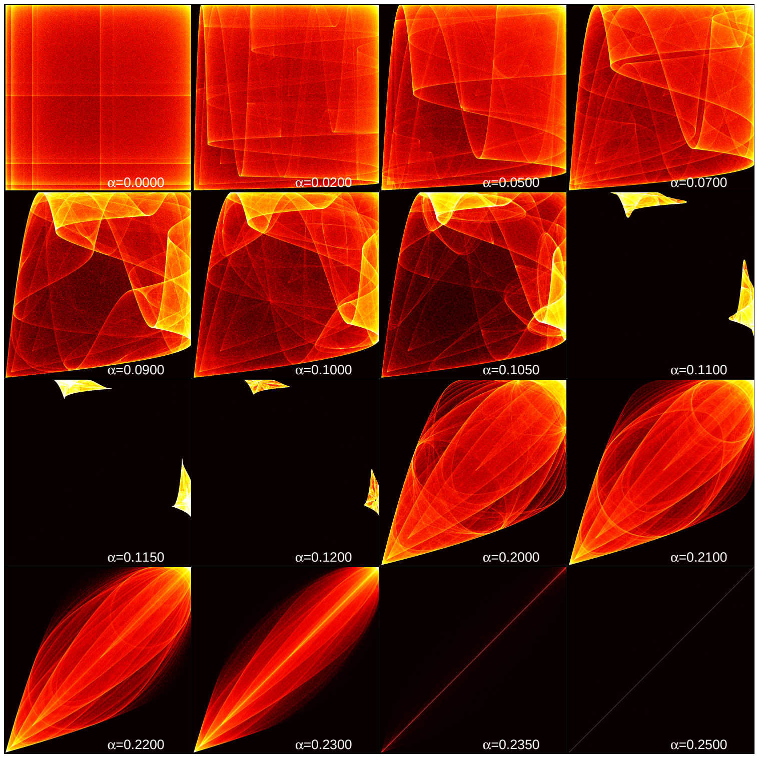


Figure 3.2 The symmetric coupling behaves similarly to the master-slave coupling. However, it completely synchronizes with $\alpha = 0.25$

3.1.3 Variable Coupling

The variable coupling has two parameters α and β. This makes classifying the complete synchronization a little different. For one value of α there is a value of β that causes complete synchronization. If α changes then β changes, and we can't give a particular value for either α or β. Looking at Fig. 3.3 we see that with $\beta = 0.1$ and $\alpha = 0.4$ the two systems have completely synchronized. We notice that $\alpha + \beta = 0.5$. This turns out to be the relationship between α and β every time the two systems synchronize.

