LOGISTIC MAPS AND SYNCHRONIZATION

by

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ABSTRACT

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Chaotic systems are frequently encountered in nature. Their continued discovery imparts relevance to the effort being made to understand the behavior of chaotic dynamics. The wide occurrence of chaos in nature shows that the chaos found in many simple mathematical models is not a trite ancillary mathematical effect but adumbrates a profound natural phenomenon. The logistic map (LM) is frequently cited as one such simple model capable of exhibiting chaotic behavior and is used widely as a pedagogical tool. It provides a proverbial stepping stone toward expanding our understanding of the path dynamical systems take toward chaos. Also, the 2-D LM serves as a convenient tool for studying the synchronization behavior encountered frequently in coupled chaotic systems. Complete synchronization can be shown to occur in the coupled logistic map for certain values of the coupling constant. Furthermore, intermittent synchronization precedes the onset of complete synchronization.

Analytic techniques can be used to precisely determine the onset of complete synchronization. A general analytic method, however, for predicting regions of synchronization remains to be found.

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Chapter 1

The logistic equation: A review

1.1 Introduction

The ubiquity of chaos in the natural world has given rise to a plentitude of fascinating science. Phenomenologically inherent to many naturally occurring systems, chaotic dynamics are encountered on the microscopic scale as well as the macroscopic scale. From the rotation of several gravitationally attracted large bodies, to the dynamics of viscous flow or population growth, chaos is widely prevalent. The importance of understanding the origin and behavior of chaos is incontrovertible; and though a relatively new science there already exists a wealth of literature on the topic.

In the present paper, chaos is explored by means of the deceptively simple *logistic map*(LM). Often cited as a canonical example of chaotic behavior and used frequently solely for didactic purposes, the LM is a natural model useful in a variety of applications. The *logistic equation* from which the LM originates is an ordinary differential equation introduced first as a demographic model in 1838 by Pierre Francois Verhulst [1]. One arrives at the typical form of the logistic equation from Verhults' by a series of simple variable changes shown in 1.2.

The deeper ramifications of the logistic model, specifically its chaotic behavior, were reviewed in a seminal paper by the biologist Robert May in 1976 [2]. Indeed, numerical and analytical efforts to understand the behavior of the LM have continued steadily since that time. The happy advent of computers has provided an invaluable tool for insight into chaotic dynamics. The present paper provides a summary and extension of a few of the many results related to the LM.

1.2 Verhults equation

The logistic equation was written in the following form by Verhulst in his population model published in 1845 [1].

$$\frac{dN}{dt} = \frac{rN(K-N)}{K} \tag{1.1}$$

Here r is called the Malthusian parameter, and K is referred to as the carrying capacity— to see why, simply let N go to K,

$$\frac{dN}{dt} = \frac{rN(K-K)}{K} \equiv 0. \tag{1.2}$$

Thus, when N equals K the population growth is zero and is at a maximum. The logistic equation, however, is most often written in a different form by letting $x \rightarrow N/K$,

$$k\frac{dx}{dt} = rx(1-x). \tag{1.3}$$

Then redefining r by $r \rightarrow r/K$,

$$\begin{aligned} \frac{dx}{dt} &= rx(1-x) \\ \dot{x} &= rx(1-x). \end{aligned} \tag{1.4}$$

I have introduced the common notation, $\dot{x} = dx/dt$. As a result of the rescaling, x is now between 0 and 1. Further, r (called the biotic constant or potential) represents the largest possible value of growth.

1.3 Sigmoid curves

With this form of the logistic equation as a starting point, the a familiar *sigmoid solutions* are readiably obtainable;

$$\frac{dx}{dt} = rx(1-x). \tag{1.5}$$

Using the partial fractions method, this is easily separable and integrable.

$$\frac{1}{rx(1-x)} = \frac{A}{rx} + \frac{B}{1-x} \tag{1.6}$$

$$A - Ax + Brx = 1 \tag{1.7}$$

$$A = 1 \quad B = \frac{1}{r} \tag{1.8}$$

$$\int_{x_0}^{x} \frac{1}{rx} + \frac{1}{r(1-x)} dx = \int_{t_0}^{t} dt \tag{1.9}$$

For convenience and without loss of generality, let $t_0 = 0$:

$$\frac{1}{r}\left([\ln(x) - \ln(1-x)] - [\ln(x_0) - \ln(1-x_0)]\right) = t \tag{1.10}$$

$$\frac{1}{r}\ln\left(\frac{x}{1-x}\frac{1-x_0}{x_0}\right) = t. \tag{1.11}$$

Solving (1.11) for x takes a little algebra but gives the following result:

$$x(t) = \frac{1}{1 + e^{-rt}(\frac{1}{x_0} - 1)}. \tag{1.12}$$

Functions of this form are called *sigmoid functions*. Sigmoid functions plotted for several initial values of x are shown in Figure 1.1.

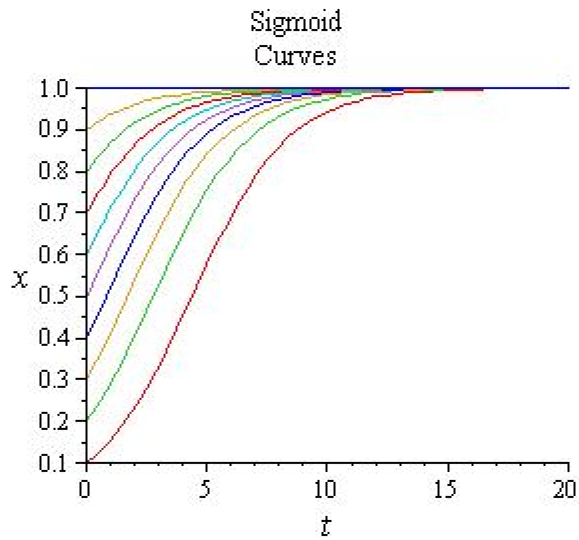

Figure 1.1 Sigmoid curves for several values of x_0. The horizontal line at $x = 1$ corresponds $x_0 = 1$

1.4 Discretization

With the logistic equation solved analytically, we can observe that the behavior is always non-chaotic. Discretization, however, introduces the possibility of chaotic behavior in the LM. The motivation behind discretization is primarily pragmatic: Numerical analysis of a *differential equation* requires the transformation into a *difference equation*. The transformation to a discretized equation, however, is not without its complications: one dimensional differential equations never display chaotic behavior, while one dimensional difference equations do so frequently. Indeed, the emergence of chaos in difference equations is often inherent in the structure, i.e., the LM.

