# OPTICAL LEVITATION OF INTENSITY-FLEEING PARITICLES IN A SINGLE-BEAM LASER TRAP

by

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## Abstract

We have discovered a way to optically trap and manipulate microscopic opaque particles, including metallic spheres. With this novel approach particles can be trapped three dimensionally in the focus of a single laser beam. Normally, opaque particles should flee the intense light of the laser focus. However, they can be caught in dark regions within the laser beam. Our trap is able to levitate a myriad of particles of various densities, sizes and shapes. Other schemes to hold intensity-fleeing particles have complicated set-ups and are limited in their trapping capabilities. This thesis explores the characteristics of the new trap and discusses different hypotheses of the mechanisms that make the levitation possible. We have shown that neither spherical aberration nor convection currents are primarily responsible for the observed trapping.

## Acknowledgements

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# Chapter 1: Introduction

Light can be used to trap small particles because it carries momentum. As light is reflected or refracted by an object, it changes direction and transfers a distinct amount of momentum to the object. Historically, the idea that light carries momentum dates back to Kepler. He supposed that the reason a comet's tail always points away from the sun is because of radiation pressure. Later Maxwell confirmed the theory that light carries momentum, but concluded that the forces due to the momentum change were tiny and lacked practical terrestrial application. With the advent of the laser higher intensity light became available, which consequently yielded stronger forces. Wolfgang Paul was the first to utilize the momentum of light to trap particles [1]. Using a combination of magnetic fields and laser light he successfully held neutrons in a magneto-optical trap. Purely optical traps have since been developed to hold particles as small as individual electrons and as large as 50 micron spheres. Laser traps now assist researchers in many fields of science.

## 1.1 Transparent Particles

In 1969, Arthur Ashkin's studies of radiation pressure led him to predict that microscopic particles can be accelerated by light [2]. This led to a simple experiment with a mildly focused Gaussian beam in a solution of water and 2-micron latex transparent spheres. The experiment verified particle motion caused by the incident laser light. However, the spheres unexpectedly gravitated to the focus of the beam and remained trapped. The force was so strong that they remained fixed even when the beam

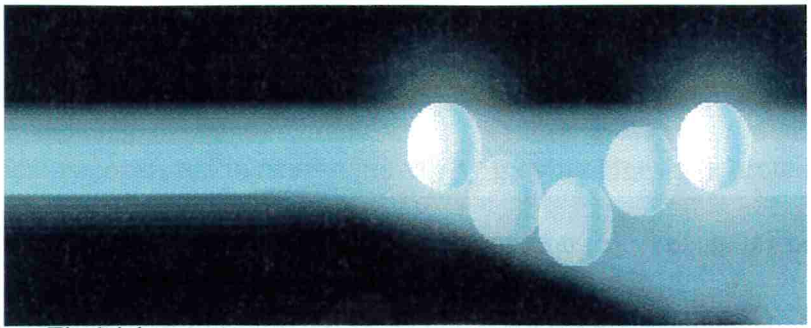


**Fig 1.1.1 Translucent particle in focus.** As a particle tries to leave the focus, it acts as a lens and bends the light downward, causing an opposite force to pull it.

was moved from side to side [3]. This trap can be explained by examining a single transparent particle in the focus of a laser. If the particle is pulled down, the displaced particle (represented in figure 1.1.1) will act as a lens that refracts the light downward. This change in the momentum of light yields a corresponding reactive force upward on the particle, which pulls the particle back into the focus. This traps the particle 2-dimensionally; it is then confined axially by the gradient of the focus, making it a 3-dimensional trap.

The particles can also be trapped 3-dimensionally with a single beam by directing the laser vertically [2]. Gravity and the axial light force balance the particle vertically, while the particle is held radially in the same way as described above. These types of traps are often referred to as optical tweezers or laser tweezers.

Optical tweezers are used in a wide range of biological applications. One of the more common uses for optical tweezers is in characterizing protein binding strength. Proteins are glued to transparent spheres with a resin. The spheres are trapped by the optical tweezer and the protein is pulled apart [4]. The force at which the protein splits, is then derived from the strength of the optical tweezer. Optical tweezers are also used in connection with laser scalpels to perform in vitro fertilization. First a laser scalpel cuts off the tail of a sperm cell and then etches a hole in an egg. An optical tweezer is next used to capture the translucent sperm cell and manually insert it into the egg [5].

Optical tweezers aid in neuron growth by moving kinesin (a protein that stimulates neuron growth) slowly away from the neuron [6]. This helps neurons grow extraordinarily long, making them useful in repairing spinal injuries. DNA studies [7] and cancer research are also facilitated with the aid of optical tweezers [8].

## 1.2 Opaque Particles

Opaque particles are more difficult to trap because they absorb light instead of refract it. Figure 1.2.1 shows how this change in momentum pushes opaque particles out

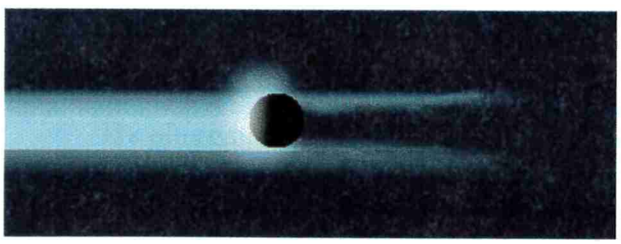


**Fig 1.2.1 Opaque particle in a focus.** A perfectly absorbing sphere is pushed out of the focus by the laser beam. Trapping is not possible.