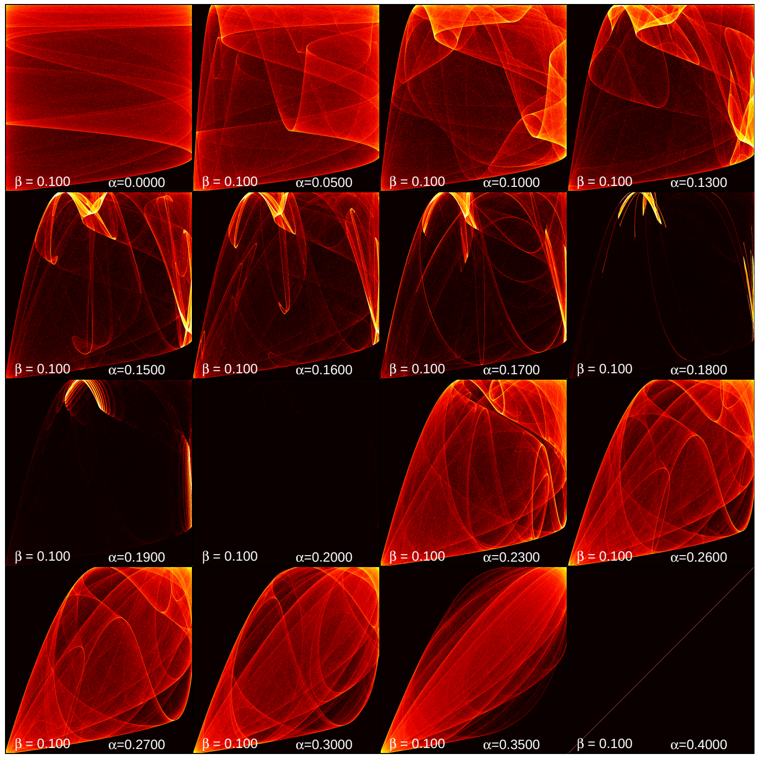


Figure 3.3 The variable coupling completely synchronizes when $\alpha + \beta = 0.5$

3.2 Conclusions

Through various computational methods we were able to observe several areas where the two coupled systems became completely synchronized. This was the case for all of the couplings that we used. When the coupling strength was strong enough the two systems became completely synchronized. We found 0.5 to be the coupling that caused complete synchronization. For the master-slave coupling $\alpha = 0.5$, for the symmetric case $\alpha = 0.25$, which we multiply by 2 because both systems contribute, and for the variable case $\alpha + \beta = 0.5$. We also observed a region where we had a periodic master driving what we would expect to be a periodic slave into chaos.

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[2] J. Bragard, G. Vidal, H. Mancini, C. Mendoza, and S. Boccaletti, "Chaos suppression through asymmetric coupling," Chaos: An Interdisciplinary Journal of Nonlinear Science **17,** 043107 (2007).

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Appendix A

Code for Bifurcation Diagrams

This code will run an animation of bifurcation diagram of the slave. It will also show the animation of the subtracted bifurcation diagram.

```
% MATLAB program for iterating the logistic map
% and making a bifurcation plot -- r vs x_inf

close all; clear all; clc

Nr = 300;     % Number of different values of r to include
r_a = 3.3;       % lower bound on r range
r_b = 4;     % upper bound on r range

r = linspace(r_a, r_b, Nr); % Assign the values of r

Nskips = 200; % Number of iterations to skip (avoid transient
               % in plot)
Nsteps = 2e3;   %Number of points in plot

Rsteps = Nsteps+Nskips; %number of points to iterate
Nbin = 300;    %Number of bins in the y-dir
ybin = linspace(0,1,Nbin);

nx = 450;
ic = .00;
fc = .50;
a = linspace(ic,fc,nx);
```

```
% data = zeros(Nsteps,Nr);  % Store the points to plot in this
% data2 = zeros(Nsteps,Nr);  % array

im1 = zeros(Nbin,Nr);     %build the arrays for the color plot
im2 = zeros(Nbin,Nr);
out1 = zeros(1,Nbin);
out2 = zeros(1,Nbin);

x = zeros(2,Nsteps);

alpha=0.00;

set(0,'defaultaxeslinewidth',1)
set(0,'defaultaxesfontsize',26)
set(0,'defaultlinelinewidth',2)
set(0,'defaulttextfontsize',26);

figure
set(gcf,'position',[100 100 1000 700])

cc = colormap('hot');
% cc = cc(size(cc,1):-1:1,:);
% cc = [cc(:,3),cc(:,2),cc(:,1)];
% colormap(cc)
a1=axes;

set(a1,'position',[0.093 .58 0.42 0.33])
p1=image(r,ybin,zeros);
title('Master')
axis([r_a r_b 0 1])
set(gca,'xtick',[],'ytick',[])
set(gca,'ydir','normal');
a2=axes;
set(a2,'position',[0.55 .58 0.42 0.33])
p2=image(r,ybin,zeros);
title('Slave')
axis([r_a r_b 0 1])
set(gca,'xtick',[],'ytick',[])
set(gca,'ydir','normal');
a3=axes;
set(a3,'position',[0.093 .105 0.88 0.4])
p3=image(r,ybin,zeros);
% title('Slave Minus Master')
```

```
xlabel('r');ylabel('x infinity');
axis([r_a r_b 0 1])
% set(gca,'ytick',[],'xtick',[])
set(gca,'ydir','normal');
t1=text(r_a+.03,.85,'alpha = ','color','w');
text(3.57,1.05,'Slave Minus Master');

for i_r = 1:Nr   % loop over r values

      x(1,1)=.71;

      for i_st = 1:Rsteps   % loop over i
        x(1,i_st+1) = r(i_r)*x(1,i_st)*(1-x(1,i_st));  % Update master

      end

      out1 = histc(x(1,Nskips:Rsteps),ybin);
      im1(:,i_r)=out1/max(out1);

end

im1=im1*255;
set(p1,'cdata',im1)

for j =1:nx

    x(1,1) = .7; % Starting value, (almost) anything works
    x(2,1) = .71;

    alpha = a(j);

    for i_r = 1:Nr   % loop over r values

      for i_st = 1:Rsteps   % loop over i
        x(1,i_st+1) = r(i_r)*x(1,i_st)*(1-x(1,i_st));  % Update master
        q = (alpha*x(1,i_st) + (1-alpha)*x(2,i_st));
        x(2,i_st+1) = r(i_r)*q*(1-q);

      end

      out1 = histc(x(1,Nskips:Rsteps),ybin);
      im1(:,i_r)=out1/max(out1);
      out2 = histc(x(2,Nskips:Rsteps),ybin);
```

```
      im2(:,i_r)=out2/max(out2);

    end
    im1=im1*255;
    im2=im2*255;

    im=im2-im1;

    set(p2,'cdata',im2)
    set(p3,'cdata',im)

    s = sprintf('alpha = %8.4f\n',alpha);
    set(t1,'string',s)
    drawnow

end
```