The LM is obtained by discretizing the logistic equation not by the common Euler method but in the following manner:

$$\begin{aligned} \dot{x}(t) &\rightarrow x_{n+1} \\ x(t) &\rightarrow x_n \end{aligned} \tag{1.13}$$

Then making these substitutions, the logistic equation becomes

$$x_{n+1} = rx_n(1 - x_n). \tag{1.14}$$

A discrete relation of the above form is also sometimes generally reffered to as a quadratic map, because it is in the form of a quadratic equation. Tabulating the first few iterations and choosing x_0 to represent the initial value yields,

$$x_1 = rx_0(1 - x_0) \tag{1.15}$$

$$x_2 = rx_1(1-x_1) = rx_0(1-x_0)(1-rx_0(1-x_0)) = r^2x_0-(r^3+r^2)x_0^2+2r^3x_0^3-r^3x_0^4 \tag{1.16}$$

$$x_3 = rx_2(1 - x_2) = r^3x_0(1 - x_0)\,(1 - rx_0(1 - x_0))\,(1 - r^2x_0(1 - x_0)(1 - rx_0(1 - x_0)) \tag{1.17}$$

As you can see, calculating the values in this manner gets complicated very quickly with the order of the polynomial doubling with each each iteration. This hints at the complexity introduced by discretizing even a simple differential equation.

Note also that the behavior of the logistic map depends on the parameter r. For certain values of r there exist exact solutions which give the value after t-iterations as a function of the initial point x_0. Two exact solutions occur at $r = 2$ and 4 and take the following form [3]

$$r = 4 \rightarrow \sin^2(2^t \sin^{-1}(\sqrt{x_0})) \tag{1.18}$$

$$r = 2 \rightarrow \frac{(1 - (1 - 2x_0)^{2t})}{2}. \tag{1.19}$$

From these exact solutions the general behavior of the logistic map is apparent: Let us look at the behavior of the map at $r = 2$ and 4 in the limit as t goes to infinity. When $r = 4$ the solution is the square of a periodic function, $\sin^2[2^t \sin^{-1}(\sqrt{x})]$, and in the limit as t goes to infinity is bounded by 0 and 1. We should note, then, that the map is chaotic at $r = 4$. When $r = 2$ however, the solution for $x \in (0, 1)$ converges; A *stationary point* thus exists.

$$\lim_{t \to +\infty} \frac{[1 - (1 - 2x)^{2t}]}{2} = \frac{1}{2} \tag{1.20}$$

From these two exact solutions we can hypothesize (correctly) that the map must evolve from non-chaotic behavior (one-period to be precise) to chaotic behavior at least by the time $r = 4$. This, of course, does not rule out the possibility of intermittent regions of non-chaotic behavior even after the map has generally become chaotic. The bifurcation diagram obtained by numerically iterating the LM shown in Fig 1.2 verifies this prediction.

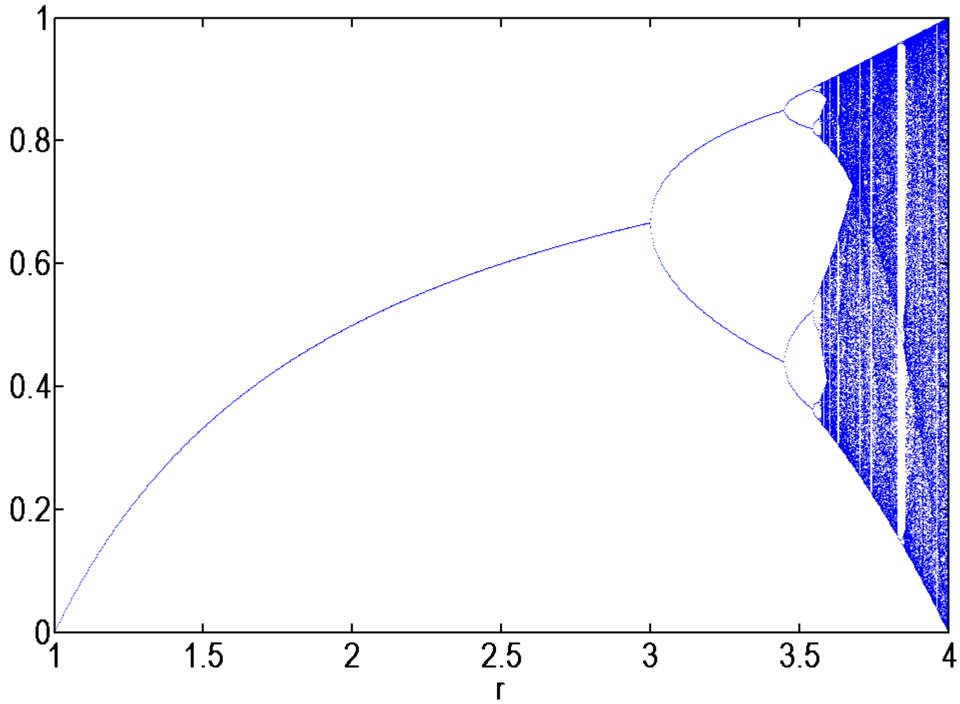

Figure 1.2 A bifurcation diagram of the logistic map. The first bifurcation occurs at $r = 3$

Chapter 2

Fixed points, bifurcation and roots

2.1 Introduction

In the case of the logistic map, we can calculate the fixed points as a function of the parameter r by noting that the condition $x_{n+a} = x_n$ must be satisfied and by substituting into the mapping function. When $a = 1$ we have,

$$x_n = rx_n(1 - x_n) \tag{2.1}$$

We may write this without the subscripts and solve for x,

$$x = rx(1 - x) \rightarrow x(1 - r + rx) \tag{2.2}$$

$$x = 0, \frac{r-1}{r}. \tag{2.3}$$

Notice that the non-zero root verifies the result presented in Chapter 1 from the exact solution, namely that the trajectories converge to 1/2 when $r = 2$.

For the $a = 2$ case we get a quartic equation resulting in two separate stationary points. This is appropriately called a two-cycled attractor because the trajectories oscillate between two stationary points.

$$x = r(rx(1 - x))(1 - rx(1 - x)) \tag{2.4}$$

$$x = 0, \frac{r-1}{r}, \frac{r+1 \pm \sqrt{r^2-2r-3}}{2r} \tag{2.5}$$

It is readily seen that $(r+1\pm\sqrt{r^2-2r-3})/2r$ are imaginary until $(r-3)(r+1)=0$. Thus, we should expect to see some behavioral difference in the map at $r = 3$. Further investigation reveals that, indeed, the stationary point found in the $a = 1$ case becomes unstable at $r = 3$, while the newly found $a = 2$ roots are stable. The change from a single stationary point to two results in a *bifurcation*. The $a = 1$ root is plotted with the $a = 2$ roots in 4.6 showing a bifurcation. This agrees exactly with the *bifurcation diagram* shown in Chapter 1.