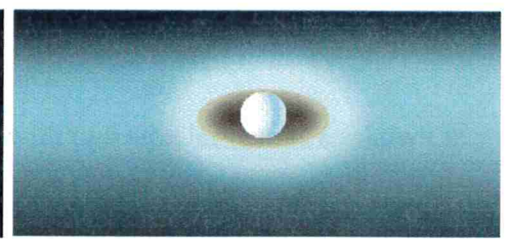


**Fig 1.2.2 Opaque particle in a dark trap.**

of the focus. For metallic particles (i.e. reflective), the forces are doubled because the change in momentum is twice as great as for absorbing particles; thus more complicated experiments are required to trap them. If Mie particles (diameter d >> $\lambda$) are surrounded by light, they may be confined in what is called a dark trap as shown in figure 1.2.2. When a particle tries to leave the dark region of the trap, light from the surrounding intense regions push the particle back into the trap. A dark trap can be created by directing a Laguerre-Gaussian beam (characterized by an on-axis intensity void) vertically and focusing tightly enough that the beam waist is smaller than the particle.

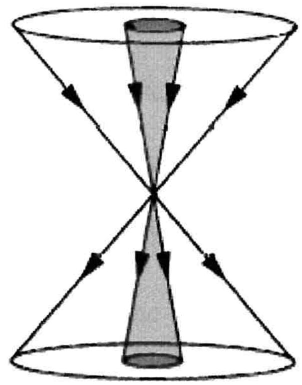


**Fig 1.2.3 Optical Vortex.** A focused Laguerre-Gaussian Beam resembling a funnel [24].

Figure 1.2.3 shows that the particle is confined radially by the surrounding light of the hollow beam, and axially by gravity and the narrowing focus. This can be compared with placing a marble in a funnel. If the marble is larger than the spout, it is trapped. Hollow beams have been created using: computer generated holograms [9], intercavity apertures [10], external mode converting optics [11], modulated wave plates [12] and obstructed laser beams [13].

By plugging a Laguerre-Gaussian beam with two smaller beams as shown in figure 1.2.4, a particle can be trapped three-dimensionally regardless of the orientation of the beam [14].

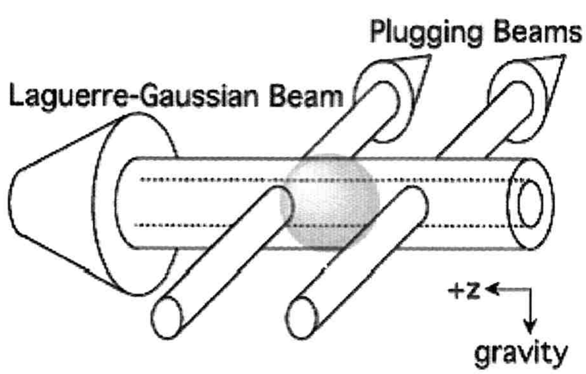



**Fig 1.2.4 Plugged beam.** A trapped particle in a plugged Laguerre-Gaussian beam [14].

Another way of creating dark regions in the focus is to have two interfering plane waves meet at the focus [15]. It is also possible to trap an opaque-particle by scanning a laser around the particle, this creates a time-averaged pocket [16]. Opaque-particle traps may have several physical applications. Because a levitated particle is a point-like source, scattering off of it leads to interesting complex spectroscopy with sharp resonances [17]. Levitation affords unique visibility, as its scattered light emerges unobstructed. Levitated particles can be used to measure applied forces on small objects, aerosols and droplets. Electric force measurements of charged levitated particles have led to the discovery of low photon emission rates as precise as a single-electron unit [18]. It may be possible to use levitated objects as laser fusion targets [19]. Optical levitation can also be used to validate present scattering theories [20].

## 1.3 Motivation

Our group in the Physics Department at BYU had been investigating a particle-levitation scheme. Dark pockets were created by imaging the shadow of a mask placed in the beam. We hoped that this would allow us to achieve a point-like scattering source [21]. Since we were working in this area, Dr. Larry Baxter of the BYU Chemical Engineering Department requested that we try to levitate his Black Liquor particles (a solvent used in the paper industry). In order to study the combustion of Black Liquor, it is useful to isolate a small sample to observe how it burns. Since extremely high temperatures are needed for combustion, the trap has to be purely optical. The Black

Liquor particles cannot be held in a magnetic trap, which requires the particle to be charged [22], because as a particle heats up for combustion it loses its charge.

