GENERAL RELATIVISTIC MODELS OF ROTATING ASTRONOMICAL OBJECTS

by

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ABSTRACT

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The problem of modeling rotating astronomical objects is one central to astrophysics. Because most, if not all, astronomical bodies are rotating, an understanding of this problem has universal application. We will be considering two rotating astronomical systems. The first system we will consider will be a rotating galaxy. We will use the van Stockum metric [1] in order to find an equation for the tangential velocity of a galaxy. There are problems associated with the van Stockum metric including the occurrence of closed timelike curves in the presence of matter. These problems prevent the van Stockum metric from being used to describe a physical system. The second system that we will consider will be a rotating neutron star, which is the type of star that creates a pulsar. Following the work of Cook, Shapiro and Teukolsky (CST) [2] we will discuss the methods used and derive the Green's function that can be used to find the coefficients of the metric. We will also derive the source terms using

the Einstein Equations. Then we will consider an extension to the problem done by CST by relaxing the assumption of circularity. We will create the corresponding Einstein tensors and other associated tensors.

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Chapter 1

Introduction

A question that has always intrigued astronomers is the origin and nature of the planets and stars. When Isaac Newton introduced his law of gravitation it became possible to begin to explain the structure and motion of these massive objects. Newton's law of gravity gave insight into the forces involved in holding a massive body together under the infulence of its own gravitiational field. It was only natural to consider the problem of how an astronomical body would be affected by its rotation. In fact this problem was first investigated in a systematic way by Isaac Newton when he considered how rotation would affect the shape of the earth. In his *Principia* he considered the earth to be a homogeneous mass with constant angular momentum. The rotation would slightly deform an otherwise spherical earth and give it a slight bulge at the equator. In Newton's investigation he considered the rotation and the resulting bulge to be small. While these initial steps were important the greatest advancements in how to model a rotating astronomical object came in the 1700's when Alexis-Claude Clairaut and Colin Maclaurin were able to first model self-gravitating spheroids that were still roughly spherical, but the breakthrough came in 1740 when Maclaurin published his book *A Treatise on Fluxions* in which he was able to model spheroids that were highly oblate. Because of his groundbreaking work these oblate spheroids that depart radically from spherical form were named the Maclaurin Spheroids.

While the work of Maclaurin and Clairaut was important for laying the groundwork of how we consider self-gravitating rotating fluids, our modern understanding and formulation of the problem comes from the work of Leonhard Euler, the Marquis Pierre-Simon de Laplace and Adrien-Marie Legendre. In 1755 Euler was able to give the conditions of hydrostatic equilibrium which describe the balancing of forces inside and at the surface of a static fluid. Later, from their investigations of the work done by Maclaurin and Clairaut, Laplace developed his equation for finding potentials and Legendre introduced his series of polynomials. Their work was followed by the work of Siméon-Denis Poisson who developed his equation to describe the potential of a body with a given distribution.

The work of these pioneers became the standard for considering how to describe a self-gravitating fluid which is the basic model for galaxies, stars and planets. For our purposes, we are interested in how these ideas apply to galaxies and stars, but we will be considering general relativistic models of stars and galaxies. Essentially we are investigating the general relativistic analogue of astrophysical models that have already been worked out in Newtonian Physics. In the case of modeling a galaxy our motivation is to see if the general relativistic model can reproduce the results of the Newtonian model and also see if it coincides more closely to observation. In the case of modeling stars we will consider the special case of neutron stars and first consider the Newtonian case for rapidly rotating neutron stars and then consider the general relativistic case.

With the introduction of general relativity in 1915 a natural avenue of investigation was to consider spherical fluids much like those investigated by Newton, Maclaurin, Laplace and all the others who contributed to the work. When Karl Schwarzschild found the first exact solution to the Einstein field equations in 1915, he did so by assuming spherical symmetry. While his solution was a convenient approximation for most astronomical calculations, the basic assumptions made by Schwarzschild prevented it from truly representing astronomical systems. While most astronomical bodies are roughly spherical, thus satisfying Schwarzschild's assumption of symmetry, all known astronomical bodies are rotating, which Schwarzschild's solution did not account for. So while his solution can be used as a good general approximation to solving problems in general relativity, it does not account for all or even a majority of possible physical configurations.

In the following years an effort was made to find other exact solutions to the Einstein field equations. One of these was a solution by Willem van Stockum with "dust" as the matter source. In 1937 van Stockum presented his exact solution to the Einstein field equations at the annual meeting of the Edinburgh Mathematical Society. His solution included the assumptions of axisymmetry, rotation and a cylindrical spacetime. The reason why his solution is referred to as a "dust" is because van Stockum also assumed that the matter in the spacetime acted as a pressureless fluid, which in general relativity is often called dust. This was the first time that a solution had been found that accounted for the rotation of the matter. But along with the introduction of his exact solution, van Stockum also introduced the possibility of closed timelike curves. This presented a problem because closed timelike curves provide for the violation of causality. Thus, this van Stockum solution may be a convenient exact solution to the Einstein field equations, but there is some question as to its usefulness in representing physical systems.

In 1963, Roy Kerr introduced the Kerr solution which assumed axisymmetry and stationarity. The Kerr solution, unlike the van Stockum solution, was asymptotically flat, which allowed for the modeling of physical systems. Because the universe is dynamic we are interested in applying these solutions to dynamic systems and figuring out how a curved spacetime would affect motion. In the following years a number of methods were developed to solve the equations of motion of rotating astronomical objects using the Einstein field equations. Some methods included expressing the equations of motion as differential equations while others employed methods of integration, such as Green's function techniques, to express the equations of motion. Either way, the goal was to find equilibrium configurations which could then be evolved through time using numerical techniques. Through these means a wider range of physical systems could be investigated. Certain assumptions and simplifications could be relaxed, such as allowing for differentially rotating matter as opposed to rigidly rotating matter. With modern computational techniques and the collective research and experience of years of investigation we may see new ways of solving these problems that allow us to see general relativistic effects never seen before. The doors may be opened to modeling more realistic systems that may account for and explain the wonders and diversity that we observe in the universe.

In this investigation we will consider a few types of axially symmetric solutions to the Einstein Equations, along with some applications. In Chapter 2 we will use the van Stockum solution to find a method for modeling a rotating galaxy, and then consider the implications and problems with using this particular solution to the field equations. In Chapters 3, 4 and 5 we will deal with methods of modeling a rotating neutron star. In Chapter 3 we will first consider the corresponding classical case and methods employed to solve the problem. The same problem will then be considered in Chapter 4 using general relativity with an intent to find similar effects to those found in the classical case. In Chapter 5 of this work we will relax some of our constraints on how the matter inside the star can move and allow for effects such as convection inside the star, while including electromagnetic effects, to find a solution to the field equations that can be used to find equilibrium configurations of a neutron star.

Chapter 2

Rotating Galaxies

2.1 Method and Motivation

One of the puzzles of modern astrophysics is resolving the discrepancy between the observed motion of galaxies and theoretical predictions of their motion. In the 1930's the astronomer Fritz Zwicky attempted to model the motion of a galaxy cluster using Newtonian mechanics. When he compared his calculations to astronomical observation of the Coma cluster of galaxies he found that his prediction was off by factor of 500 [7]. At that time he postulated that there was additional matter distributed throughout the cluster inbetween the individual galaxies. The ability to confirm this prediction through observing the rotation of an individual galaxy did not exist because of technological limitations.

It was not until the 1970's that Ostriker, Peebles and Yahil noted a discrepancy between the computed mass of galaxies using an assumed mass-to-light ratio and the calculated mass based on the observed motions of galaxies [8]. In some cases they found a difference of a factor of 10 or more. About the same time astronomers developed equipment sensitive enough to measure the rotation of individual galaxies. Rubin, Ford and Thonnard were able to make these measurements with enough precision to determine the tangential velocity of the galaxy at different radii [9]. A plot of the tangential velocity with respect to the radius is called a rotation curve for the galaxy (see for example Fig. 2.1). In a subsequent paper Rubin *et.al.* [10] presented additional evidence and showed that their work supported the claims made by Ostriker *et.al.* These observations showed that most stars in the galaxy moved with fairly uniform velocity regardless of how far they were from the galactic axis of rotation. This observation and the failure of classical models to explain it came to be known as the problem of flat rotation curves (see Fig. 2.1). This discrepancy between Newtonian theory and observation lead to the idea of dark matter. In order to bring astronomical data into agreement with their calculations, astronomers postulated that there is a large halo of dark matter surrounding the galaxies that does not interact with normal matter through electromagnetic radiation yet interacts with visible matter through gravity. This condition that exotic dark matter interacts with visible matter *only* through gravity prevents us from observing it through a telescope, and therefore presents a problem in determining what it is. Current observations [12] of the mass of dark matter put it at more than five times the mass of the visible matter in the galaxy.

Historically, the motion of individual galaxies has been modeled using only Newtonian gravity. We will explore a method of modeling a rotating galaxy in general relativity. This is done with the intent of finding the tangential velocity of a galaxy at any given radius. The purpose of this chapter is to investigate a method that can produce galactic rotation curves, especially the flat rotation curves observed in many galaxies. It was suggested by Cooperstock [13] that the flat rotation curves could be

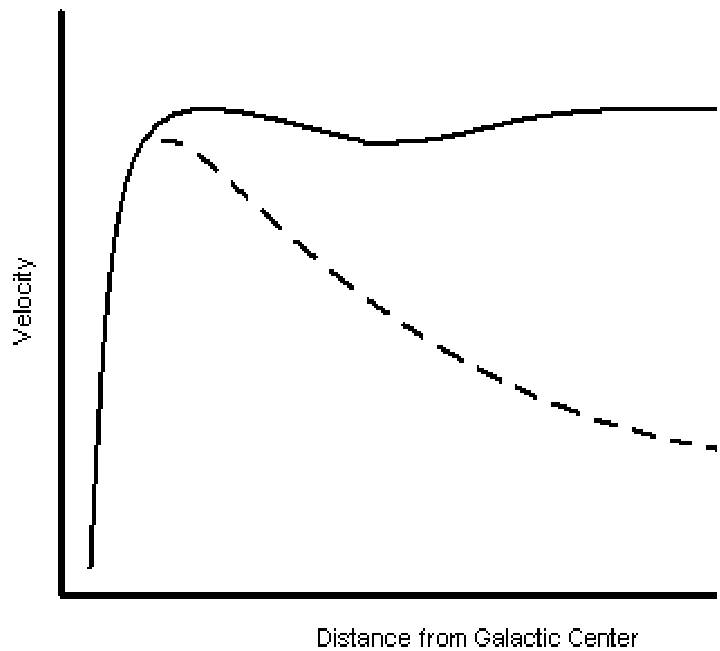

Figure 2.1 This illustrates the tangential velocity of matter in a galaxy. The solid line shows the actual velocities and the dotted line gives the Newtonian prediction without exotic dark matter.

produced without having to introduce a halo of dark matter as a correction to the theoretical prediction.

One of the two main reasons why Newtonian gravity has been used instead of general relativity is because comparable problems are linear when done using Newtonian mechanics. It should be noted that modeling the dynamical gravitational problem in Newtonian mechanics is non-linear, but the calculation of the gravitational potential can be treated as a linear problem. Kent [14] took advantage of this linearity to construct a model of a galaxy with three discrete components, consisting of a galactic bulge, the galactic disk and a spherical halo of dark matter. Van Albada, Bahcall, Begeman and Sancisi [15] used a similar approach but only considered a galactic bulge, or sphere, with an exponentially decreasing disk of matter. These approaches were both conceptually and mathematically easier than using general relativity, which does not allow for linear superposition of gravitational potentials.

The main reason why Newtonian mechanics was used in modeling galaxies was due to the weak gravitational field. In the weak gravitational limit general relativity should reduce to Newtonian gravity. Because of the low average density of a galaxy the correction due to general relativity was assumed to be so small as to be undetectable when compared with the Newtonian result. Even by assuming the existence of supermassive black holes at the galactic center to increase the average density of the galaxy, the resulting gravitational field was assumed to be too weak to see any significant departure from Newtonian predictions. Even with these considerations we may assume that the correction may be small but we will not know how it affects the overall system until we actually calculate it. It is for these reasons that we should not make any *a priori* assumptions as to the extent of the correction supplied by general relativity. Cooperstock has argued [13] that the correction arises from the inherent non-linear nature of general relativity. Thus we will consider the method proposed by Cooperstock and assess its validity and ability to reproduce the flat rotation curves of galaxies.

2.2 Derivation of the Cylindrical Metric

In this section we will show the derivation of the van Stockum metric [1]. The van Stockum metric is characterized by being stationary and also axisymmetric with cylindrical symmetry.

If we consider a privileged observer O whose worldline forms a timelike geodesic g then there exists a 3-space S consisting of all spacelike geodesics orthogonal to g.

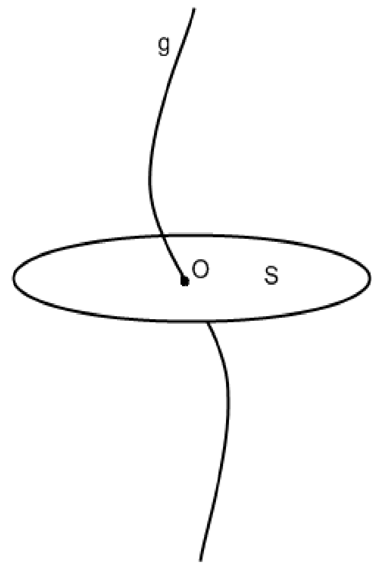

Figure 2.2 An observer O on S whose worldline forms a timelike geodesic g. The 3-space S is everywhere orthogonal to g. Time is measured the observer by moving along g.

The observer O defines his position on g by a parameter t. The universe is considered stationary if, as the observer O moves along g, he can detect no change in the geometry of S. Thus it follows that the coefficients of the metric must be independent of t. This is equivalent to defining a timelike Killing vector. The universe is said to be axially symmetric if there exists a single privileged geodesic a such that in all directions in S normal to a space is indistinguishable (i.e. there is no preferred direction). Of necessity this geodesic a must pass through every point of g and be orthogonal to it. These assumptions are equivalent to the existence of a second Killing vector, which is spacelike. Both Killing vectors are assumed to be hypersurface orthogonal with closed orbits. When we say that the Killing vectors are hypersurface orthogonal we mean that the Killing vectors define hypersurfaces to which they are orthogonal.