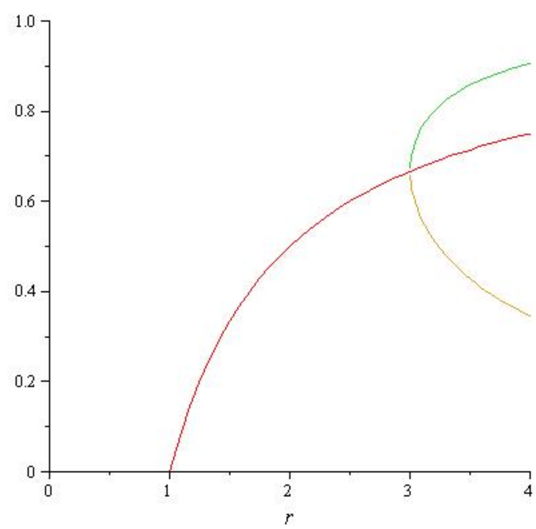

Figure 2.1 The first bifurcation of the logistic map occurs at $r = 3$. The first root becomes unstable at this point.

2.2 Root behavior of the logistic map

The results above indicate that the emergence of new roots leads to a bifurcation. We can understand in a more general way the root behavior of iterative functions by the following simple analysis. Begin by defining a *homogenous iterative function* to be any composite function of the form

$$F \circ F \circ ... \circ F(x) = 0 \tag{2.6}$$

Now let the roots of the above equation be denoted by x_r.Starting with the zeroth iteration, $F(x) = 0$, we will have roots $x_1, ..., x_n$ where n is the order of the polynomial map F. The roots are in general complex. Now let us look at F composed once with itself. This gives the second iteration,

$$F \circ F(x) = 0. \tag{2.7}$$

If we substitute a solution of $F(x) = 0$, x_r, into this we have

$$F \circ F(x_r) = 0. \tag{2.8}$$

And so by definition of polynomial roots we have

$$F(0) \stackrel{?}{=} 0 \tag{2.9}$$

which is clearly not true in general, since a general function evaluated at zero is certainly not always zero.

Now let a non-homogenous iterative function be any function of the form

$$F \circ F \circ ... \circ F(x) = g(x) \tag{2.10}$$

Roots of this relation, again denoted by $x_1, ..., x_n$, are then solutions to the following:

$$F \circ F \circ ... \circ F(x) - g(x) = 0. \tag{2.11}$$

n is determined by the order of the entire left hand side. For simplicity let us look at the above relation letting $g(x) = x$.

$$F \circ F \circ ... \circ F(x) - x = 0 \tag{2.12}$$

A root of this equation, then, can also be viewed as a stationary point of a map, $x_{n+1} = M(x_n)$. To reiterate, a stationary point x_0 is any point that satisfies $M(x) = x$ and is therefore also a solution of $M(x) - x = 0$. Starting with a stationary point x_0, let us substitute into a general iteration, $F \circ F \circ ... \circ F(x) - x = 0$.

$$F \circ F \circ ... \circ F(x_0) - x_0 \stackrel{?}{=} 0 \tag{2.13}$$

And as before, we ask whether this is true in general. By definition of a stationary point, $F(x_0) = x_0$, and so the equation reduces to what we started with.

$$F(x_0) - x_0 = 0 \Rightarrow x_0 - x_0 = 0 \tag{2.14}$$

Thus, a solution of $F(x) - x = 0$ is a general root of all iterations of the map $x_{n+1} = F(x_n)$.

We can give credence to this result with the help of a symbolic math program like Maple. Using the `@` symbol, Maple will compose functions saving us a good deal of effort. Let us find the roots of the LM for several iterations. Again we use the LM in the usual form $x_{n+1} = rx(1 - x)$. We define a function on Maple as follows,

```
F:=x->r*x*(1-x)
```

Maple can then be used to solve the map after the zeroth iteration, $rx(1 - x) = x$. The roots are 0 and $(r - 1)/r$ in agreement with what was shown in Section 2.1. Now using the `@` command, `(F@F)(x)`, Maple gives the following solution.

$$r^2 x (1 - x) (1 - rx (1 - x)) \tag{2.15}$$

Finally, using the `solve` command, `solve((F@F)(x)-x = 0, x)` yields four solutions,

$$0, \frac{r-1}{r}, \frac{1/2r + 1/2 + 1/2\sqrt{r^2 - 2r - 3}}{r}, \frac{1/2r + 1/2 - 1/2\sqrt{r^2 - 2r - 3}}{r} \tag{2.16}$$

For the second iteration $F^2(x)$, the command `(F@F@F)(x)` gives

$$r^3 x (1 - x) (1 - rx (1 - x)) \left(1 - r^2 x (1 - x) (1 - rx (1 - x))\right) \tag{2.17}$$

And we get eight roots,

$$0, \frac{r-1}{r}, \ldots$$

Even though Maple failed to explicitly solve for all eight roots, it still found the roots of the zeroth iteration in agreement with the analytical result. Let us do one more iteration, $F^3(x) - x = 0$. Solving gives,

$$0, \frac{r-1}{r}, \frac{1/2\,r + 1/2 + 1/2\,\sqrt{r^2 - 2\,r - 3}}{r}, \frac{1/2\,r + 1/2 - 1/2\,\sqrt{r^2 - 2\,r - 3}}{r}, \ldots$$

And again as expected, 0 and $(r-1)/r$ are still roots. These results verify the claim made above that the zeroth-iteration yields roots that are solutions to all iterations.

2.2.1 Higher order roots

We have only verified that the roots to the zeroth iteration are roots to all iterations. It is natural to ask what happens for higher order roots. Let us look in the general case at roots of the first iteration, $F \circ F(x) - x = 0$. There will be four solutions denoted by x_0, x_1, x_2, x_3. We know that two of these roots are solutions to the zeroth iteration equation. Thus, the only new roots are x_2 and x_3. We want to know if these new roots are indeed roots to a general iteration of the map. We might as well start with the next iteration, the second iteration: $F \circ F \circ F(x) - x = 0$. Substituting x_2 into the second iteration equation gives,

$$F \circ F \circ F(x_2) - x_2 = 0 \Rightarrow F(x_2) - x_2 \neq 0. \tag{2.18}$$

We have used the fact that $F \circ F(x_2) = x_2$ to reduce the equation. It is important to realize that x_2 is a solution of $F \circ F(x) - x = 0$ (and every even iteration) and therefore an odd iteration such as $F(x_2)$ will not in in general be equal to x_2. From this, it is clear that all even iterations of the map will return roots including x_2 and

x_3 but the odd iterations will not. This analysis can easily be extended to solutions of the nth iteration.