## 1.4 Discovery


**Fig 1.4.1** **Trapped Black Liquor particle.**

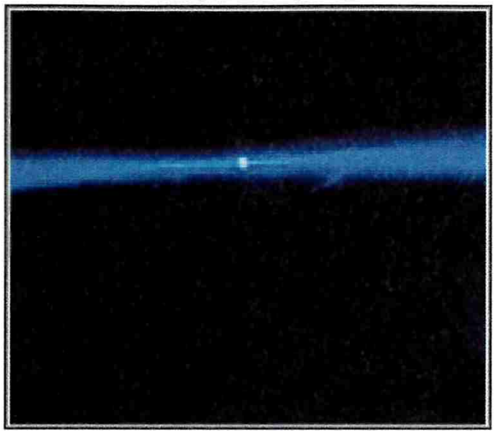


**Fig 1.4.2** **Trapped charcoal particle.**

While attempting to create a dark trap [21], I discovered a three-dimensional single beam levitation trap in October, 2001. This trap has successfully levitated opaque particles such as charcoal, graphite, Black Liquor, iron filings, nickel and magnesium oxide. It has also levitated metallic spheres including tungsten, silver and aluminum. We have held various particles for hours with ambient air pressures ranging from 647 Torr to 15 Torr. Several different alignments can be used to create the trap. Some of our set-ups favor single particle trapping while others allow for multiple particles to be trapped at one time. We have levitated particles with two different lasers and at various wavelengths ranging from 456nm-532nm. This thesis details a range of experiments, which characterize the behavior of the new trap. We postulate that the particles trapped are held in dark pockets within the laser focus.

# Chapter 2: Levitated Particles

Our most frequently used set-up directs a 5 Watt, 532 nm laser beam through a negative 50 cm lens. This creates a gradually diverging beam that we direct horizontally onto a 5 cm focal-length spherical mirror. The mirror creates a sharp focus where we levitate the particles. A similar set-up that replaces the spherical mirror with an 8 cm focal length lens is also used. Particles are loaded into the trap by dipping a sewing needle into a sample of particles. The needle is then either spun above the focus until a particle falls off the needle into the trap, or waved through the focus until a particle is knocked off the needle by the beam and becomes trapped.

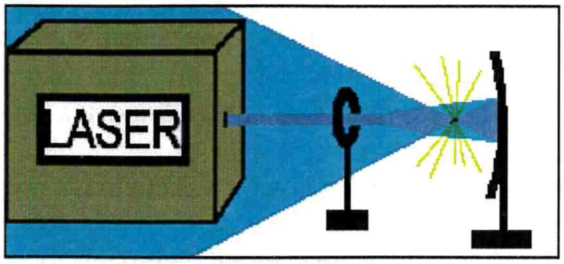



**Fig 2.0.1 Lens-mirror set-up.**

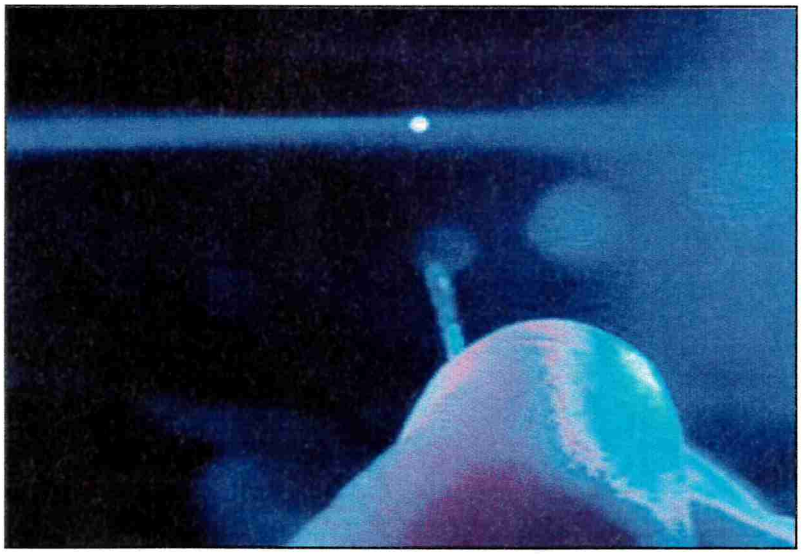


**Fig 2.0.2 Method of loading.** The authors thumb and sewing needle used to load the trap.

## 2.1 Pressure

Trapping forces are large enough to overcome gravity, but not large enough to overcome even small air currents. Our first experiments were performed on an open optics table and were heavily influenced by the surrounding air. A 3" cubic plexi-glass enclosure, resembling the glass cube used by Ashkin [17], was incorporated into the set-up to keep air from knocking the particles out of the trap. Our enclosure differed from Ashkin's, in that ours had a hole drilled in the side to allow the laser to enter the box undisturbed.

A vacuum chamber was also assembled (from existing parts) to more effectively reduce air currents and verify that the phenomenon of trapping is not a product of convection currents caused by localized particle heating (see Sect. 2.2). The chamber is capable of achieving pressures as low as 10e-6 Torr. We back-fill the chamber with helium and again with argon and found no apparent effect on the trap compared to air at the same pressure.