Appendix B

Scatter Plots Animation

```
% MATLAB program for iterating two coupled logistic
% maps, one master and one slave

close all    % Close any open plot windows
clear; clc   % Clear all variables, clear the command line
tic
% Try plotting the average distance from the diagonal as a
%function of alpha. This might show some sort of phase transition.
set(0,'defaultaxesfontsize',36) set(0,'defaulttextfontsize',36)

cc = colormap('hot');
colormap(cc)

Ngrid = 300;           %Number of points in x and y
h = 1/Ngrid;
sp = h:h:1-h;%5-1/Ngrid;
empty = zeros(Ngrid,Ngrid);

figure(1) a1=axes; set(a1,'position',[0.2 .15 0.7 0.7])
set(gcf,'position',[100 200 700 700]);
    p1=image(sp,sp,empty);
    axis([0 1 0 1])
    axis square
    title('Master-Slave')
    set(gca,'ydir','normal');

    t1=text(-.2,1.095,'\\alpha= ');
```

```
    xlabel('Master');ylabel('Slave');

Nsteps = 5e3;     % Change this number to add more steps
x = zeros(2,Nsteps); % Initilize the 'x' array (make it empty)

nalpha = 5;    %Number of frames in animation
ua = .50;      %The final value of alpha
la = .00;      %The initial value of alpha
a = linspace(la,ua,nalpha);

rr = 4;   % This is the rate of reproduction (should be between 0 & 4)
          % At this point we assume that it's the same for both
          % maps


for j =1:nalpha

    x(1,1) = rand;   %The starting value of the master
%     x(2,1) = rand; %The starting value of the slave
    x(2,1) = x(1,1)+.001;

     alpha = a(j);   % Coupling constant

% This part of the program iterates the equation (the "logistic map")
        for i = 1:Nsteps-1
            x(1,i+1) = rr*x(1,i)*(1-x(1,i));  % Update master
            q = (alpha*x(1,i) + (1-alpha)*x(2,i));
            x(2,i+1) = rr*q*(1-q);
        end

    histmat=hist2(x(1,:),x(2,:),sp,sp);
    im1 = log(histmat+1)/max(max(log(histmat+1)))*64*1.5;
    set(p1,'cdata',im1(1:end-1,1:end-1))
    s = sprintf('\\alpha=%4.3f',alpha);
    set(t1,'string',s)
    drawnow

    M(j) = getframe(gcf);
end
```

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