To sum up the analysis presented in this section section we should observe the following. The zeroth iteration yields roots that are solutions to a general iteration (all numbers are multiples of one) whereas roots of an nth iteration are solutions only to $m \times n$th iterations where m is a positive integer value. You may have noticed also the repetition of roots in the results of Sec. 2.2.0 as predicted here.

Chapter 3

Lyapunov exponents

3.1 Lyapunov exponents for one dimensional maps

The Lyapunov exponent(LE) is often used as a measure of the chaoticity of a dynamical system. Simply stated, the LE characterizes the degree to which trajectories that start very close together diverge over time. In a system with LE > 0, the trajectories will diverge exponentially and thus the system exhibits sensitive dependence on initial conditions (frequently quoted as the defining characteristic of chaotic systems). Further, if LE < 0 the system is dissipative in the sense that the trajectories converge, and if LE$= 0$, the system is conservative.

Let us start with two initially nearby trajectories $f^1(x_0)$ and $f^1(x_0+\epsilon)$ to derive the Lyapunov exponent for one dimensional maps [4]. Let us assume that the divergence Δ of the trajectories can be expressed as an exponential relation $\epsilon e^{n\lambda}$. It is clear from the figure 3.1 that $\Delta = f^n(x_0) - f^n(x_0 + \epsilon)$. Thus we have,

$$f^n(x_0) - f^n(x_0 + \epsilon) = \epsilon e^{n\lambda} \tag{3.1}$$

$$\frac{f^n(x_0) - f^n(x_0 + \epsilon)}{\epsilon} = e^{n\lambda} \tag{3.2}$$

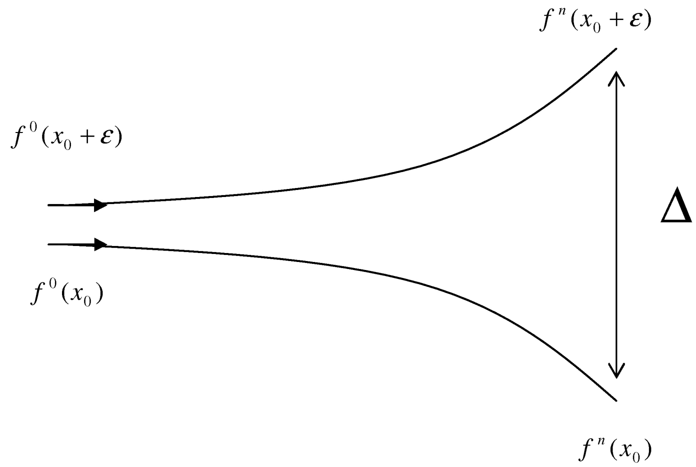

Figure 3.1 Schematic of two initially close trajectories

If take the limit as ϵ goes to zero, the left hand side of the above relation is just the derivative,

$$\left.\frac{df^n(x)}{dx}\right|_{x_0} = e^{n\lambda(x_0)} \tag{3.3}$$

Solving for λ we find

$$\lambda = \frac{1}{n}\ln\left|\left.\frac{df^n(x)}{dx}\right|_{x_o}\right| \tag{3.4}$$

As noted earlier, recursive maps are just composite functions: $f^n(x_0) = f \circ f \circ ... \circ f(x_0)$. Thus, when evaluating the derivative of $f^n(x)$ we can utilize the chain rule from calculus.

$$\left.\frac{df^n}{dx}\right|_{x_0} = \left.\frac{df}{dx}\right|_{x_{n-1}} \left.\frac{df}{dx}\right|_{x_{n-2}} \cdots \left.\frac{df}{dx}\right|_{x_0} \tag{3.5}$$

Then, due to the property of logarithms, the expression for λ can be written as a sum in the following useful form.

$$\lambda = \frac{1}{n}\sum_{i=1}^{n-1}\ln|f'(x_i)| \tag{3.6}$$

3.2 Lyapunov exponents in n-dimensions

Evaluating the LE's in n-dimensional maps is usually not a trivial task; it involves finding the solution of the variational equations of the system [5]. However, methods do exist that allow one to approximately determine the highest LE for a system of dimension greater than one.[1]. In the present paper, a method introduced by Chen and Li [6] will be used from which it is possible to evaluate the *limits* of the Lyapunov exponents of a discrete system in n-dimensions. This method is applicable to the coupled logistic map (CLM). The theorem proved by Chen and Li applies to a general nonlinear system assumed only to be continuously differentiable in the region of interest [6].

$$x_{k+1} = f(x_k), x_k = \in \Omega \subset R^n, k = 0, 1, ... \tag{3.7}$$

$$x_0\text{–given} \tag{3.8}$$

The following theorem is adapted from [6] with only minor notational modifications.

Theorem 1: Consider system (3.7)-(3.8). Suppose that $\|f'(x)\| \leq N$ and that the smallest eigenvalue of $[f'(x)]^T f'(x)$ satisfies $\lambda_{\min}([f'(x)]^T f'(x)) \geq \alpha > 0$, where $\alpha \leq N^2$ and $x \in \Omega$. Then, for any $x_0 \in \Omega$, all the Lyapunov exponents at x_0 are located inside $[0.5 \ln \alpha, \ln N]$. That is,

$$0.5 \ln \alpha \leq \lambda_i(x_0) \leq \ln N, i = 1, 2, ..., n \tag{3.9}$$

In the notation of Chen and Li, $\|\mathbf{J}\|$ denotes the *spectral norm* of a matrix $\mathbf{J}$, and $f'(x)$ is the Jacobian matrix of the map. Further, all matrices are assumed to be evaluated at the limits of the attractors of the system. For the logistic map these are, $x = y = 1$ at $r = 4$.

[1]For a comparison between methods of computing Lyapunov exponents see ref [5]

3.2.1 Limits on the LE of the CLM

In accordance with *Theorem 1* we first find the Jacobian matrix of the CLM. For this derivation, we will use the *master-slave* coupling,

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) \\ y_{n+1} &= r q (1 - q) \\ q &= a x_n + (1 - a) y_n . \end{aligned} \tag{3.10}$$

It is known that the coupled logistic map is chaotic and bounded by $x = y = 1$ at $r = 4$. The Jacobian matrix evaluated at these values is then,

$$\mathbf{J} = \begin{bmatrix} -4 & 0 \\ -4\,a & -4 + 4\,a \end{bmatrix}. \tag{3.11}$$

To implement the theorem further, we need the transpose of $\mathbf{J}$,

$$\mathbf{J}^T = \begin{bmatrix} -4 & -4\,a \\ 0 & -4 + 4\,a \end{bmatrix}. \tag{3.12}$$