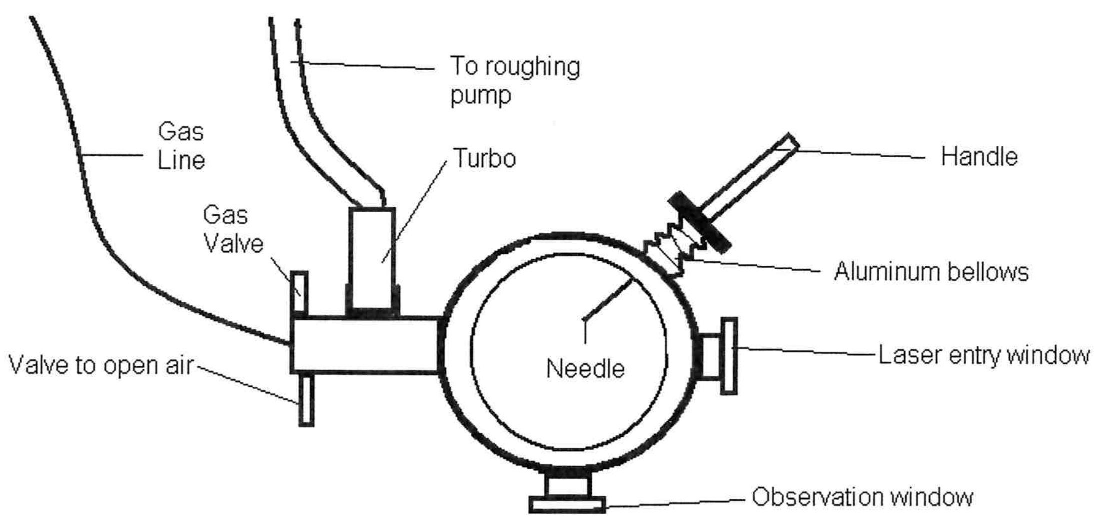



**Fig 2.1.2** **Vacuum chamber.** A top view schematic of our vacuum chamber.

Table 2.1.1 gives a list of pressures and laser powers at which we observed optical trapping. We have been able to levitate particles in pressures as low as 7 Torr to atmospheric (we have not explored pressures above atmosphere). At low pressures many interesting properties of the trap were observed. Particles were trapped on- and sometimes off-axis of the laser beam both upstream and down from the focus. Some particles could be seen orbiting around the focus. The smallest particles (graphite) would complete as many as 10 orbits before falling out of the trap, traversing as much as

| Magnesium Oxide (100 mesh) | Pressure | Starting Power | End Power | Tungsten (4-6μ) | Pressure | Starting Power | End Power |
|---|---|---|---|---|---|---|---|
| | 7 Torr | 3 Watts | 1.5 Watts | | 26 Torr | 5 Watts | .44 Watts |
| | 23 Torr | 5 Watts | .05 Watts | | 29 Torr | 5 Watts | .46 Watts |
| | 33 Torr | 3 Watts | .05 Watts | | 34 Torr | 5 Watts | .69 Watts |
| | 40 Torr | 2.5 Watts | .22 Watts | | 36 Torr | 5 Watts | .36 Watts |
| | 42 Torr | 5 Watts | .01 Watts | | 38 Torr | 5 Watts | .21 Watts |
| | 44 Torr | 2 Watts | .02 Watts | | 40 Torr | 5 Watts | .51 Watts |
| | 46 Torr | 2 Watts | .02 Watts | | 52 Torr | 5 Watts | .41 Watts |

**Table 2.1.1 Pressures and powers** at which particles were trapped.

10 cm each orbit. At both high and low pressures, particles could be seen hopping from one fixed trapping point to another. Particles would hop both upstream and down, before settling into fixed locations. As yet, we have no conclusive explanation for these phenomena. It may be due to mode variations in the laser beam.

## 2.2 Convection Currents

A number of optical traps have been designed that utilize particle heating from the laser, to create convection currents. Such convection currents can levitate the particles. One reason for investigating the trap at low pressure was to rule out convection currents as the mechanism for our trap. It is unlikely that there is enough air at 7 Torr to create convection currents that would contribute significantly to the trap. As we attempted to trap at even lower pressures, it became apparent that some dispersive medium is required to trap. This confirmed Ashkin's postulate that a dispersive medium is required to load

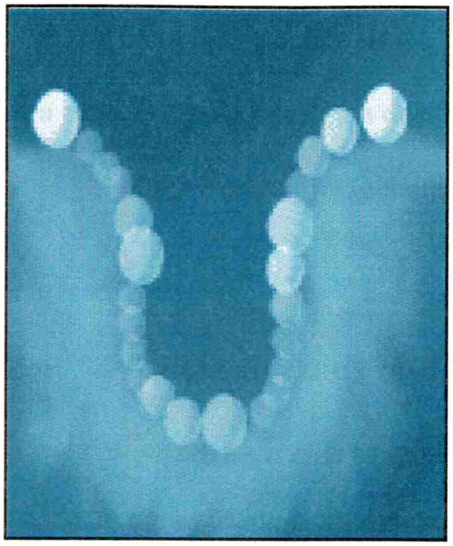


**Fig 2.2.1 Potential well.** A particle goes in and out of a frictionless potential well

the trap [17]. If a particle falls into a potential well (i.e. enters the trap), in the absence of friction all of the particle's potential energy is transformed into kinetic energy, enough to exit from the other side of the well.

A strong argument that convection currents are not responsible for our trap is found in the set-up of the experiment. In all convection current traps the laser beam is directed vertical. This allows for uniform heating of the underside of the particle and creates a stable trap. Our alignment is horizontal, and it is unlikely that the particle could heat uniformly across the bottom of the particle, owing to the propagation direction of the light.

**Fig 2.2.2 Convection currents.** Heating caused by a laser incident on a trapped particle.

## 2.3 Trapping Duration

The stability of the trap is dependent on the alignment of optics and on the absence of air currents. We have held Black Liquor particles for more than 3 hrs at 647 Torr, on an open air optics table. At 35 torr we have held particles consistently for 6 hrs at a time. Trapping duration does not seem to be particle-dependent; it is subject to the strength of the limiting factors that knock the particles out of the trap.