To define a coordinate system for the spacetime we choose the unit vector a, and two unit vectors in Σ to form an orthogonal triad (see Fig. 2.3). This triad can be

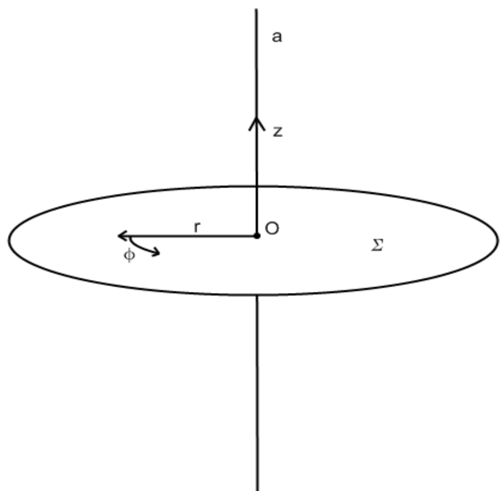

Figure 2.3 The observer O in S, observes a privileged spacelike geodesic a such that space in all directions orthogonal to a, forming a surface Σ, is indistinguishable. A coordinate system is set up in the space such that z is a measure along and parallel to a, and r and ϕ lie in Σ with r as a measure of the length from a and ϕ being an azimuthal coordinate.

used to set up a system of geodesic cylindrical co-ordinates with r being the length of a geodesic connecting any arbitrary point to O. We can define z as the measurement of length along and "parallel" to a, and ϕ can be defined as the azimuthal angle. By propagating the orthogonal triad parallelly along g we can define a co-ordinate system in every S with r, z, ϕ and t being the co-ordinates. From our assumption of axial symmetry by defining a spacelike Killing vector, we know that the coefficients of the metric must be independent of the azimuthal angle ϕ. Furthermore because of our choice of co-ordinates and how they are defined it can be seen that the t-lines must be normal to the r-lines. This can be shown by holding r, z, ϕ constant and observing the line defined by t through a point. This line is described by propagating the radial vector from g which by definition is normal to g. The curve described by propagating the t-line is everywhere orthogonal to the radial vector. Similarly from the definition of axisymmetry we can show that the t-lines are orthogonal to the z-lines. Consider a surface Σ in S formed by all points geodesically normal from a. As Σ is propagated in either t or z if there is any discernable difference in the intrinsic geometry of Σ then this would violate the basic assumption on a, that it is a privileged geodesic such that in all directions in S normal to a space is indistinguishable.

Thus from these assumptions we find that the off diagonal coefficients of the metric $g_{tr} = 0$ and $g_{tz} = 0$, and the metric takes on the general form:

$$ds^2 = H(dr^2 + dz^2) + Ld\phi^2 + 2Md\phi dt - Fdt^2 \tag{2.1}$$

Where H, L, M and F can be functions of r and z based only on the assumptions about the spacetime. Additionally, the coefficients are determined by assumptions about the presence of matter in the system and differential equations resulting from the Einstein equations.

2.3 Solutions to the Einstein Equations

Similar to the method used by Cooperstock and Tieu [13], we will be using the axially symmetric van Stockum [1] metric. This metric describes the shape of spacetime and defines a cylindrical coordinate system in t, r, z and ϕ for all calculations. Cooperstock takes the form of the metric to be:

$$ds^2 = -e^{\nu-w}(udz^2 + dr^2) - r^2e^{-w}d\phi^2 + e^w(cdt - Nd\phi)^2 \tag{2.2}$$

Here it should be noted that we do not set c to 1. Using this metric we can solve the Einstein equations to find the gravitational field of a cylindrical distribution of matter. To define the stress-engery tensor we will assume that the galaxy is a pressureless fluid, or dust. By assuming a pressureless fluid the stress-energy tensor becomes:

$$T^{ab} = \rho u^a u^b \tag{2.3}$$

u^a represents the four velocity, where ρ is the density of the fluid. It is advantageous to work in a frame that is rotating with the matter so in this case we set $u^a = \delta^a_t$. Using Maple and doing some simplifications by hand, we find that the Einstein equations can be simplified down to:

$$8\pi c^2 \rho = -\frac{N_r^2 + N_z^2}{r^2 e^\nu} \tag{2.4}$$

$$0 = N_{rr} + N_{zz} - \frac{N_r}{r} \tag{2.5}$$

$$0 = N_r^2 + N_z^2 + 2r^2(\nu_{rr} + \nu_{zz}) \tag{2.6}$$

$$0 = 2r\nu_r + N_r^2 - N_z^2 \tag{2.7}$$

$$0 = r\nu_z + N_r N_z \tag{2.8}$$

Subscripts denote differentiation with respect to the subscripted variable.

2.3.1 Solutions to the Rotation Curves

We would like to find an analytical solution to the rotation curves of a galaxy using the solutions to the Einstein Equations. To do this we now must find an expression for the angular velocity, and we do this by first introducing two Killing vectors, $X^a = \delta^a_\phi$ and $T^a = \delta^a_t$. The angular velocity can now be expressed as:

$$\begin{aligned} \omega &= \frac{\dot{\phi}}{\dot{t}} = \frac{u^\phi}{u^t} \\ &= \frac{u^a X_a}{u^c T_c} = \frac{g_{ab} u^a X^b}{g_{cd} u^c T^d} \\ &= \frac{g_{a\phi} u^a}{g_{ct} u^c} = \frac{g_{t\phi}}{g_{tt}} \\ &= \frac{-N e^w c}{e^w c^2} \\ \omega &= -\frac{N}{c} \end{aligned} \tag{2.9}$$

We see from (2.9) that the angular velocity of a galaxy is dependant on the metric coefficeint N. Equation (2.5) can be solved analytically to get a solution for N which we can use to find the angular velocity of the galaxy. Using separation of variables (2.5) becomes:

$$0 = \frac{R''}{R} + \frac{Z''}{Z} - \frac{1}{r}\frac{R'}{R} \tag{2.10}$$

where we have assumed $N = R(r)Z(z)$. We define $R(r) = r\psi$, and note that ψ is a function of r only and Z is a function of z only. Primes denote differentiation with respect to those variables. This equation can be separated out into:

$$\frac{\psi''}{\psi} + \frac{1}{r}\frac{\psi'}{\psi} - \frac{1}{r^2} = -k^2 \tag{2.11}$$

$$\frac{Z''}{Z} = k^2 \tag{2.12}$$

where k^2 is a real separation constant. These equations yield the solutions:

$$\psi(r) = AJ_1(kr) + BY_1(kr) \qquad Z(z) = C_1e^{kz} + C_2e^{-kz} \tag{2.13}$$

We can simplify this problem by applying boundary conditions that constrain the solution on the axis and also keep the solution finite in the z direction. We do this by considering a cylinder with radius a and height h. We start by requiring $\psi(r)$ to be finite on the axis. Because the Bessel function of the second kind (or the Neumann function), denoted by Y_1 blows up at $r = 0$ when we apply the boundary condition we set $B = 0$ leaving us with the Bessel function J_1. We further require that $\psi(a) = 0$. By making this assumption we are assuming that the galaxy has a finite radius and that there is no rotation at the edge of the galaxy. To satisfy this boundary condition we require $k_m = \frac{\alpha_m}{a}$ where α_m is the m^{th} zero of the J_1 function.

At this point we have our choice of boundary conditions for the z direction of the galaxy. Based on our choice of boundary conditions we will find different solutions for $Z(z)$. Cooperstock [13] chose to model the galaxy as an infinite cylinder in the z direction. This assumption gave him a solution involving a decaying exponetial in z direction. Even though this assumption may make the solution appear simpler it creates problems as will be discussed in section 2.4. Below is the equation for tangential velocity as given by Cooperstock.

$$V = -c \sum_{m=1}^{\infty} k_m B_m J_1(k_m r) e^{-k_m |z|} \tag{2.14}$$

An alternate way of finding a solution would be to assume a finite height, h, to the cylinder. If we assume that $Z(\pm h) = 0$ then we find that $C_1 = C_2 = 0$ causing $N = 0$ and thus removing the rotation from the system. If we assume that $Z(\pm h) \neq 0$ then the problem does not reduce to the trivial solution. By setting $Z(\pm h)$ equal to some constant, f, we find:

$$C_1 = C_2 = \frac{f}{2\cosh(kh)}$$

This gives the solution for $Z(z)$ as:

$$Z(z) = \frac{f}{2\cosh(kh)} \left(2\cosh(kz)\right) \tag{2.15}$$

Combining this with our solution for $\psi(r)$ we can write the solution for N.

$$N(r, z) = \frac{f}{2\cosh(kh)} \sum_{m=1}^{\infty} A_m J_1(k_m r) \left(2\cosh(kz)\right) \tag{2.16}$$

Using the orthogonality of J_1 we can find A_m to be:

$$A_m = \frac{1}{2\cosh(kh) a J_2(k_m a)^2} \int_0^a r J_1(k_m r) dr \tag{2.17}$$

Using (2.9) and the solution for N (2.16) we can find exact solutions for the tangential velocity using $V = \omega r$. This gives us the following as the solution for tangential velocity:

$$V = -\frac{f}{2\cosh(kh)} \sum_{m=1}^{\infty} A_m \frac{r}{c} J_1(k_m r) \left(2\cosh(kz)\right) \tag{2.18}$$

Equation (2.18) will allow us to find the tangential velocity of a galaxy at any given radius which then can be fitted to astronomical data. This hopefully would reproduce the flat rotation curved discussed in section 2.1, but there are some problems with this method that make the results non-physical as will be discussed in section 2.4.

2.4 Objections

There have been objections raised by some physicists [16–19] to Cooperstock's model. According to Vogt [19] and Korzynski [18] Cooperstock's model fails because according to his calculations, in the plane of the galaxy there exists a disk of infinite density. This disk results from the absolute value that appears in the exponential in equation (2.14). This, they argue, constitutes the dark matter that Cooperstock did not add into his calculations. Cooperstock argued that this was just a mathematical artifact and that had no physical interpretation. He countered that away from the galactic axis his model could accurately reproduce the rotation curves [20].

Other objections to the method proposed by Cooperstock include the claims made by Bratek *et al.* [21] that the van Stockum class of metrics requires that all solutions that are asymptotically flat be massless. This can be seen by returning to the general form of the metric (2.1). According to the derivation given by van Stockum [1] if we suppose M to be a function of r only, the coefficients of the metric become:

$$H = e^{-a^2r^2} \tag{2.19}$$

$$L = r^2(1 - a^2r^2) \tag{2.20}$$

$$M = a^3cr^4 \tag{2.21}$$

$$F = c^2(1 + a^2r^2 + a^4r^4) \tag{2.22}$$

In these equations a is a constant of integration and c is the speed of light. From (2.20) we see that it is necessary for $ar < 1$ or else the coefficient on $d\phi^2$ will be negative inside the cylinder. If the coefficient on $d\phi^2$ were negative then that would make the angular coordinate a timelike coordinate. So in order to prevent having two timelike coordinates we assume that the cylinder must have a maximum radius. We denote the maximum radius of the cylinder by R, where R is the value of r on the boundary of the cylinder. The equation for the density, which is dependant on M, can be written as:

$$\kappa\mu = 4a^2 e^{-a^2 r^2} \tag{2.23}$$

Where $\kappa = \frac{8\pi G}{c^4}$ and G is Newton's gravitational constant. Resulting from the derivation of the coefficients of the metric we find the angular velocity ω of the cylinder to be:

$$\omega = ac \tag{2.24}$$

If we define the density at the axis to be μ_0 then the density equation (2.23) becomes:

$$\frac{8\pi G}{c^4}\mu_0 = 4a^2 \tag{2.25}$$

Solving this equation for a and substituting into (2.24) we get an equation for ω that is dependent on the density on the axis μ_0.

$$\omega = \sqrt{2\pi G\mu_0} \tag{2.26}$$

Using (2.24) and the previously mentioned constraint ($ar < 1$) for the maximum radius R we can find a relation for the maximum radius of a spacetime of a given density subject to (2.26).

$$\omega R < c \tag{2.27}$$

It should be noted that R is not an invariant length, but the invariant length representing the radius can be found with the equation:

$$R' = \int_0^R e^{-\frac{1}{2}a^2 r^2} dr \tag{2.28}$$

where R' denotes the invariant radius of the spacetime. It is important to point out here that this relation defines a maximum radius for the spacetime if there is any matter present. In other words if there is any matter present (i.e. $\mu_0 \neq 0$) in the spacetime then the exterior solution does not go to an asymptotically flat solution at infinity. If there is no matter present (i.e. $\mu_0 = 0$) then the case is trivial and the metric reduces down to Minkowski space in cylindrical coordinates. But if matter *is* present then the relationship (2.27) holds and determines a maximum radius R. As a note, the upper limit of R is the same for a rotating cylinder in the special theory of relativity. To understand the problems of a spacetime that is not asymptotically flat it is useful to consider the motion of a particle, traveling either with the rotation of the spacetime or contrary to the rotation of the spacetime. The motion of this particle lies along a ϕ-line where with a certain angular velocity this ϕ-line describes a geodesic in the spacetime. We start out by writing the Lagrangian as:

$$\frac{d}{ds}\frac{\partial T}{\partial x'^i} - \frac{\partial T}{\partial x^i} = 0$$

where

$$T = H(r'^2 + z'^2) + L\phi'^2 + 2M\phi' t' - Ft'^2$$

and $i = 1, 2, 3, 4$ and primes denote differentiation with respect to some arc s. If we assume $r = constant$ and $z = constant$ the Lagrangian reduces to

$$\frac{d}{ds}\frac{\partial T}{\partial r'} - \frac{\partial T}{\partial r} = 0$$

This produces a quadratic formula for $d\phi/dt$ which is the definition of angular velocity ω.

$$L_{,r} d\phi^2 + 2M_{,r} d\phi dt - F_{,r} dt^2 = 0 \tag{2.29}$$

where subscripts indicate differentiation. Using solutions for L, M and F ((2.20), (2.21) and (2.22) respectively) we can solve the quadratic and we get two roots, namely:

$$\omega_1 = ac \tag{2.30}$$

$$\omega_2 = -\frac{1+2a^2r^2}{1-2a^2r^2}ac \tag{2.31}$$

The first root produces the angular velocity of the rotating cylinder. In other words, a particle following a geodesic traveling with the rotation of the spacetime will travel with the same angular velocity of the spacetime. The second root describes the motion or angular velocity of a particle that is traveling contrary to the rotation of the spacetime, or the angular velocity that it must have for it to travel contrary to the rotation and follow a geodesic.

The tangent vectors of the worldlines of these particles must be timelike and in order to check this we consider surfaces of constant r and z. In other words we are considering the null directions in ϕ and t. Accordingly the metric reduces to:

$$Ld\phi^2 + 2Md\phi dt - Fdt^2 = 0 \tag{2.32}$$

Again we have a quadratic for $d\phi/dt$ which we can solve to produce:

$$\Omega_1 = \frac{1-a^3r^2}{1-a^2r^2}\frac{ac}{ar} \tag{2.33}$$

$$\Omega_2 = -\frac{1+a^3r^2}{1-a^2r^2}\frac{ac}{ar} \tag{2.34}$$

Here we can interpret Ω_1 as the angular velocity of a photon along a surface of constant r and z in the direction of the rotation of the spacetime. Likewise Ω_2 represents the angular velocity of a photon traveling contrary to the rotation of the spacetime. If we take the limit of (2.33) as $ar \to \infty$ we find that it approaches ac from above. This means that $\omega_1 < \Omega_1$ for all values of r. Because the tangent vector of the particle in the null direction is always greater than the angular velocity of the spacetime the worldline of a particle moving *with* the rotation is always timelike. If we consider the motion of a particle moving contrary to the rotation of the cylinder as described by (2.31), with (2.34) we can find that if $ar = \frac{1}{2}$ then $\Omega_2 = \omega_2$. This means that as $ar \to \frac{1}{2}$ then the velocity of the particle as it moves contrary to the motion of the cylinder approaches the speed of light. Beyond this value the particle cannot follow a ϕ-line in this sense, it will be dragged along with the cylinder.