Finally, $\mathbf{J}$ and $\mathbf{J}^T$ are multiplied according to matrix multiplication,

$$\mathbf{J}\mathbf{J}^T = \begin{bmatrix} 16 & 16\,a \\ 16\,a & 16\,a^2 + (-4 + 4\,a)^2 \end{bmatrix}. \tag{3.13}$$

Below are the two eigenvalues of the above matrix. The *spectral norm* is calculated with the largest..

$$\begin{bmatrix} 16 - 16\,a + 16\,a^2 + 16\,\sqrt{2\,a^2 - 2\,a^3 + a^4} \\ 16 - 16\,a + 16\,a^2 - 16\,\sqrt{2\,a^2 - 2\,a^3 + a^4} \end{bmatrix}. \tag{3.14}$$

The spectral norm is,

$$\|\mathbf{J}\| = 4\sqrt{1 - a + a^2 + \sqrt{2\,a^2 - 2\,a^3 + a^4}} = N. \tag{3.15}$$

The smallest eigenvalue gives α,

$$16 - 16\,a + 16\,a^2 - 16\sqrt{2\,a^2 - 2\,a^3 + a^4} \triangleq \alpha \tag{3.16}$$

It then follows that the Lyapunov exponents at $r = 4$ lie within

$$0.5\ln(\alpha) \leq \lambda_1(a) \leq \lambda_2(a) \leq \ln N \tag{3.17}$$

A plot of the upper and lower bounds varying with the coupling constant a is shown below.

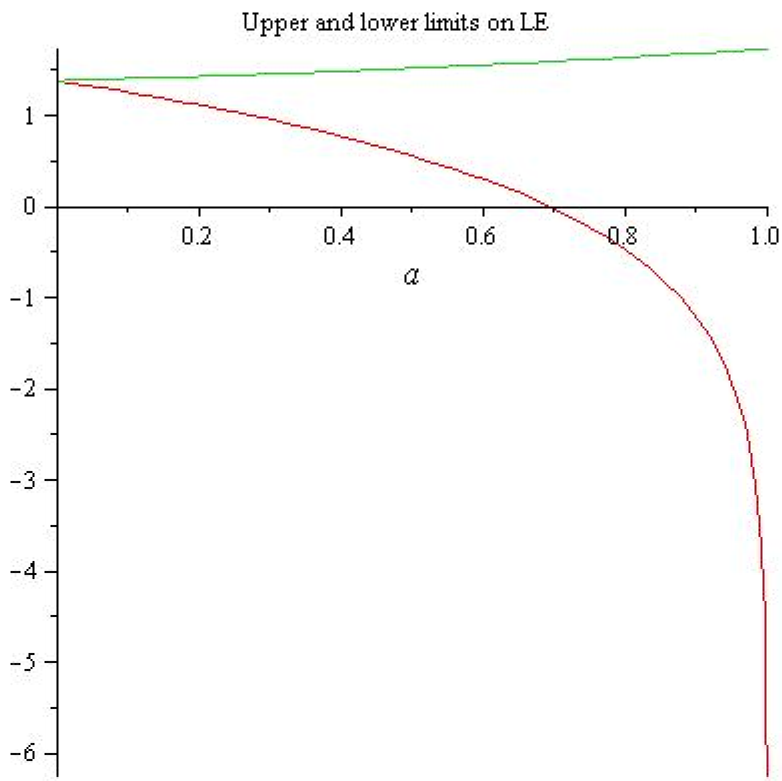

Figure 3.2 The upper and lower limits of the LE plotted as a function of a at $r = 4$. For many values of a both limits lie above the horizontal indicating *hyper-chaoticity*. The value at which the lower limit crosses below the horizontal is 0.692

Chapter 4

Synchronization

4.1 Characterization of Synchronization in Chaotic Maps

The trajectories of the LM in the chaotic regime can conveniently be thought of as chaotic oscillations. This paradigm lends itself to a characterization of synchronization based on the properties of waves. We may use *frequency*, *amplitude*, and *phase*, to quantify the behavior of the trajectories which may in turn be used to identify synchronization: If the phase, amplitude and frequency agree in a coupled system, the system will exhibit *complete synchronization* (CS). Since we are dealing with chaotic oscillations, there is no well defined frequency, phase, or amplitude. A *mean frequency*, *mean phase*, and *mean amplitude* must be defined.

Mean frequency

By defining a **mean frequency** [7] we partially characterize the oscillations,

$$\langle f \rangle = \frac{N_\tau}{\tau} \tag{4.1}$$

where N_τ is the number of cycles within a time τ. Two systems with identical (or near to it) mean frequencies are defined to be *frequency synchronized*.

Mean Phase

Analogous to the definition of a mean frequency, we can define the **mean phase** for two oscillating trajectories.

$$\phi_i \equiv |x_{i+1} - x_i| \Rightarrow \langle \phi_\tau \rangle = \frac{\sum_{i=1}^{N_\tau} \phi_i}{N_\tau} \tag{4.2}$$

x_i correspond to the x-values of local maxima. Two systems with equivalent mean phase or a constant difference in phase are *phase synchronized*. Note also that $\phi_i = const. \, \forall \, i$ when the oscillations are a sine, cosine, or triangle wave.

Mean amplitude

Amplitude, phase, and frequency completely define an oscillation. Using a **mean amplitude** defined to be

$$\langle A \rangle = \frac{\sum_{i=1}^{N_\tau} A_i}{N_\tau} \tag{4.3}$$

we can completely characterized the oscillations. As in the previous two cases, *Amplitude synchronization* occurs when the mean amplitude is identical in the coupled systems. Note also that by virtue of their respective definitions, the mean amplitude and phase should be qualitatively similar.

4.2 Application to the CLM

The above quantities can be found in a fairly straightforward manner for the CLM presented in Sec. 3.2.1. We must first, however, define the frequency for a discrete system. By choosing local maxima in the discrete trajectories of the CLM to act as maxima of an oscillation, the frequency can be determined easily. Local maximums are found by evaluating a forward and backward difference for every point in the trajectory and storing only those points where the forward and backward differences are larger than one. The Matlab code in Appendix A evaluates local maxima. Further running the code in Appendix A produces Fig 4.6-4.4. It is apparent from Figure