## 2.4 Particle Species and Size

We have been able to trap all species of opaque particles that we have introduced into the trap, including metallic reflective particles, though not all sizes. Particle shape has not been investigated thoroughly, but it does not appear to be important as some of the particles levitated were poorly characterized dust samples. It is unlikely that they were spherical or symmetrical, which is necessary for some types of traps [9, 14, 15]. We have trapped charcoal, black liquor, graphite, nickel dust, iron filings and magnesium oxide. We have also trapped spherically engineered metallic particles that include: tungsten, aluminum, and silver.

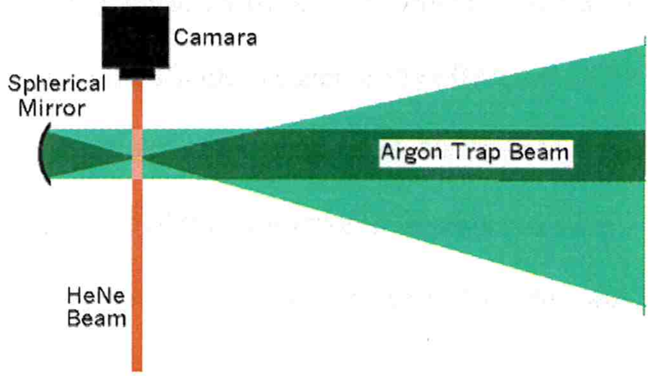



**Fig 2.4.1 Size detection set-up.** Top view of the size detection setup. A HeNe beam incident on a trapped particle.

| Particle | Size | Trapped | Unable to Trap |
|---|---|---|---|
| Silver | 1.3-2.3 $\mu$ | X | |
| Tungsten | 1-5 $\mu$ | X | |
| Tungsten | 4-6 $\mu$ | X | |
| Tungsten | 12 $\mu$ | | X |
| Aluminum | 10-14 $\mu$ | X | |
| Aluminum | 17-30 $\mu$ | | X |
| Nickel | 7-9 $\mu$ | X | |
| Iron | 162 mesh | X | |
| Magnesium Oxide | 100 mesh | X | |
| Charcoal | unknown | X | |
| Graphite | 325 mesh | X | |
| Black Liquor | unknown | X | |

**Table 2.4.1 Particle Sizes.** Trap performance with particles of differing make and size.

To view the size of particles while in the trap, a HeNe laser beam was directed through the trap focus. We hoped to determine the size of the particles from a diffraction pattern caused by the particle, but none was observed. We determined that the Black Liquor particles that we trapped were significantly smaller than 10 microns because no diffraction pattern was observed in the emerging probe laser beam, even when HeNe light was observed to be scattered from the particle. This upper limit on the particle size was confirmed by observation with a microscope. Because our HeNe beam was roughly 1 mm in diameter, a 10 micron particle could only block less than .01% of the beam, making it difficult to observe any effect.

We tried focusing the HeNe beam to a smaller beam waist so that a higher percentage of the beam would be scattered away. We found it extremely difficult to align the beam so that the focus of the HeNe beam was incident on the trapped particle. When it appeared that they were aligned, the particle was knocked out. This difficulty in characterizing the particles, led us to purchase well characterized particles from Bioworld. Table 2.4.1 shows the particle sizes that were trapped.

## 2.5 Rayleigh Particles

A Rayleigh particle is one that is much smaller than a wavelength (d << $\lambda$). It is possible that the trapped particles from the poorly characterized samples (e.g. Black Liquor, graphite), were small enough to act as Rayleigh particles. Rayleigh particles are trapped in a different manner than Mie particles. One way to verify if the particles trapped are Rayleigh, is to see if light scattered from the trapped particle is directional. When placed in the polarized laser, Rayleigh particles oscillate along the electric field axis. This oscillation should cause the scattered light to be directed perpendicular to the electric field polarization of the laser beam. If the particles are Rayleigh we should not be able to see any light scattered in the direction parallel to the field of polarization. However, the observed scattered light from all of our particle samples did not vary in intensity with beam polarization orientation; subsequently the trapped particles cannot be Rayleigh particles.

# Chapter 3: Outlook

## 3.1 Spherical Aberration

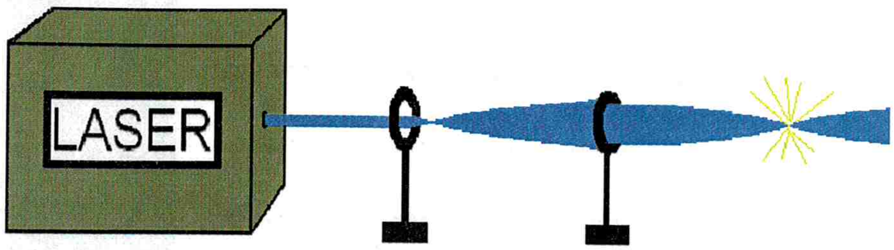



**Fig 3.1.1 Original setup.** Horizontal beam on open optics table is expanded by first mirror and tightly focused by second lens. The focus of the second lens is the location of the trap.

We first hypothesized that spherical aberration was the mechanism causing our trap. Our initial set-up was outside of the vacuum chamber and used a 200 mW beam. The laser beam was passed through an 8 cm lens that focused the light then expanded the beam until it nearly filled a 5 cm lens. The lens then focused the light creating our trap. When we removed the first lens we found that we could not trap. It appeared that a complete filing of the second lens was necessary to make the trap, in which case there would be significant spherical aberration present. To show that spherical aberration was the reason the particles were being trapped, all optics were then removed; and an infinity corrected microscope objective that would limit spherical aberration was inserted in the beam path. A similar f# to that which we had in our original system was created.