Again returning to (2.31) we see that as $ar \to \frac{1}{\sqrt{2}}$ then the angular velocity, ω_2, goes to infinity. The ϕ-lines represent the circumference of the spacetime at a given radius. If we calculate the invariant length of a given ϕ-line using:

$$l = \int_0^{2\pi} r(1 - a^2r^2)^{\frac{1}{2}} d\phi = 2\pi r(1 - a^2r^2)^{\frac{1}{2}} \tag{2.35}$$

we see from this equation that the length achieves a maximum when $ar = \frac{1}{\sqrt{2}}$. Above this value the length, meaning the length of the ϕ-lines, decreases until $ar \to 1$ then $l \to 0$. In this case if we consider a surface of constant z then all geodesics normal to the axis will converge again at the boundary. In other words, at the boundary the geodesics reduce to a line, an antipole to the axis of symmetry. From this we see that for the van Stockum metric, if there is any matter present in the spacetime then it cannot be asymptotically flat.

2.5 Conclusion

While this approach offers a different explanation of dark matter it has some difficulties in presenting a physical system. The objections raised must be considered, especially that if there is any matter present then the spacetime does not admit an asymptotically flat solution. The idea that general relativity, or even using a proper configuration of matter, can compensate for at least some of the additional mass needed for calculations to agree with observation warrants further investigation. In 1985 van Albada, Bahcall and Begeman [15] offered a possible Newtonian model for galaxies that did not have to include a halo of dark matter, but this author has not been able to investigate the validity of this claim. It is for these reasons that additional investigation of this idea might prove beneficial. It is possible that a correction to the estimate of how much dark matter there is could be made but it seems unlikely that a method could be found which does away with dark matter all together.

Chapter 3

A Rotating Star in the Classical Regime

3.1 Introduction and Background

In 1968 Antony Hewish and Jocelyn Bell Burnell [22] announced their discovery of astronomical objects that emit pulsed radio signals, which have since been named pulsars. That same year Thomas Gold proposed that the source of these periodic radio signals were rotating neutron stars [23] [24]. He came to this conclusion by noting that the period of the radio signal coincided with the expected period of rotation for a neutron star. A year later Ostriker and Gunn [25] followed up the research of Gold, by exploring the implications of his proposal. They found that due to the rotation of the neutron star with a magnetic field, the star will emit large amounts of magnetic-dipole radiation and also they assumed through gravitational radiation, which can accelerate charged particles to relativistic energies. Because of the magnetic-dipole radiation, the accelerated particles will concentrate into a beam along the axis of the dipole, and that due to the orientation of the beam relative to the earth, coupled with the precession of the magnetic axis the radio signal would appear to pulse (see Fig. 3.1). It was the success of these initial calculations that prompted further investigation into modeling rotating neutron stars.

In the classical case, rotating stars are generally treated as Maclaurin spheroids. Maclaurin spheroids are self gravitating fluids described by solving the Navier-Stokes equations for the case of an incompressible, homogenous and uniformly rotating fluid. The resulting shapes are aptly called spheroids because while the basic topology may be spherical, rotation causes the body to deform and become oblate. The ellipticity of a spheroid is calculated by the following equation:

$$\epsilon = \frac{R_e - R_p}{R_e}$$

In the above equation R_e is the equatorial radius, R_p is the polar radius and ϵ is the ellipticity or eccentricity of the spheroid, $0 \leq \epsilon \leq 1$. For slowly rotating bodies the oblateness may be small as in the case of the earth, which has an ellipticity of .0033529.

In the case of a rapidly rotating star the rotation may cause severe deformation of the star. The spin will cause it to bulge out at the equator. This bulging would correspond to an increased ellipticity. When the ellipticity is large (i.e. close to 1) we begin to see effects that are of particular interest. It is in the case of a rapidly rotating star, as reported by Shapiro, Teukolsky and Nakamura [26], that as the star loses angular momentum J, the star's angular velocity Ω will actually *increase*. Normally, when a rotating body loses angular momentum its angular velocity will decrease and as the star loses energy, the star will begin to spin down. But in the case of large ellipticity the exact opposite can occur. This is due to fact that the moment of inertia I of the star depends indirectly on its rotation. A highly oblate

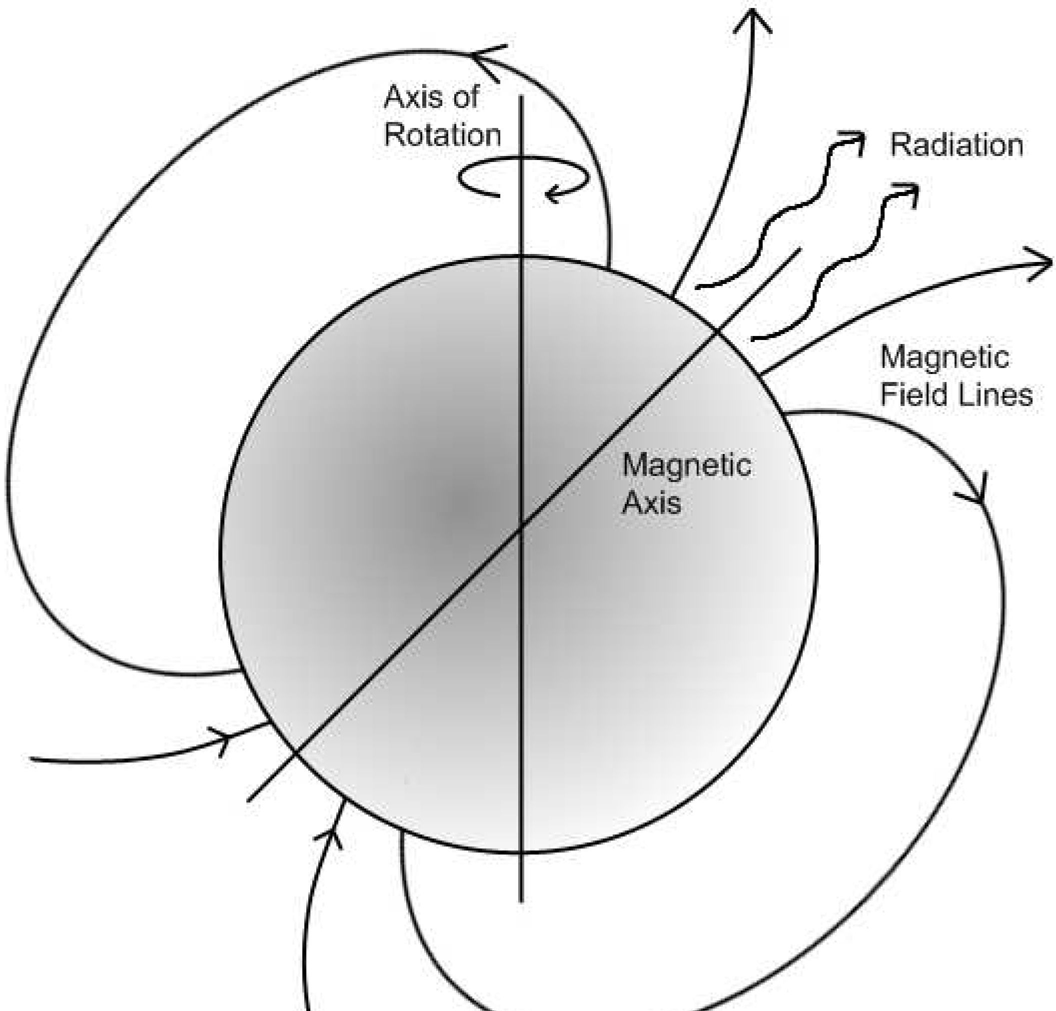

Figure 3.1 For astronomical objects, the axis of rotation and the magnetic axis do not align. In the case of a strongly magnetized neutron star a beam of radiation is created along the magnetic axis. Due to the rotation of the star this beam will sweep out an arc in space. For an observer on the Earth the signal would appear to pulse on and off as the rotation brings it both into and out of alignment with the Earth.

star will have a different moment of inertia than a more spherical star of the same mass. As a rotating star loses energy, and therefore angular momentum, its ellipticity will decrease. This redistribution of matter will change the moment of inertia of the star. For a slowly rotating star a decrease in the angular momentum will not significantly change the shape and moment of inertia of the star. But for a rapidly rotating star, due to its high oblateness, a decrease in J can significantly decrease I. In this case Ω may increase, meaning that as the star loses angular momentum it will spin-*up*.

3.2 Spin-Up of a Rapidly Rotating Neutron Star

Ostriker and Gunn [25] proved that for a uniformly rotating star that slowly loses energy the energy loss E is related to the loss of angular momentum in the following way:

$$\frac{dJ}{dt} = \frac{1}{\Omega}\frac{dE}{dt} \tag{3.1}$$

This is based on the assumption that the neutron star is rotating almost rigidly, or that the physical configuration of the star changes very little in the period of one rotation. The actual mechanism of energy loss could be through electromagnetic or gravitational radiation. Shapiro, Teukolsky and Nakamura [26] used the above relation to show the spin-up of neutron stars with a specific adiabatic index. To show this, it is advantageous to rewrite (3.1) in a form that can indicate whether the spin of the star is increasing or decreasing independent of the actual mechanism of energy loss. Considering J to be a function of Ω, using the chain rule we can rewrite (3.1) as:

$$\frac{dE}{dt} = \Omega\frac{dJ}{d\Omega}\frac{d\Omega}{dt}$$

If we assume conservation of baryon mass M and entropy S, we can use the following relationship to rewrite the above equation.

$$d\Omega = \left(\frac{\partial \Omega}{\partial J}\right)_{M,S} dJ$$

The subscripted M and S denote holding the mass and entropy constant. Using this we can rewrite (3.1) in the following form:

$$\frac{dE}{dt} = \frac{1}{(\partial\Omega/\partial J)_{M,S}} \Omega \frac{d\Omega}{dt} \tag{3.2}$$

Where it is evident that the sign of $d\Omega/dt$ depends on the sign of $(\partial\Omega/\partial J)_{M,S}$. dE/dt is negative because we are considering energy loss. Writing (3.1) in this form is also useful in that the derivative in the denominator does not depend on the actual mechanism of energy loss but only on an equilibrium sequence of evolution.

Finn and Shapiro [27] were able to show that for polytropic stars with indices approaching $n = 3$ (that is, an adiabatic index of 4/3 where the index $\gamma = 1 + 1/n$) the derivative, $(\partial\Omega/\partial J)_{M,S}$, became negative, where the negative sign corresponds to the spin-up of neutron stars as they lose energy.

In [26] the authors mention that it is convenient to convert the equations of hydrostatic equilibrium in a dimensionless form in order to solve them numerically. They cite James [28] where he defines the dimensionless quantities:

$$\begin{aligned} v &= \frac{\Omega^2}{2\pi G \rho_c} \\ \tilde{M}(v) &= \frac{M}{4\pi\alpha^3\rho_c} \\ \tilde{I}(v) &= \frac{I}{\alpha^5\rho_c} \end{aligned} \tag{3.3}$$

where G is Newton's gravitational constant, ρ_c is the density at the center of the spheroid and α is a unit of length defined as:

$$\alpha = \left[\frac{(n+1)K}{4\pi G}\rho_c^{-1+1/n}\right]^{1/2} \tag{3.4}$$

and

$$K = \frac{P}{\rho^{1+1/n}} \tag{3.5}$$

where K is the polytropic constant and P is the pressure.

The equations (3.3) can be rewritten in an alternative set of dimensionless parameters in order to study the evolutionary sequences of neutron stars along curves of constant mass and entropy. Following [26] we introduce the parameters:

$$\Omega^* = \frac{\Omega}{[2\pi G\rho_c(0)]^{1/2}} = \left[\frac{\tilde{M}(0)}{\tilde{M}(v)}\right]^{n/(3-n)} v^{1/2} \tag{3.6}$$

$$J^* = \frac{J}{M\alpha^2(0)[2\pi G\rho_c(0)]^{1/2}} = \frac{\tilde{I}(v)v^{1/2}}{4\pi\tilde{M}(v)}\left[\frac{\tilde{M}(0)}{\tilde{M}(v)}\right]^{(2-n)/(3-n)} \tag{3.7}$$

Where the (0) denotes the spherical, nonrotating body on the sequence with equivalent values of M, K and n. With this notation the evolution of the stars can be plotted in the $\Omega^* - J^*$ plane as they slowly lose J. In this plane the stars will follow a unique equilibrium curve parameterized by v and the dimensionless quantities (3.6) (3.7). Returning to (3.2) we can find the sign on the differential $(\partial\Omega/\partial J)_{M,S}$ from (3.6) (3.7) in the following way:

$$\frac{J}{\Omega}\left(\frac{\partial\Omega}{\partial J}\right)_{M,S} = \frac{d\ln\Omega^*}{d\ln J^*}$$
$$= \left(-\frac{n}{3-n}\frac{d\ln\tilde{M}}{dv} + \frac{1}{2v}\right)\Big/\left(\frac{d\ln\tilde{I}}{dv} + \frac{2n-5}{3-n}\frac{d\ln\tilde{M}}{dv} + \frac{1}{2v}\right) \tag{3.8}$$

If we take the limit as $n \to 3$ we get:

$$\lim_{n\to 3}\frac{d\ln\Omega^*}{d\ln J^*} = -\frac{n}{2n-5} = -3 \tag{3.9}$$

Thus for values of n approaching 3, corresponding to an adiabatic index of near 4/3, the sign of the differential becomes negative making the change in the angular velocity positive with a negative change in the energy or angular momentum. But this relationship only holds for values of n sufficiently close to 3, otherwise the star will become unstable before the sign of the differential changes.

3.3 Conclusions

An important aspect of this analysis is to understand the mechanism that might lead to a rapidly rotating pulsar. Because of the method used we demand that the star in question be homogeneous, incompressible and uniformly rotating. A good candidate to satisfy all three of these conditions is a neutron star, thus we can assume that a the most likely candidate for a pulsar is a neutron star. Once we find a relationship between angular momentum and energy (3.1), it is advantageous to rewrite it in a manner (3.2) that determines whether or not the star will spin up or spin down independent of the mechanism of energy loss. We can now introduce several dimensionless quantities which can make computation easier and more intuitive. These quantities will depend on the polytropic index n which determines the adiabatic constant γ. Thus for a constant mass and a constant entropy we can find the change in the spin of the star for a specific value of n.

We see from (3.9) that if the polytropic index is sufficiently close to 3 and for a rapidly rotating system, the star can spin *up* while losing energy. For smaller values of n the star will become unstable through excessive rotation, before it reaches the point where it can spin up while losing energy.