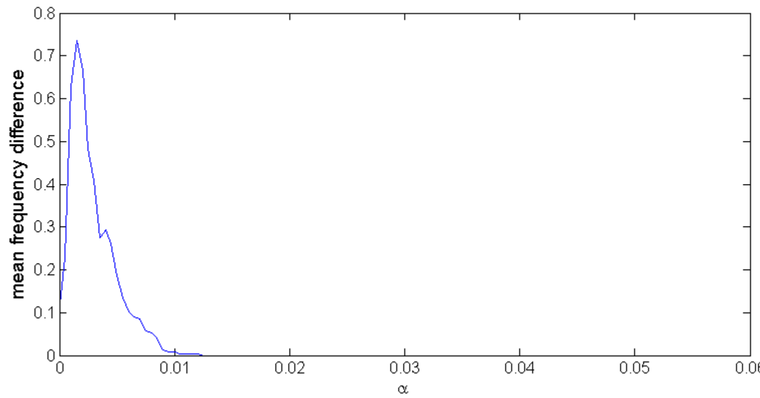

Figure 4.1 The mean frequency difference goes to zero at $x = 0.012$

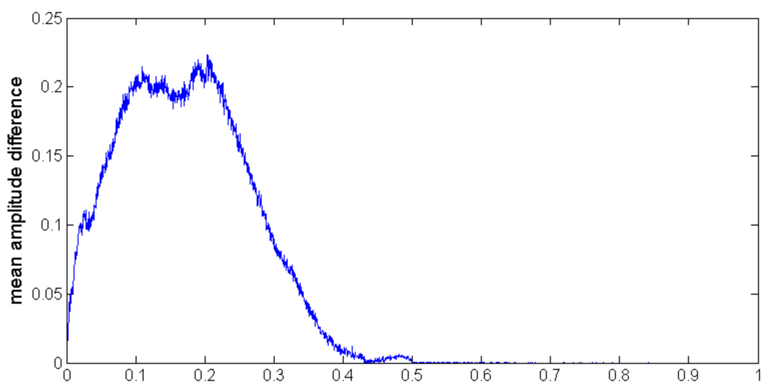

Figure 4.2 The mean amplitude goes to zero at around $a = 0.42$ upon which there is a region of intermittent synchronization. The onset of complete synchronization occurs at $a = 0.5$

4.1 and 4.2 that frequency synchronization occurs considerably earlier than phase or amplitude synchronization. The onset of complete synchronization occurs at $a =$

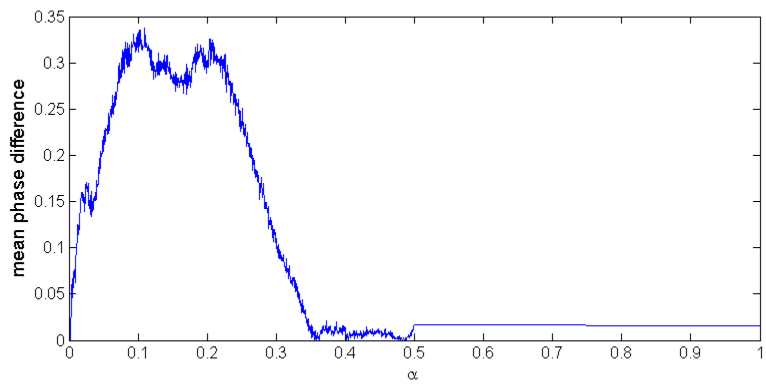

Figure 4.3 The mean phase equals a constant at $a = 0.5$

0.5. By inspection, one can also observe a region of intermittent synchronization starting at about $a = 0.4$ in the amplitude plot. In other words, a region exists before complete synchronization in which the averaged amplitude values are near to or exactly zero. Figure 4.4 shows the region of intermittent synchronization for the averaged amplitudes.

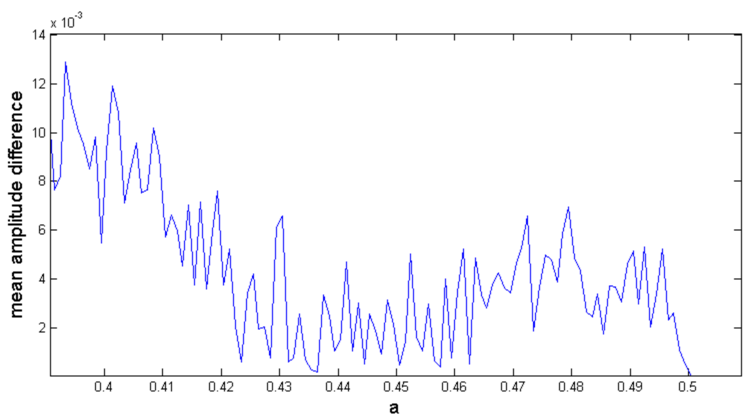

Figure 4.4 Close up of intermittent region

4.3 Intermittent synchronization in the CLM

To analyze the region of intermittent amplitude synchronization in greater detail requires we first find the distribution of amplitudes in the CLM. Amplitudes are averaged over lengths of time τ for some large number n of iterations and are then counted. For $\tau = 100$ and $n = 1 \times 10^4$, the gaussian-like distribution, 4.5 is obtained for the master. The slave similarly exhibits gaussian behavior. The width of the slave

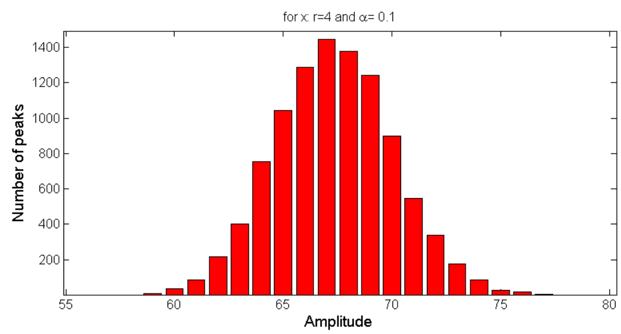

Figure 4.5 The distribution of averaged amplitudes for the master.

distribution, however, is considerably larger indicating a greater range of amplitudes in the oscillations. Using the distributions of amplitudes, intermittent synchroniza-

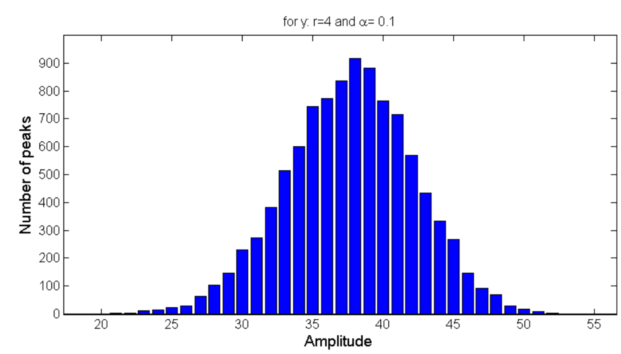

Figure 4.6 The distribution of averaged amplitudes for the master.

tion is defined to begin when the maximum of the slave-distribution first corresponds to the maximum of the master-distribution, that is, when the most probable value in

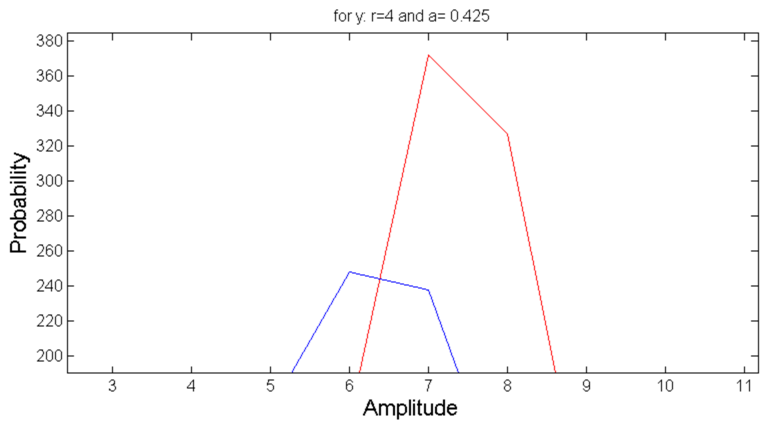

Figure 4.7 Close up of amplitude distributions just before intermittent synchronization occurs.