**Fig 3.1.2 Microscope objective set-up.** Infinity corrected microscope focusing the horizontal beam.

However, we found that we were unable to trap particles in the focus. This led us to postulate that the interference pockets created by spherical aberration were responsible for the trapping phenomenon (see figure 3.1.1).

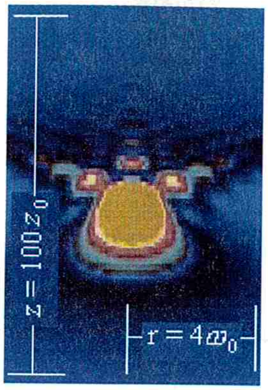



**Fig 3.1.3 Spherical aberration plot.** Intensity plot of a focus with spherical aberration using f# 4. The 532nm light propagates from top to bottom.

However, a comparison of trapping locations found in our trap with computational predictions of where the interference pockets should be, showed that spherical aberration is not the primary mechanism responsible for trapping. Interference pockets caused by spherical aberration are calculated to occur only upstream, as shown in figure 3.1.1. However, experimentally trapping locations appear both upstream and downstream. We repeated the experiment with the infinity-corrected microscope objective, this time inside the vacuum chamber with the laser power increased to 5 Watts. In this case, we were able to trap particles. We concluded that while spherical aberration may play a role in loading the trap, it cannot be the primary mechanism of the trap.

## 3.2 Interference From a Circular Aperture

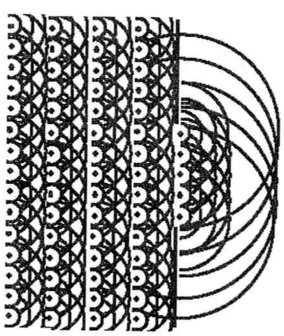


**Fig 3.2.1 Huygens's wavelet interpretation.** Plane wave through an aperture [23].

Our present hypothesis is that the optical trapping is due to dark pockes resulting from Fresnel diffraction from the hard aperture of the laser cavity. Light can be thought of as tiny wavelets that interfere to create new wave fronts (see figure 3.2.1). A wave front sent through an aperture will oscillate in

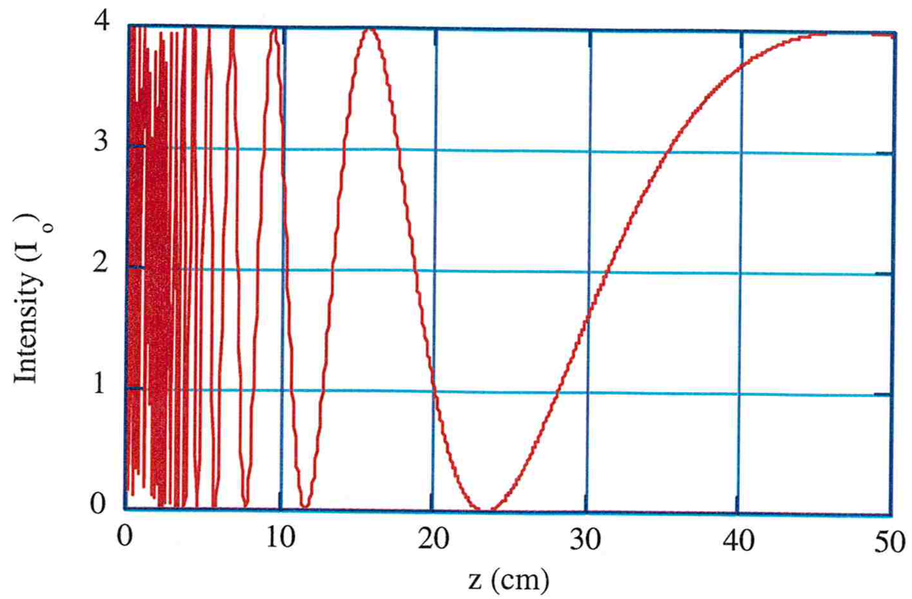



**Fig 3.2.2 Fresnel diffraction intensity plot.** A monochromatic plane wave incident on a 1mm cylindridcal aperture.

intensity along the axis of propagation. Above is a plot of the on-axis intensity of a laser beam passing through an aperture as a function of distance from the aperture (see Appendix A). As is evident, many regions of low intensity occur after the aperture. The dark regions are presumably imaged to locations near the focus when a short-focal-length lens is inserted into the beam.

In future work, we will check whether particles are being trapped in the dark regions caused by the interference from the laser aperture. Calculations must be compared with experimental data to see if they agree.