Chapter 4

A Rotating Star in General Relativity

4.1 Introduction and Background

About the same time that astrophysicists were considering rotating neutron stars in the classical regime they also began to consider them in general relativity. Because mechanics is more complex in general relativity than in Newtonian physics it took a few more years to build a foundation from which to work. In their paper published in 1971 Bardeen and Wagoner [29] considered methods for modeling rotating astronomical objects in general relativity. Their analysis assumed a rapidly rotating, pressureless fluid that had been flattened out into a disk. Later Wilson [30] introduced numerical methods for solving the Einstein equations with the inclusion of pressure, but his methods were criticized [31] because in order to make his system asymptotically flat he had to make some Newtonian approximations. Later Bonazzola and Schneider [32] developed numerical techniques that allowed analysis of rapidly rotating bodies but were able to do it with a nonzero pressure and not necessarily confined to the flattened disks that were studied by Bardeen and Wagoner. An important result of the work of Bonazzola and Schneider was that they introduced a method of solving the equations of motion by placing them in integral form. This will prove to be an especially important result for our consideration because it allows the equations to be solved using Green's functions.

In 1975 Butterworth and Ipser [31] and then in 1976 Butterworth [33] published papers explaining numerical methods of modeling rapidly rotating stars treated as fluids. They expressed the equations of motion in differential form as opposed to the integral form of Bonazzola and Schneider, and they were also successful in defining proper boundary conditions, which had been problematic for Wilson. Butterworth and Ipser used methods developed for classical mechanics by Stoeckly [34] and James [28]. Later in 1986 Friedman, Ipser and Parker [35] were able to construct models of rapidly rotating neutron stars which were based on equations of state. Komatsu, Eriguchi and Hachisu [36] revisited the method of Bonazzola and Schneider using a similar integral notation for their equations of motion to obtain polytropic models of rapidly, uniformly rotating neutron stars. Three years later Cook, Shapiro and Teukolsky (CST) [2] published their work on differentially rotating neutron stars that extended the work done by Komatsu *et al.*, and which correlated to the work done by Shapiro, Teukolsky and Nakamura (STN) [26], which was explained in chapter 3. In this chapter we will consider the method developed by CST in modeling a differentially rotating neutron star. Essentially we are considering the general relativistic analog to the work of STN.

4.2 General Relativistic Hydrodynamics

For the following consideration we assume stationarity and axisymmetry. We achieve stationarity by defining a timelike Killing vector field. We likewise get axisymmetry by defining a spacelike Killing vector field that vanishes on the axis of rotation, and has closed orbits. We also assume that the Killing vectors are hypersurface orthogonal. Additionally we make the assumption of circularity, which restricts the motion of the fluid to latitudinal motion (i.e. no meridional motion).

Simply put, circularity means that the matter in the star travels parallel to the equator in a circle. This means that there is no motion in the radial or azimuthal directions (i.e. the r and θ directions). A representation of the flow of matter with the assumption of circularity is shown in figure 4.1. By assuming circularity we can simplify the equations so that they become relatively easier to solve. To see how this simplifies our equations we note that the assumption of circularity restricts the motion of matter to the t and ϕ directions. This is accomplished by the previously mentioned introduction of two Killing vectors which are hypersurface orthogonal. Thus the motion of the matter is restricted to the t and ϕ directions and the components of the metric are restricted to the r and θ directions. This orthogonality results in cancelation of terms that will be noted in the following derivations.

Similar to the form of the metric given by Komatsu *et al.* [36] we have:

$$ds^2 = -e^{\gamma+\rho}dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{\gamma-\rho}r^2 \sin^2\theta(d\phi - \omega dt)^2 \tag{4.1}$$

In the metric ρ, γ, α and ω are functions of r and θ only. We also set $G = 1$ and $c = 1$. We define the stress energy tensor as:

$$T^{ab} = [\rho_0 + \rho_i + P]u^a u^b + g^{ab}P$$

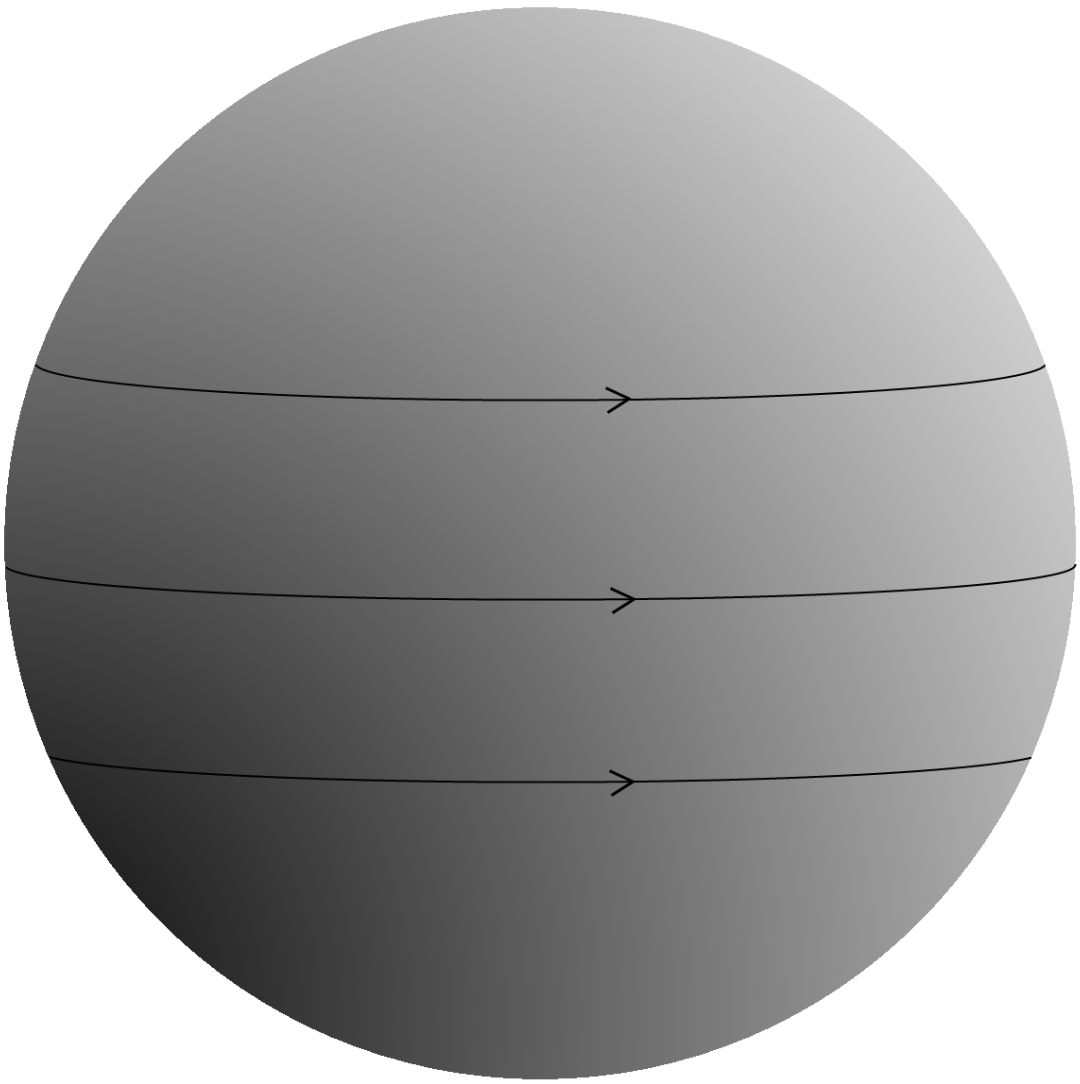

Figure 4.1 The basic assumption of circularity is that the matter inside the star moves in a circular direction parallel to the equator. There is no motion in the r and θ directions. The only parameters that determine motion are t and ϕ.

In the stress energy tensor ρ_0 is the rest energy density, ρ_i is the internal energy density, P is the fluid pressure and u^a is the matter four velocity. We can define the proper velocity of the matter with respect to a zero angular momentum observer (cf. [37]) by:

$$v = (\Omega - \omega) r \sin\theta e^{-\rho} \tag{4.2}$$

Here we define Ω as the angular velocity as measured from infinity. That is $\Omega = d\phi/dt = (d\phi/d\tau)/(dt/d\tau) = u^\phi/u^t$. From this we can find the fluid four velocity which is:

$$u^a = \frac{e^{-(\rho+\gamma)/2}}{(1-v^2)^{1/2}}[1, 0, 0, \Omega] \tag{4.3}$$

4.3 The Equation of Hydrostatic Equilibrium

In this section we will give the derivation of the equation of hydrostatic equilibrium. The equation of hydrostatic equilibrium is important because it describes the equilibrium configurations of the matter, based on the density, rotation and pressure of the fluid. The equations for the rotation and density will be found using the Einstein equations which we can then use with the equation of hydrostatic equilibrium to find the pressure. For our present case we will assume for the system the condition of circularity. Thus we will need to find a differential equation that involves the density ρ_0, the internal energy ρ_i, the pressure P, the four velocity u^μ and the angular velocity Ω as all of these either are part of or are related to the equation of state. Using the vanishing of the divergence of the stress-energy tensor, the derivation of the equation of hydrostatic equilibrium is as follows:

$$\begin{aligned} 0 &= \nabla_a T^{ab} \\ 0 &= \nabla_a \big[[\rho_0 + \rho_i + P]u^a u^b + g^{ab} P\big] \end{aligned} \tag{4.4}$$

In equation (4.4) above, the covariant derivative can be brought through the density, internal energy and pressure terms because of our assumption of circularity and the introduction of a spacelike and a timelike Killing vectors. Circularity makes the four velocity (4.3) have components only in the t and ϕ directions and the introduction of Killing vectors makes ρ_0, ρ_i and P to be functions of r and θ only. Thus $u^a\nabla_a[\rho_0 + \rho_i + P] = 0$, thereby eliminating one term resulting from the product rule. This gives us:

$$0 = [\rho_0 + \rho_i + P]\left[\nabla_a u^a \cdot u_b + u^a \nabla_a u_b\right] + \nabla_b P \tag{4.5}$$

A careful examination of equations (4.4) and (4.5) will reveal that the index on the last term in (4.5) is lowered while the corresponding term in (4.4) is raised. This is because the pressure, P, is a scalar and is invariant under transformations. If we consider the first term in (4.5), it gives us:

$$\nabla_a u^a \cdot u_b = \frac{1}{\sqrt{-g}} \cdot \partial_a(\sqrt{-g}u^a) \cdot u_b = 0$$

As mentioned previously this depends on our assumption of circularity which involves the introduction of the two Killing vectors. The metric coefficients are functions of r and θ only and those are the only derivatives that survive, but when contracted with the four velocity it is identically zero. We will simplify the second term in (4.5) below.

$$\begin{aligned} u^a\nabla_a u_b &= u^a\partial_b u_b - u^a\Gamma^c_{ab}u_c \\ &= -u^a u_c \cdot \frac{1}{2}g^{cd}[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}] \\ &= -u^a u^d \frac{1}{2}[\partial_b g_{ad}] \end{aligned} \tag{4.6}$$

We have again used circularity and the Killing vectors to cancel out the first and last terms in the above equation. Note that the minus sign in (4.6) will not be carried through after (4.6), but it is reintroduced in (4.9). This is simply a convenience for deriving the equation. We use the inverse product rule to expand the result into:

$$\begin{aligned} u^a u^d \frac{1}{2}[\partial_b g_{ad}] &= -\frac{1}{2}\partial_b\big(g_{ad}u^a u^d\big) + \frac{1}{2}\partial_b u^a \cdot g_{ad}u^d + \frac{1}{2}\partial_b u^d \cdot g_{ad}u^a \\ &= \frac{1}{2}\big(\partial_b u^a \cdot u_a + \partial_b u^a \cdot u_a\big) = u_a \partial_b u^a \end{aligned}$$

Here we separate the above result into the t and ϕ components and use the fact that $u^\phi = \Omega u^t$ to rewrite the result.

$$\begin{aligned} u_a\partial_b u^a &= u_t\partial_b u^t + u_\phi \partial_b u^\phi \\ &= u_t\partial_b u^t + u_\phi\partial_b(\Omega u^t) \\ &= u_t\partial_b u^t + u_\phi \cdot \partial_b\Omega \cdot u^t + u_\phi \cdot \Omega\partial_b u^t \\ &= (u_t + u_\phi\Omega)\partial_b u^t + u^t u_\phi \cdot \partial_b\Omega \end{aligned} \tag{4.7}$$

We need to now consider the coefficient, $(u_t + u_\phi\Omega)$, on the first term in order to simplify (4.7). Recalling the definition of $\Omega = \frac{u^\phi}{u^t}$

$$\begin{aligned} &= (u_t + u_\phi\Omega) \\ &= (u_t + u_\phi\frac{u^\phi}{u^t}) \\ &= \frac{1}{u^t}(u^t u_t + u_\phi u^\phi) \\ &= -\frac{1}{u^t} \end{aligned} \tag{4.8}$$

The simplification to the last line above is done because $u_a u^a \equiv -1$ and $u_a u^a = u_t u^t + u_r u^r + u_\theta u^\theta + u_\phi u^\phi$. But $u_r u^r = u_\theta u^\theta = 0$ and thus, $u_a u^a = u_t u^t + u_\phi u^\phi$. Putting the result (4.8) back into (4.7) we can continue with the derivation for the equation of hydrostatic equilibrium.

$$\begin{aligned} u_a \partial_b u^a &= -\frac{1}{u^t}\partial_b u^t + u^t u_\phi \cdot \partial_b \Omega \\ &= -\partial_b(\ln u^t) + u^t u_\phi \cdot \partial_b \Omega \\ &= -\big(-\partial_b(\ln u^t) + u^t u_\phi \cdot \partial_b \Omega\big) \qquad (4.9) \\ u^a \nabla_a u_b = u_a \partial_b u^a &= \partial_b(\ln u^t) - u^t u_\phi \cdot \partial_b \Omega \qquad (4.10) \end{aligned}$$

The last line is the result of simplifying the second term in (4.5). All together the derivation and the result becomes:

$$\begin{aligned} 0 &= \nabla_a T^{ab} \\ 0 &= [\rho_0 + \rho_i + P]\big[u^a \nabla_a u_b\big] + \nabla_b P \\ 0 &= -[\rho_0 + \rho_i + P]\big(\partial_b(\ln u^t) - u^t u_\phi \cdot \partial_b \Omega\big) + \nabla_b P \end{aligned}$$

In the last line the $\nabla_b P$ can be written as $\partial_b P$ because P is a scalar. To get this into the form given in CST we note that it can be rewritten in the following form.

$$\partial_b P = [\rho_0 + \rho_i + P]\big(\partial_b(\ln u^t) - u^t u_\phi \cdot \partial_b \Omega\big)$$

From this we write two separate equations replacing the index b with r and θ repectively.

$$\begin{aligned} \partial_r P &= [\rho_0 + \rho_i + P]\big(\partial_r(\ln u^t) - u^t u_\phi \cdot \partial_r \Omega\big) \\ \partial_\theta P &= [\rho_0 + \rho_i + P]\big(\partial_\theta(\ln u^t) - u^t u_\phi \cdot \partial_\theta \Omega\big) \end{aligned}$$

We can insert these into the definition of the absolute derivative of P to get:

$$\begin{aligned} dP &= \partial_r P dr + \partial_\theta P d\theta \\ dP &= [\rho_0 + \rho_i + P]\Big[\big(\partial_r(\ln u^t) - u^t u_\phi \cdot \partial_r \Omega\big)dr + \big(\partial_\theta(\ln u^t) - u^t u_\phi \cdot \partial_\theta \Omega\big)d\theta\Big] \end{aligned}$$

This last form of the equation can then be rewritten in the following form again using the definition of the absolute derivative and bringing dP back to the right hand side.

$$0 = dP - (\rho_0 + \rho_i + P)[d \ln u^t - u^t u_\phi d\Omega] \tag{4.11}$$

This corresponds to equation (12) in [2] with the exception of a minus sign on the last term. In the final form of the equation it can be integrated to find the pressure of the fluid based on the rotation of the the fluid.