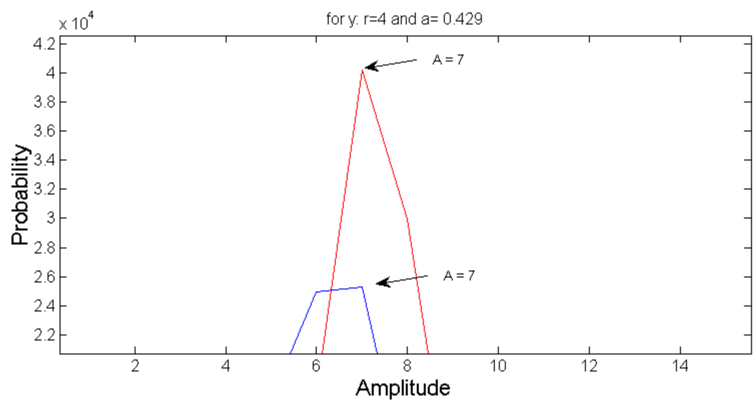

Figure 4.8 Approximate onset of intermittent synchronization. The intermittent synchronization onset exhibits a slight dependence on the number of values averaged over in determining the distributions indicating some ambiguity in the method.

the distribution of the slave is equivalent to the most probable amplitude of the master. The CLM becomes completely synchronized at $a = 0.5$, and thus it is of interest to look at the amplitude distributions near this value. Figures 4.7 and 4.8 demonstrate the onset of intermittent amplitude synchronization occurring approximately at $a = 0.425$. It is important to note that synchronization is largely dependent upon its definition. Although there exists a precise point at which complete synchronization occurs (one-to-one correspondence between trajectories), the behaviors of the individual time averaged frequencies, phases, and amplitudes vary notably. Furthermore, these quantities were only obtained by averaging non-convergent values over long time scales. Thus, the quantitative behavior naturally depends somewhat on N_τ. Qualitative behavior remains largely unchanged, however, and the prediction of complete synchronization is independent of N_τ.

Due to this dependence on N_τ, the physical significance of any solitary synchronization onset is questionable. Whether or not the early onset of frequency synchronization or the intermittent amplitude synchronization as defined in this section can be observed physically is unknown.

4.4 Analytic derivation of CS onset in the CLM

The onset of complete synchronization can be found by observing that stationary points in the coupled system occur when

$$\begin{aligned} y_{n+1} &= x_n \\ x_{n+1} &= y_n. \end{aligned} \tag{4.4}$$

Replacing x_{n+1} and y_{n+1} with their respective maps gives,

$$\begin{aligned} y_n &= r x_n(1 - x_n) \\ x_n &= r q(1 - q) \\ q &= a x_n + (1 - a) y_n. \end{aligned} \tag{4.5}$$

Solving the simultaneous set of equations when $r = 4$ for x and y gives three solutions, with the first two agreeing with the previously obtained one dimensional results:

$$x = 0, y = 0 \tag{4.6}$$

$$x = \frac{3}{4}, y = \frac{3}{4} \tag{4.7}$$

$$x = \frac{1}{4}\frac{A^{\pm}}{a - 1}, y = \frac{1}{4}\frac{aA^{\pm} - 4\,a + 5 + A^{\pm}}{1 - 2\,a + a^2} \tag{4.8}$$

$A^{\pm}$ is one of two possible roots,

$$A^{\pm} = 3/2\,a - 5/2 \pm 1/2\,\sqrt{9\,a^2 - 14\,a + 5}. \tag{4.9}$$

Let $\pm$ correspond to the plus and minus solutions respectively. Only the third solution (4.8) is a-dependent. As long as this set of stationary points is stable, a necessary condition for complete synchronization is then,

$$\frac{1}{4}\frac{A^{\pm}}{a - 1} = \frac{1}{4}\frac{aA^{\pm} - 4\,a + 5 + A^{\pm}}{1 - 2\,a + a^2}. \tag{4.10}$$

If we use the plus-solution, 4.9 becomes after subtracting one side from the other,

$$-\frac{1}{4}\,\frac{\left(1+\sqrt{9\,a^2-14\,a+5}\right)a}{\left(a-1\right)^2}=0 \tag{4.11}$$

which yields only the trivial solution of $a = 0$. If we consider the minus-solution, we get the following:

$$\frac{1}{4}\,\frac{a-\sqrt{9\,a^2-14\,a+5}}{\left(a-1\right)^2}=0 \tag{4.12}$$

Solving for a gives two solutions, $a = 1/2$ and $a = 5/4$. The first of these verifies the numerical result. The second is larger than one and can therefore be disregarded on physical grounds. The same analysis can be applied to the symmetrically coupled logistic map giving a first solution of $a = 1/4$. The result is not shown here. For values of r other than 4, higher order iterations must be solved for stationary points that are stable solutions of the map. The analysis can then be used to determine complete synchronization for a general r.