## 3.3 Future Work

In addition to testing whether the on-axis intensity of the beam oscillates to form the trapping regions, future research will include analysis of the light scattered from the trapped particles. Focusing the scattered light from the trapped particle onto a CCD

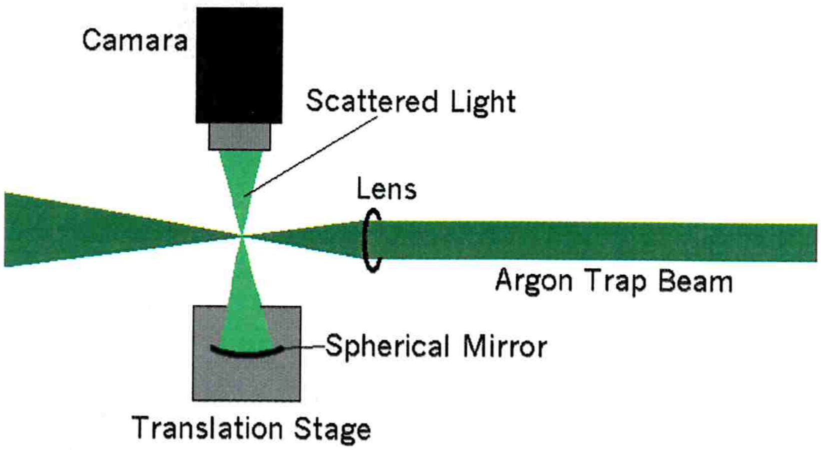



**Fig 3.3.1 Scattered light set-up.** Scatter ed light from a trapped particle is focused back on itself and interference is detected with a CCD camera.

camera could allow us to establish the particle's size. Observing fringe patterns caused by the interference of scattered light could indicate the kinetic dynamics of the trapped particle; if the particle moves around in the trap, the fringe patterns presumably would 'wash out.' We set up a spherical mirror (focal length = 5 cm) perpendicular to the trapping beam. The mirror was placed on a translation stage so the scattered light could be focused back onto the trapped particle. We predicted that this would cause interference that could be detected by the camera. Ring patterns (fig 3.3.2) as well as vertical stripes (fig 3.3.3) were observed with

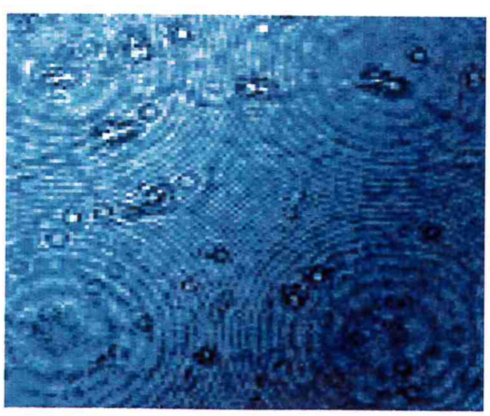
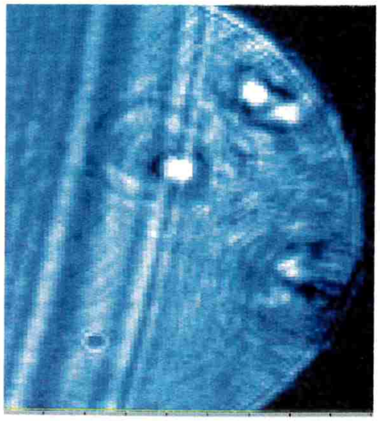


**Fig 3.3.2 Ring patterns.**



careful alignment, but we were unable to determine if this was an artifact of a thin glass plate that had been placed in front of the camera to protect it from dust. In order to obtain useful data, it will be necessary to perform the experiment again without the glass plate in place.

**Fig 3.3.3** **Fringe patterns.**

## 3.4 Conclusion

We have discovered a simple optical levitation trap. We have shown that this trap can levitate both metallic and opaque particles. Shape, symmetry and specie of particles are not critical to trapping performance. We have proven that neither convection currents nor spherical aberration are the primary mechanism by which the particles are trapped.

Our current hypothesis is that the trap is caused by Fresnel diffraction due to a hard aperture of in the laser cavity. More work is needed to more fully characterize the trap and the mechanism by which the particles are levitated. Applications for the trap may include point-like scattering and single particle analysis. This trap is unique and simple and will undoubtedly lead to other applications.

## References

[1] W. Paul, Angew. Chem. Int. Ed. Engl. **29**, 739-741 (1990).

[2] A. Ashkin, "Optical trapping and manipulation of neutral particles using lasers," Science **94,** 4853-4860 (1997).

[3] A. Ashkin, "Acceleration and trapping of particles by radiation pressure," Phys. Rev. Lett. **24**, 156-159 (1970).

[4] K. Svoboda, P.P. Mitra, S. M. Block, "Fluctuation analysis of motor protein movement and single enzyme kinetics," *Proceedings of the National Academy of Sciences of the U.S.A.* , (Dec 6,1994), Vol. 91, pp. 11782-11786.

[5] K. Schütze, and A. Clement-Sengewald, "Catch and move - cut or fuse," Nature **368**, 667-670 (1994).

[6] C. M. Coppin, *et al.*, "Detection of sub-8-nm movements of Kinesin by high-resolution optical-trap microscopy," *Proceedings of the National Academy of Sciences of the U.S.A.,* (Mar. 5, 1996), Vol. 93, pp. 1913-1917.

[7] C. G. Bauman, *et al.,* "Ionic effects on the elasticity of single DNA molecules," *Proceedings of the National Academy of Sciences of the U.S.A*, (June 10, 1997), Vol. 94, pp. 6185-6190.

[8] X. Wei, *et all,* "Mapping the sensitivity of T-Cells with an optical trap," *Proceedings of the National Acadmey of Sciences of the U.S.A*, (July 20, 1999), Vol. 96, pp 8471-8476.