4.4 Green's Functions

As mentioned in the introduction to this chapter, an important result of the method established by Bonazzola and Schneider [32] is putting the Einstein equations into integral form, allowing them to be solved using Green's function technique.[1] The Green's function technique can be used in solving cartain boundary value problems. In our case we are considering a two dimensional Green's function problem. We will be seeking a solution to the differential equation given below, which involves the general form of the differential operator that appears in all our elliptic equations.

$$\left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right]\xi = S(r,\mu) \tag{4.12}$$

In the above equation $S(r,\mu)$ represents the source terms. The actual source terms are not important for this discussion but will be discussed in section 4.5. We are also using the transformation $\mu = \cos\theta$ as used in CST. Here ∇^2 is the flat space Lapacian in spherical coordinates. When we apply the trasformation $\mu = \cos\theta$ the Lapacian is written as:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{2\mu}{r}\frac{\partial}{\partial \mu}$$

[1] This section was originally worked out by my colleague Steve Taylor who explained it to me.

For simplicity we will use L to signify the differential operator we are using in our problem. It is convenient to use the Green's function technique because the Green's function only depends on the differential opperator used and the boundary conditions. The Green's function is independent of the source terms. A Green's function is defined as the function such that when the operator L is applied to it, it results in delta function source terms.

$$LG(x,x') = \delta(x-x')$$

We want to find a general solution for ξ. To do this we start with the divergence theorem from which we can derive Green's identity.

$$\int_V \nabla \cdot \vec{A} dV = \oint_S \vec{A} \cdot \hat{n} da$$

We will define $\vec{A} \equiv (p \cdot u \nabla v - p \cdot v \nabla u)$ where u, v and p are arbitrary funtions which will be defined later. Putting this value in for $\vec{A}$ we have:

$$\int_V \nabla \cdot (p \cdot u \nabla v - p \cdot v \nabla u) dV = \oint_S (p \cdot u \nabla v - p \cdot v \nabla u) \cdot \hat{n} da$$

This form of Stokes' theorem is Green's second identity. From this form we expand the left hand side and simplify to write the above equation in the form:

$$\int_V \big[u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u) \big] dV = \oint_S (p \cdot u \nabla v - p \cdot v \nabla u) \cdot \hat{n} da$$

We will make the assumption that our differential operator L is of the form $L = \nabla \cdot (p \nabla)$.With this assumption the above equation can be written as follows.

$$\int_V [uLv - vLu] dV = \oint_S (p \cdot u \nabla v - p \cdot v \nabla u) \cdot \hat{n} da$$

Because we have not yet defined u and v we can set $v = \xi$ and $u = G$. Using the fact that $L\xi = S$ and $LG = \delta(x-x')$ we rewrite the equation as:

$$\int_V \big[GS - \xi \delta(x-x') \big] dV = \oint_S (p \cdot G \nabla \xi - p \cdot \xi \nabla G) \cdot \hat{n} da$$

We can rearrange this into a convenient form to get:

$$\xi(x') = \int_V GSdV - \oint_S (p \cdot G\nabla\xi - p \cdot \xi\nabla G) \cdot \hat{n}da$$

In this form we see we have a solution for ξ based on the Green's function, the source term and boundary conditions. We will assume Dirichlet boundary conditions where ξ is a known function at the boundary. Because we want our soution to be asymptotically flat we set $\xi = 0$ and $G = 0$ on the boundary. This reduces our soution for ξ down to:

$$\xi(r', \mu') = \int G(r, r', \mu, \mu')S(r, \mu)drd\mu$$

In this general form of the solution we have inserted the variables r and μ which we are using for our current problem.

If we are to solve for a function, ξ, using the Green's function technique then ξ must satisfy certain boundary conditions which we describe below. It should be noted that $-1 \leq \mu \leq 1$, and a subscript denotes a derivative with respect to that variable. The first two boundary conditions, given below, indicate that the function and its first derivative in μ should be equivalent at $\mu = 1$ and $\mu = -1$. This means that there can be no discontinuities in the function or in the first derivative on the boundaries for which the Green's function will be used.

$$\xi(r, 1) = \xi(r, -1) \qquad \xi_\mu(r, 1) = \xi_\mu(r, -1)$$

The next two boundary conditions respectively constrain the function with respect to r at infinity and on the axis. That is, this prevents the function from blowing up at infinity and on the axis when the operator is applied to it.

$$|\xi(0, \mu)| < \infty \qquad \lim_{r \to \infty} \xi(r, \mu) = 0$$

Earlier we made the claim that our operator is of the form $L = \nabla\cdot(p\nabla)$. As expressed in (4.12) the operator is not of this form. But it can be put in to this form if we multiply it by the arbitrary function p. That is, if we define the operator as $\tilde{L} = pL$, where $\tilde{L}$ is the operator that will be used in the Green's function technique and L is the operator as it appears in (4.12). This places a constraint on p which we must now solve for. We do this by having the operators act on an arbitrary function f and set them equal to each other.

$$\tilde{L}f = pLf \tag{4.13}$$

In the above equation we insert $\tilde{L} = \nabla\cdot(p\nabla)$ and $L = \left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right]$ and now we can solve for p.

$$\begin{aligned}
\nabla\cdot(p\nabla)f &= p\left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right]f \\
p\cdot\nabla^2 f + \nabla p\cdot\nabla f &= p\nabla^2 f + p\frac{n}{r}\partial_r f - p\frac{n\mu}{r^2}\partial_\mu f \\
\nabla p\cdot\nabla f &= p\frac{n}{r}\partial_r f - p\frac{n\mu}{r^2}\partial_\mu f \\
(\partial_r p)(\partial_r f) + \frac{1-\mu^2}{r^2}(\partial_\mu p)(\partial_\mu f) &= p\frac{n}{r}\partial_r f - p\frac{n\mu}{r^2}\partial_\mu f
\end{aligned}$$

In the last line we simply expanded the lefthand side taking in mind the transformation $\mu = \cos\theta$. We can equate the coefficient terms in front of the $\partial_r f$ and the $\partial_\mu f$, and equate them in the following way.

$$\partial_r p = p\frac{n}{r} \tag{4.14}$$

$$\frac{1-\mu^2}{r^2}\partial_\mu p = -p\frac{n\mu}{r^2} \tag{4.15}$$

The first equation, (4.14) can easily be solved to get $p = c(\mu)r^n$, where $c(\mu)$ is an unknown function of μ. We put this solution into the second equation, (4.15), and solve to get the constraint on p.

$$p = (1-\mu^2)^{n/2}r^n \tag{4.16}$$

Using this result we can write the operator $\tilde{L}$ which we can use to solve the Green's function problem. Thus the full operator becomes:

$$\tilde{L} = (1-\mu^2)^{n/2} r^n \left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right] \tag{4.17}$$

Returning to the definition of a Green's funtion $LG(x,x') = \delta(x-x')$ and using (4.17) we can write the Green's function problem as:

$$(1-\mu^2)^{n/2} r^n \left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right] G^{(n)}(r,\mu) = \frac{1}{r^2}\delta(r-r')\delta(\mu-\mu') \tag{4.18}$$

Because the Green's function is dependent on the value of n found in the operator we will denote this dependence with a superscript (n) on all associated functions. Now that we have the form of the operator, the next step is to solve for the Green's function $G^{(n)}$ which will be used to find the functions from which we can solve for the coefficients of the metric. This topic will be covered in Section 4.5. We can solve the above equation using separation of variables, but to make the problem simpler we will rewrite the equation in the following form:

$$\frac{\partial}{\partial r}\left(r^{n+2}\frac{\partial G^{(n)}}{\partial r}\right) + \frac{r^n}{(1-\mu^2)^{n/2}}\frac{\partial}{\partial \mu}\left[(1-\mu^2)^{1+n/2}\frac{\partial G^{(n)}}{\partial \mu}\right] = \frac{1}{(1-\mu^2)^{n/2}}\delta(r-r')\delta(\mu-\mu') \tag{4.19}$$

We let $G^{(n)} = R^{(n)}(r)P^{(n)}(\mu)$ and we can separate (4.19) in the following way.

$$R^{(n)\prime\prime} + \frac{n+2}{r}R^{(n)\prime} - \frac{\lambda^2}{r^2}R^{(n)} = 0 \tag{4.20}$$

$$(1-\mu^2)P^{(n)\prime\prime} - 2\mu(1+n/2)P^{(n)\prime} + \lambda^2 P^{(n)} = 0 \tag{4.21}$$

Here we introduce λ^2 as the separation constant. The first equation has solutions of the form:

$$R^{(n)}(r) = r^{\left(-n-1\pm\sqrt{(n+1)^2+4\lambda^2}\right)/2} \tag{4.22}$$

We see that (4.21) is a Legendre type equation and can be solved using the Frobenius method. This method entails assuming a solution of the form:

$$P^{(n)}(\mu) = \sum_{l=0}^{\infty} a_l \mu^l \tag{4.23}$$

Putting this into (4.21) we get:

$$0 = \sum_{l=-2}^{\infty}(l+2)(l+1)a_{l+2}\mu^l + \sum_{l=0}^{\infty}[\lambda^2 - l(l-1) - l(n+2)]a_l\mu^l$$

From this we can find the recursion relation:

$$a_{l+2} = \frac{\lambda^2 - l(l+n+1)}{(l+2)(l+1)}a_l \tag{4.24}$$

This relation is subject to the following limitation in order for it to converge.

$$\lim_{l\to\infty}\left|\frac{a_{l+2}\mu^{l+2}}{a_l\mu^l}\right| < 1$$

If we put in the recursion relation in (4.24) this will reduce to:

$$\lim_{n\to\infty}\left|\frac{\lambda^2 - l(l+n+1)}{(l+2)(l+1)}\mu^2\right| < 1 \tag{4.25}$$

This relation can be satified if $|\mu| < 1$ but previously we mentioned that $|\mu| \leq 1$ that is μ must include 1 as a possible value. Thus we conclude that this series cannot be infinite and must be finite. In order to limit the series we see that if we set $\lambda^2 = l(l+n+1)$ that at this value for λ^2 the recursive coefficeints become zero and thus setting a maximum limit for the series. This means that for the system to converge there must be a maximum l given by the condition:

$$\lambda^2 = l(l+n+1)$$

The maximum value for λ^2 will be given by l and the solution will be expressed as $P_l^{(n)}$. The full solution for $P_l^{(n)}$ will become:

$$P_l^{(n)}(\mu) = \sum_{k=0}^{l}\frac{l(l+n+1)\Big[1-\big(kn+(k!+1)\big)\Big]^k}{(2k)!}\mu^l \tag{4.26}$$

The normalization for $P_l^{(n)}$ we will express as $N_l^{(n)}$ and is given below.

$$N_l^{(n)} = \int_{-1}^{1}\left|P_l^{(n)}(\mu)\right|^2 d\mu \tag{4.27}$$

With the relation for λ^2 we can return to equation (4.22) and put in the condition for λ and simplify the result to find:

$$R^{(n)}(r) = r^{\left(-n-1\pm(2l+n+1)\right)/2}$$

The $\pm$ in the exponetial give us two possible solutions, namely:

$$R^{(n)}(r) = \begin{cases} r^{l} \\ r^{-(l+n+1)} \end{cases}$$

We want the solution for the radial part, $R^{(n)}(r)$ of the Green's function, $G^{(n)}(r,\mu)$, to be bounded at 0 and at ∞. We acomplish this by using the nature of the Green's function where we divide the range of the radial part into two parts using the parameter a, one from $0 \leq r < a$ and the other from $a < r \leq \infty$. We will then solve the homogeneous equation in both parts and then match them at a. The general solution for the radial part is given below.

$$R^{(n)}(r) = \begin{cases} c_1 r^{l} + c_2 r^{-(l+n+1)} & 0 \leq r < a \\ c_3 r^{l} + c_4 r^{-(l+n+1)} & a < r \leq \infty \end{cases}$$

In the equation above c_1, c_2, c_3 and c_4 are constants. To constrain the solution when $r = 0$ we set $c_2 = 0$ and to constrain the solution when $r = \infty$ we set $c_3 = 0$. This leaves us with the following as the solution to the radial part.

$$R^{(n)}(r) = \begin{cases} c_1 r^{l} & 0 \leq r < a \\ c_4 r^{-(l+n+1)} & a < r \leq \infty \end{cases} \tag{4.28}$$

From this we can find c_1 and c_4 by using the fact that the Green's function must be continuous at $r = a$ and the first derivative must be discontinuous at $r = a$. In this case the discontinuity in the first derivative is equal to $\frac{1}{p(a)}$. We solve for c_1 and c_4 and find:

$$c_1 = -\frac{a^{-(l+n+1)}}{A} \qquad c_4 = -\frac{a^{l}}{A} \tag{4.29}$$

In the above equations A is the Wronskian of $a^{-(l+n+1)}$ and a^l multiplied by $p(a)$. A is given below.

$$A = \frac{(1-\mu^2)^{n/2}(2l+n+1)}{a^2}$$

Putting this in to (4.29) c_1 and c_4 become:

$$c_1 = -\frac{a^2}{(1-\mu^2)^{n/2}(2l+n+1)}a^{-(l+n+1)} \qquad c_4 = -\frac{a^2}{(1-\mu^2)^{n/2}(2l+n+1)}a^l \tag{4.30}$$

Putting (4.30) back into (4.28) we find:

$$R^{(n)}(r) = \begin{cases} -\frac{a^2}{(1-\mu^2)^{n/2}(2l+n+1)}a^{-(l+n+1)}r^l & 0 \leq r < a \\ -\frac{a^2}{(1-\mu^2)^{n/2}(2l+n+1)}a^l r^{-(l+n+1)} & a < r \leq \infty \end{cases} \tag{4.31}$$

With this we have the general solution for the Green's function subject to the solution for $R^{(n)}(r)$ given in (4.31) and the solution for $P_l^{(n)}(\mu)$ given in (4.26) and the normalization, $N_l^{(n)}$ of $P_l^{(n)}(\mu)$ given in (4.27).

$$G^{(n)}(r,\mu) = \sum_{l=0}^{\infty} \frac{1}{N_l^{(n)}} P_l^{(n)}(\mu) R^{(n)}(r) \tag{4.32}$$

This will be the function that we will integrate to solve for the unknown variables in the metric, as will be explained in the next section.

4.5 Solutions

Using the general form of the differential operator given in section 4.4 we can set up field equations which can be used to solve for ρ, γ and ω which are functions of the metric (4.1). We build the differential operators out of the Einstein equations by combining different components. The remaining terms not used in the differential operator will constitute the source terms. All non-zero components on left hand side of the Einstein equations are given in Appendix A. Using the general differential operator (4.12), we set $n = 0, 1, 2$ to get three differential operators which will be used to solve for ρ, γ and ω respectively. Thus the field equations for ρ, γ and ω become:

$$\begin{aligned} \nabla^2[\rho e^{\gamma/2}] &= S_\rho(r,\mu) && (4.33)\\ \Big(\nabla^2 + \frac{1}{r}\partial_r - \frac{\mu}{r^2}\partial_\mu\Big)[\gamma e^{\gamma/2}] &= S_\gamma(r,\mu) && (4.34)\\ \Big(\nabla^2 + \frac{2}{r}\partial_r - \frac{2\mu}{r^2}\partial_\mu\Big)[\omega e^{(\gamma-2\rho)/2}] &= S_\omega(r,\mu) && (4.35) \end{aligned}$$

where S_ρ, S_γ and S_ω are defined as follows:

$$\begin{aligned} S_\rho(r,\mu) &= e^{\gamma/2}\Big\{8\pi e^{2\alpha}(\rho_0+\rho_i+P)\frac{1+v^2}{1-v^2} + \frac{1}{r}\gamma_{,r} - \frac{\mu}{r^2}\gamma_{,\mu}\\ &\quad + \frac{\rho}{2}\Big[16\pi e^{2\alpha}P - \gamma_{,r}\Big(\frac{1}{2}\gamma_{,r} + \frac{1}{r}\Big) - \frac{1}{r^2}\gamma_{,\mu}\Big(\frac{1-\mu^2}{2}\gamma_{,\mu} - \mu\Big)\Big]\\ &\quad + r^2(1-\mu^2)e^{-2\rho}\Big(\omega_{,r}^2 + \frac{1-\mu^2}{r^2}\omega_{,\mu}^2\Big)\Big\} && (4.36)\\ S_\gamma(r,\mu) &= e^{\gamma/2}\Big[16\pi e^{2\alpha}P + \frac{\gamma}{2}\Big(16\pi e^{2\alpha}P - \frac{1}{2}\gamma_{,r}^2 - \frac{1-\mu^2}{2r^2}\gamma_{,\mu}^2\Big)\Big] && (4.37)\\ S_\omega(r,\mu) &= e^{(\gamma-2\rho)/2}\Big\{\omega\Big[\frac{\mu}{r^2}\Big(2\rho_{,\mu} + \frac{1}{2}\gamma_{,\mu}\Big) + \frac{1}{4}(4\rho_{,r}^2 - \gamma_{,r}^2) + \frac{1-\mu^2}{4r^2}(4\rho_{,\mu}^2 - \gamma_{,\mu}^2)\\ &\quad - r^2(1-\mu^2)e^{-2\rho}\Big(\omega_{,r}^2 + \frac{1-\mu^2}{r^2}\omega_{,\mu}^2\Big) - \frac{1}{r}\Big(2\rho_{,r} + \frac{1}{2}\gamma_{,r}\Big)\\ &\quad - 8\pi e^{2\alpha}\frac{1}{1-v^2}\big((1+v^2)(\rho_0+\rho_i) + 2v^2P\big)\Big]\\ &\quad - 16\pi e^{2\alpha}\frac{\Omega-\omega}{1-v^2}(\rho_0+\rho_i+P)\Big\} && (4.38) \end{aligned}$$

The subscripts with a comma denote differentiation with respect to that variable. We build the differential operators by combining terms in the Einstein equations in the following manner. The equation for ρ (eqn. (4.33)) is made by combining terms in the following way:

$$G_\phi^\phi - G_t^t - 2\omega G_\phi^t + \frac{1}{2}\rho(G_\mu^\mu + G_r^r) = 8\pi\big(T_\phi^\phi - T_t^t - 2\omega T_\phi^t + \frac{1}{2}\rho(T_\mu^\mu + T_r^r)\big) \qquad (4.39)$$

The equation for γ (eqn. (4.34)) is made by combining:

$$G_r^r + G_\mu^\mu = 8\pi(T_r^r + T_\mu^\mu) \qquad (4.40)$$

The equation for ω (eqn. (4.35)) is made by combining:

$$
\begin{aligned}
G^{\phi}_{t} - \frac{r^2}{4}(1-\mu^2)\omega\big[G^r_r + G^{\mu}_{\mu} + 2(G^{\phi}_{\phi} - G^t_t + 2\omega G^t_{\phi})\big] & \\
= 8\pi\left[T^{\phi}_{t} - \frac{r^2}{4}(1-\mu^2)\omega\big[T^r_r + T^{\mu}_{\mu} + 2(T^{\phi}_{\phi} - T^t_t + 2\omega T^t_{\phi})\big]\right] & \qquad (4.41)
\end{aligned}
$$

Using the components of the Einstein tensor as given in Appendix A, the above equations will reduce down (with a little rearranging) to the desired differential equations and source terms so that we can solve for the variables, ρ, γ and ω, of the metric using the Green's function technique discussed in section 4.4 above. We can find solutions of the differential equations (4.33)–(4.35) using the general solution of the Green's function (4.32). This produces:

$$\rho e^{\gamma/2} = \int G^0(r, r', \mu, \mu')S(r, \mu)drd\mu \qquad (4.42)$$

$$\gamma e^{\gamma/2} = \int G^1(r, r', \mu, \mu')S(r, \mu)drd\mu \qquad (4.43)$$

$$\omega e^{(\gamma-2\rho)/2} = \int G^2(r, r', \mu, \mu')S(r, \mu)drd\mu \qquad (4.44)$$

The superscripts on G denote the value of n used in the general form of the operator (4.12). These can be integrated to solve for the different variables ρ, γ and ω respectively.

Chapter 5

A Rotating Star Without Assuming Circularity

5.1 Introduction

In Chapter 4 we made certain assumptions about the spacetime which simplified the problem but also restricted the possible motion of matter in the star, and thus could not represent a complete picture of the physical system. The three assumptions made in previous general relativistic considerations of rotating bodies have been stationarity, axisymmetry and circularity. A stationary spacetime is defined by a timelike Killing vector. An axisymmetric spacetime is defined by a compact spacelike Killing vector field. The circularity condition constrains the motion of the matter to circular or latitudinal motion parallel to the equatorial plane. In this chapter we will still assume the existence of two Killing vectors but we will relax the circularity assumption to allow that the fluid velocity *not* be restricted to latitudinal motion. This will allow for motion such as convection and motion towards and away from the equator along with poloidal motion (that is, motion from pole to pole). A representation of this is given in figure 5.1. This type of motion is closer to the actual motion of matter that we would expect to find in a star. Thus by relaxing the circularity condition we will be able to create a more realistic model of a neutron star.

Using the two Killing vectors we will perform a double Kaluza-Klein decomposition on the spacetime. In considering the two Killing vectors on our spacetime we will not assume hypersurface orthogonality. Thus when we apply one of the Killing vectors onto an arbitrary metric the result will be a congruence of curves defined by the Killing vector and a three dimensional projection operator that can also function as a three dimensional metric. The application of the second Killing vector will leave us with a two dimensional metric.

From the resulting metric we will be able to solve the Einstein equations and find the equations of motion similar to what was done in Chapter 4. By relaxing the circularity assumption and also by including electromagnetic effects our equations will be more complicated but we will still be able to put them into a form so that the equations can be solved using the Green's function technique.

5.2 The Metric and Relevant Tensors

We start by defining two Killing vectors, a timelike Killing vector defined as $X^\mu = (1, 0, 0, 0)$ and a spacelike Killing vector defined as $N^\mu = (0, 0, 0, 1)$. The normalization of the vectors we will define as follows:

$$X^\mu X_\mu = s^2 \qquad N^\mu N_\mu = -q^2$$

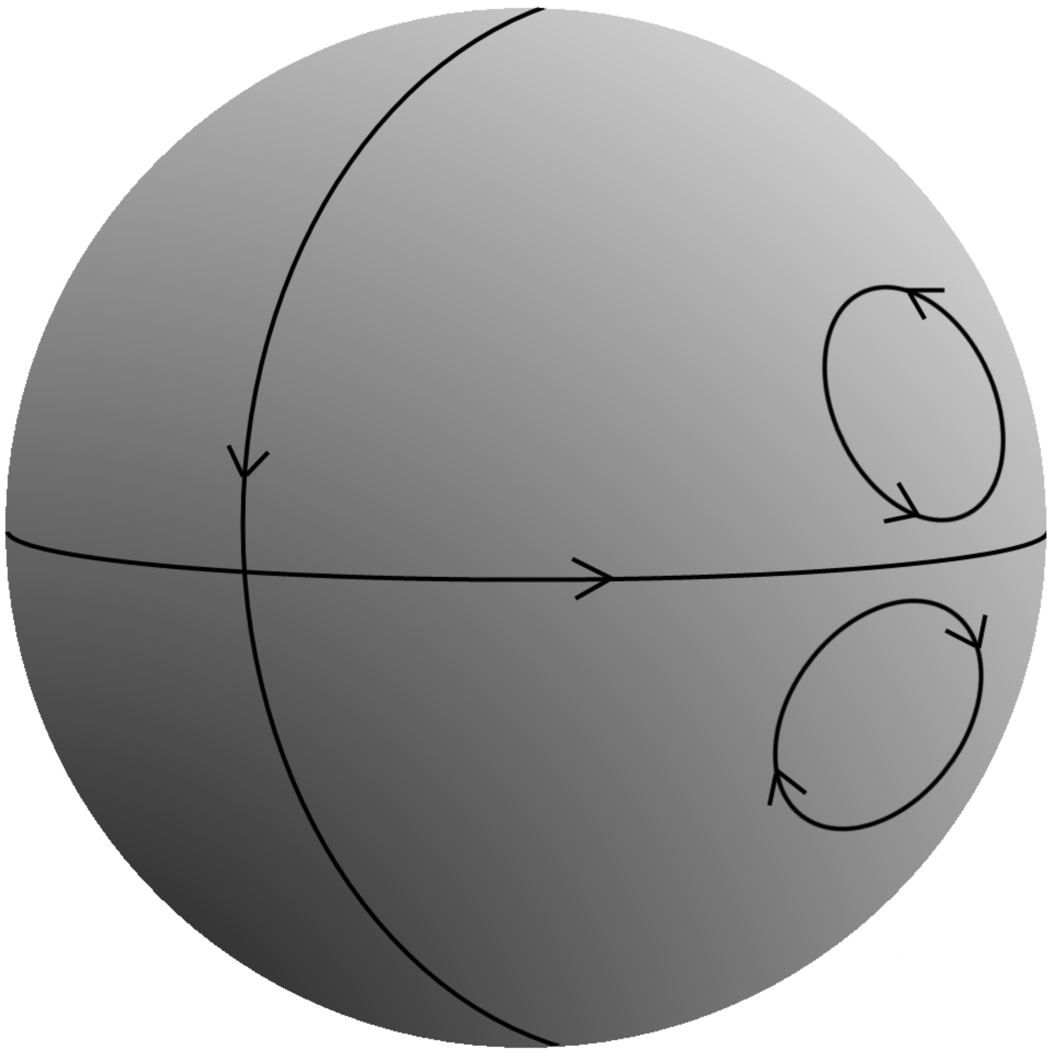

Figure 5.1 If we do not assume circularity then matter is free to move towards and away from the poles and is not constrained to move parallel to the equitorial plane. The matter still moves axisymmetrically, but now matter can also move in the θ and r directions. This allows for convenction and poloidal motion (motion from pole to pole).

For convenience we will rescale the timelike Killing vector in the following way:

$$Y^\mu = \frac{X^\mu}{s^2}$$

We will now apply the Killing vector X^μ to the 4-metric and perform the timelike decomposition of the 4-metric $\gamma^{\mu\nu}$. When we do this we can define a projection operator $g_{\mu\nu}$ which also serves as a 3-metric.

$$g_{\mu\nu} = \gamma_{\mu\nu} - s^2 Y_\mu Y_\nu \tag{5.1}$$

Using (5.1) as a projection operator we can define a three dimensional projection of the spacelike Killing vector, N_α, as follows:

$$^3N_\mu = g^\alpha_\mu N_\alpha = \gamma^\alpha_\mu N_\alpha - s^2 Y^\alpha Y_\mu N_\alpha = N_\mu - N_\phi Y_\mu = \left(1, 0, 0, -\frac{N_\phi}{s^2}\right)$$

As a comment on notation, to indicate a three dimensional object we will use a pre-superscript of 3, and to indicate a two dimensional object we will use a presuperscript of 2. We can now find the normalization of $^3N_\mu$, which we will define as $-Q^2$.

$$^3N^{\mu 3}N_\mu = -\left[q^2 + \left(\frac{N_\phi}{s^2}\right)^2\right] = -Q^2$$

Just as we did for the timelike Killing vector we will introduce a rescaling of $^3N_\mu$ and define it as:

$$M^\mu = \frac{^3N^\mu}{Q^2}$$

We can now perform the second Kaluza-Klein decomposition with $\sigma^{\mu\nu}$ representing the two dimensional projection operator and which will also function as a 2-metric. The fully decomposed metric becomes:

$$\gamma^{\mu\nu} = {}^2\sigma^{\mu\nu} - Q^2 M^\mu M^\nu + s^2 Y^\mu Y^\nu \tag{5.2}$$

Using the metric (5.2) we can rewrite all relevant tensors (i.e. the Einstein and stress-energy tensors and other related tensors) and tensor-like objects (the Christoffel symbol) in a two dimensional form with additional pieces resulting from the two killing vectors M^{μ} and Y^{μ}. The rewritten tensors and other objects we will refer to as decomposed tensors or objects, because we are breaking the four dimensional object down into a projected two dimensional object. The derivations of the tensors and tensor-like objects necessary for our problem are given in Appendix C. The bulk of our work for this chapter is represented in the derivations and tensors given in the appendicies. Some identities and definitions used in the derivations are given in Appendix B.

5.3 Conclusion

As we mentioned at the begining of this chapter we are considering the same problem as in Chapter 4 while relaxing some of our assumptions. Thus the purpose of this work is to find solutions that can be solved using Green's function technique similar to the method used in Chapter 4. The next steps for this research are to see if we can use the same combinations of the Einstein equations to reproduce the differential operators and corresponding source terms given in Chapter 4. At the time of this writing this is our current emphasis of investigation.

Chapter 6

Conclusion

To put this work in perspective we need to consider the history of modeling astronomical objects. Starting with Newton's theory of gravity it became possible to mathematically model rotating astronomical objects. The contribution of Maclaurin, Laplace, Poisson and so many others gave us the fundamental methods of how to model self-gravitating fluids. With the advent of general relativity a natural avenue of investigation was to consider how to model systems in general relativity that previously had been worked out in Newtonian gravity. The simplest cases, such as the Schwarzschild and Kerr metrics, could be considered as basic models of astronomical objects. But these simpler models could not explain the full range of phenomena in the universe. Thus to develop a system that is closer to physical reality we must relax some of our assumptions and consider a more complex system.

In this particular work we are considering the corresponding general relativistic models of systems previously developed in Newtonian gravity. In Chapter 2 we were considering the general relativistic case to see if the results from general relativity give different results from the Newtonian model. In our investigation of the rotating neutron star we were trying to get agreement between the general relativistic model and the Newtonian model. We did this with the intent to establish a basis from which we can work and then change some of our assumptions in order to consider different possible models of rotating astronomical objects. Essentially this work is the next step in the history of how to model rotating astronomical objects.