[9] H. He, *et al.*, "Optical particle trapping with higher-order doughnut beams produced using high efficiency computer generated holograms," J. of Mod. Opt., **42**, 217-223 (1995).

[10] S. Sato, M. Ishigure, and H. Inaba, "Applications of higher-order mode Nd:Yag laser beam for manipulation and rotation of biological cells," *Conference of lasers and Electro-optics*, Vol. **10**, pp. 280-281 (1991).

[11] N. B. Simpson, *et al.*, "Mechanical equivilence of spin and orbital angular momentum of light: an optical spanner," Optics Letters **22**, 52-54 (1997).

[12] J. L. Chaloupka, Y. Fisher, T. J. Kessler, and D. D. Meyerhofer, "Single-beam, ponderomotive-optical trap for free electrons and neutral atoms," Optics Letters **22**, 1021-1023 (1997).

[13] M. Gu, and D. Morrish, "Three dimensional trapping of Mie metallic particles by the use of obstructed laser beams," J. of Appl. Physics **91**, 1606-1612 (2002).

[14] T. Kuga, *et al.,* "Novel Optical trap of Atoms with a doughnut beam," Phys. Rev. Lett. **78**, 4713-4716 (1997).

[15] K. Sasaki, *et al.*, "Optical trapping of a metal particle and a water droplet by a scanning laser beam," Appl. Phys. Lett. **60**, 807-809 (1992).

[16] M.P. MacDonald, *et al.*, "Trapping and manipulation of low-index particles in a two dimensional interferometric optical trap," Optics Letters **26**, 863-865 (2001).

[17] A. Ashkin, "Applications of laser radiation pressure," Science **210**, 1081-1088 (1980).

[18] J. Arnold, Journal of Aerosol Science **10**, (1979).

[19] G. Roosen, *et al.*, Optical Communication **24**, (1978).

[20] L. Khaykovich, *et al.*"Ultrasensitve two-photon spectroscopy based on long spin - relaxation tie in a dark optical trap," Europhysics Letters **50**, 454-459 (2000).

[21] B. A. Belleville, "Single-beam laser traps for opaque or reflective particles," Senior Thesis, Brigham Young University (2002).

[22] L. Baxter, (private communication).

[23] Justin Peatross, "Physics of Light and Optics," On-line text 2001, Chapter 10, optics.byu.edu.

[24] K.T. Gahagan, and G. A. Swartzlander, "Optical vortex tapping of particles," Optics letters **21**, 827-829 (1996).

## Appendix A Fresnel Calculations

In this appendix we calculate the on-axis intensity via the Fresnel-Kirchhoff diffraction formula of a monochromatic plane wave with intensity $I_o$ incident on a circular aperture:

$$E(0,0,d) = \frac{-i}{\lambda} \iint E(x',y',0) \frac{e^{ik\sqrt{x'^2+y'^2+d^2}}}{\sqrt{x'^2+y'^2+d^2}} dx'dy' \qquad \text{(A.1)}$$

We first employ a change of variables:

$$\begin{aligned} x' &= \rho' \cos\theta \\ y' &= \rho' \sin\theta \\ x'^2 + y'^2 &= \rho'^2 (\cos^2\theta + \sin^2\theta) = \rho'^2 \\ dx' &= -\rho' \sin(\theta) d\theta' d\vartheta' \\ dy' &= \rho' \cos(\theta) d\theta' d\vartheta' \\ \sqrt{dx'^2 + dy'^2} &= \rho' d\rho' d\theta' \end{aligned} \qquad \text{(A.2)}$$

This yields the following integral:

$$E(0,0,d) = \frac{-iE_0}{\lambda} \int_0^{2\pi} d\theta' \int_0^{\ell/2} \rho' \frac{e^{ik\sqrt{\rho'^2+d^2}}}{\sqrt{\rho'^2+d^2}} d\rho' \qquad \text{(A.3)}$$

where the diameter of the aperture is $\ell$.

Another change of variables

(A.4)

$$\zeta = \sqrt{\rho'^2 + d^2}$$
$$\rho' = \sqrt{\zeta^2 - d^2}$$
$$d\rho' = \frac{\zeta}{\sqrt{\zeta^2 - d^2}} d\zeta$$

gives,

(A.5)

$$E(0,0,d) = \frac{-iE_0}{\lambda} \int_0^{2\pi} d\theta' \int_d^{\sqrt{(l/2)^2 + d^2}} \frac{\zeta\sqrt{\zeta^2 - d^2}}{\zeta\sqrt{\zeta^2 - d^2}} e^{ik\zeta} d\zeta = e^{ik(\zeta, \sqrt{(l/2)^2 - d^2} - d)}$$

Which gives us the on-axis intensity

(A.6)

$$\psi = \sqrt{(l/2)^2 - d^2} - d$$
$$I = n\varepsilon_0 c E_0' \cdot E_0^*[e^{ik(\psi - d)} e^{-ik(\psi - d)}] = \frac{n\varepsilon_0 c}{2}[2 + e^{ik(\psi - d)} + e^{-ik(\psi - d)}]$$
$$I = 2n\varepsilon_0 \overline{E_0} \cdot \overline{E_0^*}[1 - \cos(k(\psi - d))] = 2I_0[1 - \cos(k(\sqrt{(l/2)^2 - d^2} - d))]$$