Appendix A

Solutions of the Einstein Tensor

We present here solutions to the left hand side of the Einstein equations using the metric in [2]. These solutions are used in Chapter 4.

$$G^t_t = \frac{1}{2}\frac{1}{r^2 e^{2\alpha}}\Big[r^2\gamma_{,rr} + 2r^2\alpha_{,rr} + (1-\mu^2)\gamma_{,\mu\mu} - (1-\mu^2)\rho_{,\mu\mu} - r^2\rho_{,rr} + \frac{r^2}{2}\gamma^2_{,r}$$
$$- r^2\rho_{,r}\gamma_{,r} + 3r\gamma_{,r} + \frac{1}{2}(1-\mu^2)\rho^2_{,\mu} - (1-\mu^2)\gamma_{,\mu}\rho_{,\mu} + 3\mu\rho_{,\mu} + \frac{r^2}{2}\rho^2_{,r}$$
$$- 3r\rho_{,r} - 2\mu\alpha_{,\mu} + \frac{1}{2}(1-\mu^2)\gamma^2_{,\mu} - 3\mu\gamma_{,\mu} + 2(1-\mu^2)\alpha_{,\mu\mu} + 2r\alpha_{,r}\Big]$$
$$+ \frac{1}{4}\frac{(1-\mu^2)}{e^{2(\alpha+\rho)}}\Big[(1-\mu^2)\omega^2_{,\mu} + r^2\omega^2_{,r}\Big] - \omega G^t_\phi$$

$$G^\phi_\phi = \frac{1}{2}\frac{1}{r^2 e^{2\alpha}}\Big[r^2\gamma_{,rr} + 2r^2\alpha_{,rr} + (1-\mu^2)\gamma_{,\mu\mu} + (1-\mu^2)\rho_{,\mu\mu} + r^2\rho_{,rr}$$
$$+ \frac{r^2}{2}\gamma^2_{,r} + r^2\rho_{,r}\gamma_{,r} + r\gamma_{,r} + \frac{1}{2}(1-\mu^2)\rho^2_{,\mu} + (1-\mu^2)\gamma_{,\mu}\rho_{,\mu} - \mu\rho_{,\mu} + \frac{r^2}{2}\rho^2_{,r}$$
$$+ r\rho_{,r} - 2\mu\alpha_{,\mu} + \frac{1}{2}(1-\mu^2)\gamma^2_{,\mu} - \mu\gamma_{,\mu} + 2(1-\mu^2)\alpha_{,\mu\mu} + 2r\alpha_{,r}\Big]$$
$$- \frac{3}{4}\frac{(1-\mu^2)}{e^{2(\alpha+\rho)}}\Big[(1-\mu^2)\omega^2_{,\mu} + r^2\omega^2_{,r}\Big] + \omega G^t_\phi$$

$$G^r_r = \frac{1}{4}\frac{1}{r^2 e^{2\alpha}}\Big[4(1-\mu^2)\gamma_{,\mu\mu} + r^2\gamma_{,r}^2 + 4r^2\alpha_{,r}\gamma_{,r} + 6r\gamma_{,r} + (1-\mu^2)\rho_{,\mu}^2 + 2\mu\rho_{,\mu} + r^2\rho_{,r}^2 + 2r\rho_{,r} - 4(1-\mu^2)\gamma_{,\mu}\alpha_{,\mu} + 4\mu\alpha_{,\mu} + 3(1-\mu^2)\gamma_{,\mu}^2 - 10\mu\gamma_{,\mu} + 4r\alpha_{,r}\Big] - \frac{1}{4}\frac{(1-\mu^2)}{e^{2(\alpha+\rho)}}\Big[(1-\mu^2)\omega_{,\mu}^2 - r^2\omega_{,r}^2\Big]$$

$$G^\mu_\mu = -\frac{1}{4}\frac{1}{r^2 e^{2\alpha}}\Big[-4r^2\gamma_{,rr} - 3r^2\gamma_{,r}^2 + 4r^2\alpha_{,r}\gamma_{,r} - 6r\gamma_{,r} + (1-\mu^2)\rho_{,\mu}^2 + 2\mu\rho_{,\mu} - r^2\rho_{,r}^2 + 2r\rho_{,r} - 4(1-\mu^2)\gamma_{,\mu}\alpha_{,\mu} + 4\mu\alpha_{,\mu} - (1-\mu^2)\gamma_{,\mu}^2 + 2\mu\gamma_{,\mu} + 4r\alpha_{,r}\Big] + \frac{1}{4}\frac{(1-\mu^2)}{e^{2(\alpha+\rho)}}\Big[(1-\mu^2)\omega_{,\mu}^2 - r^2\omega_{,r}^2\Big]$$

$$G^r_\mu = -\frac{1}{2}\frac{1}{e^{2\alpha}}\Big[-2\alpha_{,\mu}\gamma_{,r} + \gamma_{,\mu}\gamma_{,r} - \frac{\mu}{1-\mu^2}\gamma_{,r} + \rho_{,\mu}\rho_{,r} - \frac{1}{r}\rho_{,\mu} + \frac{\mu}{1-\mu^2}\rho_{,r} - \frac{2}{r}\alpha_{,\mu} - 2\gamma_{,\mu}\alpha_{,r} - \frac{1}{r}\gamma_{,\mu} + \frac{2\mu}{1-\mu^2}\alpha_{,r} + \gamma_{,r\mu}\Big] + \frac{r^2(1-\mu^2)}{2e^{2(\alpha+\rho)}}(\omega_{,r}\omega_{,\mu})$$

$$G^t_\phi = -\frac{1}{2}\frac{(1-\mu^2)}{e^{2(\alpha+\rho)}}\Big[r^2\omega_{,r}\gamma_{,r} - 2(1-\mu^2)\omega_{,\mu}\rho_{,\mu} - 2r^2\omega_{,r}\rho_{,r} + (1-\mu^2)\omega_{,\mu}\gamma_{,\mu} + 4r\omega_{,r} - 4\mu\omega_{,\mu} + r^2\omega_{,rr} + (1-\mu^2)\omega_{,\mu\mu}\Big]$$

Appendix B

Identities

These are identities used in the derivations show in Appendix C.

$$^4X^a = (0,0,0,1) \qquad ^4N^a = (1,0,0,0) \qquad Y^a = \frac{X^a}{s^2} \qquad M^a = \frac{^3N^a}{Q^2}$$

$$X^\mu X_\mu = s^2 \qquad ^4N^{\mu 4}N_\mu = -q^2 \qquad Z_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu \qquad W_{\mu\nu} = \partial_\mu M_\nu - \partial_\nu M_\mu$$

$$^3N^\mu = N^\mu - N_\phi Y^\mu = (1,0,0,-\frac{N_\phi}{s^2}) \qquad ^3N^{\mu 3}N_\mu = -Q^2 = -\left[q^2 + \left(\frac{N_\phi}{s^2}\right)^2\right]$$

$$^3X_\mu = {}^4X_\mu + \frac{1}{q^2}{}^4N_\mu \cdot {}^4X_t \qquad Y^\alpha \cdot \partial_\mu \sigma_{\nu\alpha} = 0 \qquad ^3N_\mu Y^\mu = 0 \qquad Y^\mu \partial_\mu = 0$$

$$X^\alpha \sigma_{\nu\alpha} = 0 \qquad ^3N^\alpha \sigma_{\nu\alpha} = 0 \qquad ^3N_\alpha \sigma^{\nu\alpha} = 0 \qquad X_\alpha \sigma^{\nu\alpha} = 0$$

$$Y_\alpha \partial_\mu \sigma^{\beta\alpha} = 0 \qquad M_\alpha \partial_\mu \sigma^{\beta\alpha} = 0 \qquad \sigma^{\alpha\beta}\partial_\mu X_\alpha = 0 \qquad \sigma^{\alpha\beta}\partial_\mu {}^3N_\alpha = 0$$

$$^3N^\alpha \cdot \partial_\mu \sigma_{\nu\alpha} = 0 \qquad Y^\mu W_{\mu\nu} = 0 \qquad M^\mu W_{\mu\nu} = 0 \qquad X^\mu M_\mu = 0$$

$$^3N^\mu \partial_\mu = 0 \qquad Y^\nu Z_{\mu\nu} = 0 \qquad Y_\phi = 1 \qquad Y_t = \frac{N_\phi}{s^2}$$

$$M_\mu N^\mu = M_t \qquad \partial_\alpha Y^\alpha = \partial_\phi \frac{1}{s^2} \qquad \partial_\alpha M^\alpha = \partial_\phi \left(-\frac{N_\phi}{Q^2 s^2}\right)$$

$$-{}^3N^\alpha \partial_\mu \left(\frac{^3N_\nu}{Q^2}{}^3N_\alpha\right) = \partial_\mu({}^3N_\nu) \qquad ^3N^\alpha \cdot s^2 Y_\nu \partial_\mu Y_\alpha = s^2 Y_\nu \cdot \partial\left(\frac{N_\phi}{s^2}\right)$$

Appendix C

Derivations

C.1 Four Christoffel

Below we show the decomposed four Christoffel. That is, we will break the four Christoffel down into a two Christoffel and other terms resulting from the double Kaluza-Klein decomposition. We will insert the decomposed 4-metric 5.2 into the definition of the Christoffel and expand. As a reminder on notation, a tensor or other comparable object with indices that has a presuperscript of 2 or 3 indicates that it is a projected object into the two or three space. A presuperscript of 4 indicates the full four dimensional object.

$$ {}^4\Gamma^{\lambda}_{\mu\nu} = {}^2\Gamma^{\lambda}_{\mu\nu} + \frac{1}{2}\sigma^{\lambda\alpha}\Xi_{\mu\nu\alpha} + \frac{s^2}{2}Y^{\lambda}\Sigma_{\mu\nu} - \frac{Q^2}{2}M^{\lambda}\Omega_{\mu\nu} $$

In defining the four Christoffel we introduced the objects $\Xi_{\mu\nu\alpha}, \Sigma_{\mu\nu}$ and $\Omega_{\mu\nu}$ that we can use to simplify our calculation of the Ricci tensor. They have no physical meaning, but are useful for simplifying subsequent derivations. They are defined below:

$$\begin{aligned}
\Xi_{\mu\nu\alpha} &= \Big[s^2(Y_\mu Z_{\nu\alpha} + Y_\nu Z_{\mu\alpha}) - Q^2(M_\mu W_{\nu\alpha} + M_\nu W_{\mu\alpha}) \\
&\qquad + M_\mu M_\nu \partial_\alpha(Q^2) - Y_\mu Y_\nu \partial_\alpha(s^2)\Big] \\
\Sigma_{\mu\nu} &= \Big[\partial_\mu(s^2 Y_\nu) + \partial_\nu(s^2 Y_\mu)\Big] \\
\Omega_{\mu\nu} &= \left[\partial_\mu M_\nu + \partial_\nu M_\mu + s^2 Y_\nu \partial_\mu\left(\frac{N_\phi}{s^2}\right) + s^2 Y_\mu \partial_\nu\left(\frac{N_\phi}{s^2}\right)\right]
\end{aligned}$$

Here we give the derivation of the four Christoffel used in Chapter 5.

$$\begin{aligned}
{}^4\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\gamma^{\lambda\alpha}(\partial_\mu \gamma_{\nu\alpha} + \partial_\nu \gamma_{\mu\alpha} - \partial_\alpha \gamma_{\mu\nu}) \\
&= \frac{1}{2}(\sigma^{\lambda\alpha} - Q^2 M^\lambda M^\alpha + s^2 Y^\lambda Y^\alpha)\Big[\partial_\mu(\sigma_{\nu\alpha} - Q^2 M_\nu M_\alpha + s^2 Y_\nu Y_\alpha) \\
&\quad + \partial_\nu(\sigma_{\mu\alpha} - Q^2 M_\mu M_\alpha + s^2 Y_\mu Y_\alpha) - \partial_\alpha(\sigma_{\mu\nu} - Q^2 M_\mu M_\nu + s^2 Y_\mu Y_\nu)\Big] \\
&= {}^2\Gamma^\lambda_{\mu\nu} + \frac{1}{2}\sigma^{\lambda\alpha}\Big[s^2 Y_\nu \partial Y_\alpha - Q^2 M_\nu \partial_\mu M_\alpha - Q^2 M_\mu \partial_\nu M_\alpha \\
&\qquad + s^2 Y_\mu \partial_\nu Y_\alpha - \partial_\alpha\big[s^2 Y_\mu Y_\nu - Q^2 M_\mu M_\nu\big]\Big] \\
&\quad - \frac{Q^2}{2} M^\lambda \Big[M^\alpha \partial_\mu\big[\sigma_{\nu\alpha} - Q^2 M_\nu M_\alpha\big] + M^\alpha s^2 Y_\nu \partial_\mu Y_\alpha \\
&\qquad + M^\alpha \partial_\nu\big[\sigma_{\mu\alpha} - Q^2 M_\mu M_\alpha\big] + M^\alpha s^2 Y_\mu \partial_\nu Y_\alpha\Big] \\
&\quad + \frac{s^2}{2} Y^\lambda \Big[Y^\alpha \partial_\mu\big[\sigma_{\nu\alpha} + s^2 Y_\nu Y_\alpha\big] - Y^\alpha Q^2 M_\nu \partial_\mu M_\alpha \\
&\qquad + Y^\alpha \partial_\nu\big[\sigma_{\mu\alpha} + s^2 Y_\mu Y_\alpha\big] - Y^\alpha Q^2 M_\mu \partial_\nu M_\alpha\Big]
\end{aligned}$$

$$\begin{aligned}
&= {}^2\Gamma^\lambda_{\mu\nu} + \frac{1}{2}\sigma^{\lambda\alpha}\Big[s^2(Y_\mu Z_{\nu\alpha} + Y_\nu Z_{\mu\alpha}) - Q^2(M_\mu W_{\nu\alpha} + M_\nu W_{\mu\alpha}) \\
&\qquad + M_\mu M_\nu \partial_\alpha(Q^2) - Y_\mu Y_\nu \partial_\alpha(s^2)\Big] \\
&\quad - \frac{Q^2}{2} M^\lambda \left[\partial_\mu M_\nu + \partial_\nu M_\mu + s^2 Y_\nu \partial_\mu\left(\frac{N_\phi}{s^2}\right) + s^2 Y_\mu \partial_\nu\left(\frac{N_\phi}{s^2}\right)\right] \\
&\quad + \frac{s^2}{2} Y^\lambda \Big[\partial_\mu(s^2 Y_\nu) + \partial_\nu(s^2 Y_\mu)\Big] \\
&= {}^2\Gamma^\lambda_{\mu\nu} + \frac{1}{2}\sigma^{\lambda\alpha}\Xi_{\mu\nu\alpha} + \frac{s^2}{2} Y^\lambda \Sigma_{\mu\nu} - \frac{Q^2}{2} M^\lambda \Omega_{\mu\nu}
\end{aligned}$$

In the last line we introduced the objects $\Xi_{\mu\nu\alpha}, \Sigma_{\mu\nu}$ and $\Omega_{\mu\nu}$ that we can use to simplify our calculation of the Ricci tensor. They have no physical meaning, but are useful for simplifying subsequent derivations. They are defined below.

$$\begin{aligned}
\Xi_{\mu\nu\alpha} &= \Big[s^2(Y_\mu Z_{\nu\alpha} + Y_\nu Z_{\mu\alpha}) - Q^2(M_\mu W_{\nu\alpha} + M_\nu W_{\mu\alpha}) \\
&\qquad + M_\mu M_\nu \partial_\alpha(Q^2) - Y_\mu Y_\nu \partial_\alpha(s^2)\Big] \\
\Sigma_{\mu\nu} &= \Big[\partial_\mu(s^2 Y_\nu) + \partial_\nu(s^2 Y_\mu)\Big] \\
\Omega_{\mu\nu} &= \left[\partial_\mu M_\nu + \partial_\nu M_\mu + s^2 Y_\nu \partial_\mu\left(\frac{N_\phi}{s^2}\right) + s^2 Y_\mu \partial_\nu\left(\frac{N_\phi}{s^2}\right)\right]
\end{aligned}$$

In preparation of decomposing the four Ricci we will show a simplification of a Christoffel symbol with two identical indicies.

$$\begin{aligned}
{}^4\Gamma^\lambda_{\mu\lambda} &= {}^2\Gamma^\lambda_{\mu\lambda} + \frac{1}{2}\sigma^{\lambda\alpha}\Big[s^2 Y_\mu Z_{\lambda\alpha} - Q^2 M_\mu W_{\lambda\alpha}\Big] \\
&\qquad -\frac{1}{2}M^\lambda \partial_\mu(Q^2 M_\lambda) + \frac{1}{2}Y^\lambda \partial_\mu(s^2 Y_\lambda) \\
-\frac{1}{2}M^\lambda \partial_\mu(Q^2 M_\lambda) &= -\frac{1}{2}\partial_\mu(Q^2 M^\lambda M_\lambda) + \frac{1}{2}Q^2 M_\lambda \partial_\mu M^\lambda \\
&= \frac{Q^2}{2} M_\lambda \partial_\mu\left(\frac{1}{Q^2}\right) \cdot {}^3N^\lambda + \frac{1}{Q^2}\partial_\mu\left(-\frac{N_\phi}{s^2}\right) X^\lambda \\
&= -\frac{1}{2}Q^2\left(-\frac{2}{Q^3}\right)\partial_\mu Q \\
&= \partial_\mu \ln(Q)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}Y^\lambda \partial_\mu(s^2 Y_\lambda) &= \frac{1}{2s^2}\partial_\mu s^2 \\
&= \partial_\mu \ln(s) \\
{}^4\Gamma^\lambda_{\mu\lambda} &= {}^2\Gamma^\lambda_{\mu\lambda} + \partial_\mu \ln(Q) + \partial_\mu \ln(s) \\
&= \partial_\mu \ln\sqrt{\sigma} + \partial_\mu \ln(Q) + \partial_\mu \ln(s) \\
{}^4\Gamma^\lambda_{\mu\lambda} &= \partial_\mu \ln(\sqrt{\sigma} \cdot sQ) \qquad \text{(C.1)}
\end{aligned}$$

C.2 Four Ricci

In this section we give the decomposed four Ricci which is used in Chapter 5. A useful derivation of the Christoffel symbol with two identical indices is given above.

$$
\begin{aligned}
{}^4R_{\mu\nu} \;=\; & Y_\mu Y_\nu \Big[-\frac{1}{2}\, {}^2\triangle_\alpha\, {}^2\triangle^\alpha s^2 - \frac{1}{2}\, {}^2\triangle_\alpha \ln(sQ)\, {}^2\triangle^\alpha s^2 + \frac{s^4}{4}\, {}^2Z_{\alpha\beta}\, {}^2Z^{\alpha\beta} \\
& \quad + \frac{s^2}{2}\, {}^2\triangle_\alpha \ln(s^2)\, {}^2\triangle^\alpha \ln(s^2) - \frac{1}{2}\frac{s^4}{Q^2}\, {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right) {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \Big] \\
& + M_\mu M_\nu \Big[\frac{1}{2}\, {}^2\triangle_\alpha\, {}^2\triangle^\alpha Q^2 + \frac{1}{2}\, {}^2\triangle_\alpha \ln(sQ)\, {}^2\triangle^\alpha Q^2 + \frac{Q^4}{4}\, {}^2W_{\alpha\beta}\, {}^2W^{\alpha\beta} \\
& \quad - \frac{Q^2}{2}\, {}^2\triangle_\alpha \ln(Q^2)\, {}^2\triangle^\alpha \ln(Q^2) - \frac{1}{2} s^2\, {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right) {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \Big] \\
& + (Y_\mu M_\nu + M_\mu Y_\nu) \Big[\frac{s^2}{2}\, {}^2\triangle_\alpha \ln(sQ)\, {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) + \frac{1}{2}\, {}^2\triangle_\alpha \left(s^2\, {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right)\right) \\
& \quad - \frac{Q^2 s^2}{4}\, {}^2Z_{\alpha\beta}\, {}^2W^{\alpha\beta} - \frac{s^2}{2}\, {}^2\triangle_\alpha \ln(Q^2)\, {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \Big] \\
& + (\sigma^\gamma_\mu Y_\nu + \sigma^\gamma_\nu Y_\mu) \Big[\frac{s^2}{2}\, {}^2\triangle^\alpha \ln(sQ)\, {}^2Z_{\gamma\alpha} + \frac{1}{2}\, {}^2\triangle^\alpha \left(s^2\, {}^2Z_{\gamma\alpha}\right) \Big] \\
& + (\sigma^\gamma_\mu M_\nu + \sigma^\gamma_\nu M_\mu) \Big[-\frac{Q^2}{2}\, {}^2\triangle^\alpha \ln(sQ)\, {}^2W_{\gamma\alpha} - \frac{1}{2}\, {}^2\triangle^\alpha \left(Q^2\, {}^2W_{\gamma\alpha}\right) \Big] \\
& + \sigma^\gamma_{\ \mu} \sigma^\kappa_{\ \nu} \Big[{}^2R_{\gamma\kappa} + \frac{s^2}{2}\, {}^2Z_{\gamma\alpha}\, {}^2Z^\alpha_{\ \kappa} - \frac{Q^2}{2}\, {}^2W_{\gamma\alpha}\, {}^2W^\alpha_{\ \kappa} \\
& \quad - \frac{1}{4}\, {}^2\triangle_\gamma \ln(s^2)\, {}^2\triangle_\kappa \ln(s^2) - \frac{1}{4}\, {}^2\triangle_\gamma \ln(Q^2)\, {}^2\triangle_\kappa \ln(Q^2) \\
& \quad + \frac{1}{2}\frac{s^2}{Q^2}\, {}^2\triangle_\gamma \left(\frac{N_\phi}{s^2}\right) {}^2\triangle_\kappa \left(\frac{N_\phi}{s^2}\right) - {}^2\triangle_\gamma\, {}^2\triangle_\kappa \ln(sQ) \Big]
\end{aligned}
$$

Related to the four Ricci is the Ricci scalar, given below.

$$
\begin{aligned}
{}^4R \;=\; & {}^2R - {}^2\triangle_\alpha \left(\frac{1}{sQ}\, {}^2\triangle^\alpha sQ \right) - \frac{1}{sQ}\, {}^2\triangle_\alpha \left({}^2\triangle^\alpha sQ \right) - \frac{s^2}{4}\, {}^2Z_{\alpha\beta}\, {}^2Z^{\alpha\beta} + \frac{Q^2}{4}\, {}^2W_{\alpha\beta}\, {}^2W^{\alpha\beta} \\
& - \frac{1}{s^2}\, {}^2\triangle_\alpha s\, {}^2\triangle^\alpha s - \frac{1}{Q^2}\, {}^2\triangle_\alpha Q\, {}^2\triangle^\alpha Q + \frac{1}{2}\frac{s^2}{Q^2}\, {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right) {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right)
\end{aligned}
$$

C.3 The Einstein Tensor

Using $R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R$ we can write down the Einstein tensor which is used in Chapter 5.

$$\begin{aligned}
{}^4G_{\mu\nu} = \; & {}^2G_{\mu\nu} + Y_\mu Y_\nu \Bigg[\frac{s^2}{Q}\; {}^2\Box Q + \frac{3}{8}s^4\; {}^2Z_{\alpha\beta}\; {}^2Z^{\alpha\beta} - \frac{3}{4}\frac{s^4}{Q^4}\; {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right)\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \\
& \qquad - \frac{s^2}{2}\; {}^2R - \frac{1}{8}s^2Q^2\; {}^2W_{\alpha\beta}\; {}^2W^{\alpha\beta} \Bigg] \\
& + M_\mu M_\nu \Bigg[- \frac{Q^2}{s}\; {}^2\Box s + \frac{3}{8}Q^4\; {}^2W_{\alpha\beta}\; {}^2W^{\alpha\beta} - \frac{1}{4}s^2\; {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right)\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \\
& \qquad + \frac{Q^2}{2}{}^2R - \frac{1}{8}s^2Q^2\; {}^2Z_{\alpha\beta}\; {}^2Z^{\alpha\beta} \Bigg] \\
& + \left(Y_\mu M_\nu + M_\mu Y_\nu\right) \Bigg[\frac{s^2}{2}\; {}^2\triangle_\alpha \ln(sQ)\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) + \frac{1}{2}\; {}^2\triangle_\alpha \left(s^2\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right)\right) \\
& \qquad - \frac{Q^2s^2}{4}\; {}^2Z_{\alpha\beta}\; {}^2W^{\alpha\beta} - \frac{s^2}{2}\; {}^2\triangle_\alpha \ln(Q^2)\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \Bigg] \\
& + (\sigma^\gamma_\mu Y_\nu + \sigma^\gamma_\nu Y_\mu) \Bigg[\frac{s^2}{2}\; {}^2\triangle^\alpha \ln(sQ)\; {}^2Z_{\gamma\alpha} + \frac{1}{2}\; {}^2\triangle^\alpha \left(s^2\; {}^2Z_{\gamma\alpha}\right) \Bigg] \\
& + (\sigma^\gamma{}_\mu M_\nu + \sigma^\gamma{}_\nu M_\mu) \Bigg[\frac{Q^2}{2}\; {}^2\triangle^\alpha \ln(sQ)\; {}^2W_{\gamma\alpha} - \frac{1}{2}\; {}^2\triangle^\alpha \left(Q^2\; {}^2W_{\gamma\alpha}\right) \Bigg] \\
& + \sigma^\gamma{}_\mu \sigma^\kappa{}_\nu \Bigg[\frac{s^2}{2}\; {}^2Z_{\gamma\alpha}\; {}^2Z^\alpha{}_\kappa - \frac{Q^2}{2}\; {}^2W_{\gamma\alpha}\; {}^2W^\alpha{}_\kappa - \frac{1}{4}\; {}^2\triangle_\gamma \ln(s^2)\; {}^2\triangle_\kappa \ln(s^2) \\
& - \frac{1}{4}\; {}^2\triangle_\gamma \ln(Q^2)\; {}^2\triangle_\kappa \ln(Q^2) + \frac{1}{2}\frac{s^2}{Q^2}\; {}^2\triangle_\gamma \left(\frac{N_\phi}{s^2}\right)\; {}^2\triangle_\kappa \left(\frac{N_\phi}{s^2}\right) \\
& - {}^2\triangle_\gamma\; {}^2\triangle_\kappa \ln(sQ) - \frac{1}{2}\sigma_{\gamma\kappa} \Bigg\{ - {}^2\triangle_\alpha \left(\frac{1}{sQ}\; {}^2\triangle^\alpha sQ\right) - \frac{1}{sQ}\; {}^2\triangle_\alpha \left({}^2\triangle^\alpha sQ\right) \\
& \quad - \frac{s^2}{4}\; {}^2Z_{\alpha\beta}\; {}^2Z^{\alpha\beta} + \frac{Q^2}{4}\; {}^2W_{\alpha\beta}\; {}^2W^{\alpha\beta} - \frac{1}{s^2}\; {}^2\triangle_\alpha s\; {}^2\triangle^\alpha s \\
& \quad - \frac{1}{Q^2}\; {}^2\triangle_\alpha Q\; {}^2\triangle^\alpha Q + \frac{s^2}{Q^2}\; {}^2\triangle_\alpha \left(\frac{N_\phi}{s^2}\right)\; {}^2\triangle^\alpha \left(\frac{N_\phi}{s^2}\right) \Bigg\} \Bigg]
\end{aligned}$$

C.4 Stress Energy Tensor

Similarly we decompose the stress energy tensor to get:

$$\begin{aligned}
T_{\mu\nu} &= {}^2T_{\mu\nu} + s^2 Y_\mu Y_\nu \left[P + \frac{1}{2}b_\alpha b^\alpha + \frac{1}{s^2}\left(u_\phi^2(H + b_\alpha b^\alpha) - b_\phi^2\right)\right] \\
&\quad - Q^2 M_\mu M_\nu \left[P + \frac{1}{2}b_\alpha b^\alpha - \frac{1}{s^2}\left[\left(u_t - \frac{N_\phi}{s^2}u_\phi\right)(H + b_\alpha b^\alpha) - \left(b_t - \frac{N_\phi}{s^2}b_\phi\right)\right]\right] \\
&\quad + (Y_\mu M_\nu - Y_\nu M_\mu)\left[\left(u_t - \frac{N_\phi}{s^2}u_\phi\right)(H + b_\alpha b^\alpha)u_\phi - \left(b_t - \frac{N_\phi}{s^2}b_\phi\right)b_\phi\right] \\
&\quad - (\sigma_\mu^\gamma M_\nu + \sigma_\nu^\delta M_\mu)\left[(u_\gamma + u_\delta)\left(u_t - \frac{N_\phi}{s^2}u_\phi\right)(H + b_\alpha b^\alpha) - (b_\gamma + b_\delta)\left(b_t - \frac{N_\phi}{s^2}b_\phi\right)\right] \\
&\quad + (\sigma_\mu^\gamma Y_\nu + \sigma_\nu^\delta Y_\mu)\left[(u_\gamma + u_\delta)(H + b_\alpha b^\alpha)u_\phi - (b_\gamma + b_\delta)b_\phi\right]
\end{aligned}$$

The magnetic component, $b_\alpha b^\alpha$, can be decomposed as follows:

$$\begin{aligned}
b_\alpha b^\alpha &= \gamma_{\beta\alpha} b^\beta b^\alpha \\
&= \sigma_{\beta\alpha} b^\beta b^\alpha - \frac{1}{Q^2}\left(b_t - \frac{N_\phi}{s^2}b_\phi\right)^2 + \frac{1}{s^2}(b_\phi)^2
\end{aligned}$$

We have defined the two dimensional stress energy tensor as:

$$ {}^2T_{\mu\nu} = \sigma_\mu^\gamma \sigma_\nu^\delta \left[u_\gamma u_\delta (H + b_\alpha b^\alpha) - b_\gamma b_\delta\right] + \sigma_{\mu\nu}\left[P + \frac{1}{2}b_\alpha b^\alpha\right]$$

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