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Appendix A

Matlab code

The code for determining local maximums is simply a loop over all points with two conditions.

```
for k=1:Nsteps-2
       b_diffx(k+1)=datax(k+1)-datax(k);
       f_diffx(k+1)=datax(k+1)-datax(k+2);
       b_diffy(k+1)=datay(k+1)-datay(k);
       f_diffy(k+1)=datay(k+1)-datay(k+2);
   if b_diffx(k+1)>0 && b_diffx(k+1)>0
       maxx(k) =1;
   if b_diffy(k+1)>0 && b_diffy(k+1)>0
       maxy(k) =1;
   end
   end
end
```

Mean frequency, phase, and amplitude are found with Matlab in the following manner. The code above is included but in a slightly different form.

```
clear all; close all;
Nsteps =20000;    %the 'time' over which we will take averages.
r=4;  %parameter in the chaotic regime
%amax = input('maximum value of coupling constant---');
Nskip = 1000;
N = 2000;
a=linspace(0,1,N);
%initialize the arrays for the plots
    fx_array=zeros(N,1);
    fy_array=zeros(N,1);
    phi_x_array=zeros(N,1);
    phi_y_array=zeros(N,1);
    A_x_array=zeros(N,1);
    A_y_array=zeros(N,1);

x=.1; %initial x value
y = x+.1;

datax = zeros(Nsteps,1);
datay = zeros(Nsteps,1);

for n=1:length(a)
for i_st = -Nskip:Nsteps   % loop over i

    x = r*x*(1-x);  % Compute next x
    q = a(n)*x+(1-a(n))*y;
```

```
    y = r*q*(1-q);
if i_st>1
datax(i_st-1)=x;
datay(i_st)=y;
end
  end
%  subplot(2,1,1)
% plot(datax)
% ylim([0 1.1])
% ylabel('x_{trajectory}','fontsize',12)
% xlabel('t_{iteration}','fontsize',12)
%
% subplot(2,1,2)
% plot(datay)
% ylim([0 1.1])
% ylabel('y_{trajectory}','fontsize',12)
% xlabel('t_{iteration}','fontsize',12)
                                %finding the mean frequency
     %first we calculate a very crude 'derivative'
tau=Nsteps-10;

crude_db = zeros(tau,1);
crude_df = zeros(tau,1);

for i=1:tau
b=datax(i+1)-datax(i);
```

```
f=datax(i+1)-datax(i+2);

crude_db(i)=b;

crude_df(i)=f;


end


% the idea here is that if we have a maximum then the forward difference

% and the backward difference are going to be positive.

maxpnts_x = zeros(tau,1);

xval = zeros(tau,1);

yval = zeros(tau,1);

for i=1:tau

if crude_db(i)>0 && crude_df(i)>0

  maxpnts_x(i+1) = datax(i+1);

   xval(i+1)=i+1;

  bln_x(i+1) = 1; %#ok<AGROW>

end

end


%now we do the same thing for the y-trajectories


crude_db = zeros(tau,1);

crude_df = zeros(tau,1);


for i=1:tau

b=datay(i+1)-datay(i);
```

```
f=datay(i+1)-datay(i+2);
crude_db(i)=b;
crude_df(i)=f;

end

maxpnts_y = zeros(tau,1);

for i=1:tau
if crude_db(i)>0 && crude_df(i)>0
  maxpnts_y(i+1) = datay(i+1);
  yval(i+1)=i+1;
  bln_y(i+1) = 1; %#ok<AGROW>
end
end

%now with this data, we can find the mean frequency,
%mean amplitude, and the mean phase; two peaks corresponds
%to one wavelength or one cycle.
number_of_cycles_in_x = sum(bln_x)/2; %total number of cycles in x
number_of_cycles_in_y = sum(bln_y)/2; %total number of cycles in y

%calculate the mean frequency

 fx =  number_of_cycles_in_x/tau;
 fy =  number_of_cycles_in_y/tau;
```

```
fx_array(n) =  number_of_cycles_in_x/tau;

fy_array(n) =  number_of_cycles_in_y/tau;

%  %calculated the mean phase

Ty=find(maxpnts_y==0);
for j=2:length(Ty)
    rawphasey(j)=Ty(j)-Ty(j-1);
end

phasey(n)=sum(rawphasey)/number_of_cycles_in_y;



Tx=find(maxpnts_x==0);
for j=2:length(Tx)
    rawphasex(j)=Tx(j)-Tx(j-1);
end

phasex(n)=sum(rawphasex)/number_of_cycles_in_x;

% %calculate the mean amplitude

A_x_array(n) = sum(maxpnts_x)/number_of_cycles_in_x;
A_y_array(n) = sum(maxpnts_y)/number_of_cycles_in_y;
```

```
end
            %plot the mean values to see synchronization
subplot(3,1,1)
plot(a,fx_array,a,fy_array)
 %ylim([.3 .7]) %you may have to find different ranges than mine
 xlim([0 a(end)])
xlabel('mean frequency')

subplot(3,1,2)
plot(a,phi_x_array,a,phi_y_array)
  %ylim([-.4 0])
  xlim([0 a(end)])
xlabel('mean phase')

subplot(3,1,3)
plot(a,A_x_array,a,A_y_array)
 %ylim([.3 .8])
 xlim([0 a(end)])
xlabel('mean amplitude')

figure(2)
 %since in practice the differences may actually be more useful to look at,
 %I will include them also.

plot(a,100*abs(fx_array-fy_array))
```

```
xlabel('a')
ylabel('mean frequency difference')
xlim([0 .06])

figure(3)
% plot(a,abs(phi_x_array-phi_y_array))
plot(a,abs(phasex-phasey))
xlabel('a')
ylabel('mean phase difference')

figure(4)
plot(a,abs(A_x_array-A_y_array))
xlabel('a')
ylabel('mean amplitude difference')
```

The following code calculates the distribution of amplitudes used to study intermittent synchronization.

```
clear all; close all;

Nsteps = 1e6;
r=4;

Nskip = 1000;

a=.1;
f_diffx=zeros(Nsteps-1,1);
b_diffx=zeros(Nsteps-1,1);
```

```
f_diffy=zeros(Nsteps-1,1);
b_diffy=zeros(Nsteps-1,1);

x=.1;
y =x+.1;
maxx=zeros(Nsteps,1);
maxy=zeros(Nsteps,1);
datax = zeros(Nsteps,1);
datay = zeros(Nsteps,1);

for i_st = -Nskip:Nsteps
    x = r*x*(1-x);
    q = a*x+(1-a)*y;
    y = r*q*(1-q);

    if i_st>1
datax(i_st-1)=x;
datay(i_st)=y;
end
end

    for k=1:Nsteps-2
        b_diffx(k+1)=datax(k+1)-datax(k);
        f_diffx(k+1)=datax(k+1)-datax(k+2);
        b_diffy(k+1)=datay(k+1)-datay(k);
        f_diffy(k+1)=datay(k+1)-datay(k+2);
```

```
if b_diffx(k+1)>0 && b_diffx(k+1)>0
    maxx(k) =1;
if b_diffy(k+1)>0 && b_diffy(k+1)>0
    maxy(k) =1;

end
end
    end
c=0;
for w=1:100:Nsteps-102;
c=c+1;
    sumy = 0;
    sumx = 0;
for n=w:w+100
    sumx = maxx(n) + sumx;
    sumy = maxy(n) + sumy;
end
arx(c)=sumx;
ary(c)=sumy;
end

for nmb=1:100
cn=0;
for i=1:length(arx);
    if arx(i)<=nmb && arx(i)>=nmb
```

```
        cn=cn+1;
    elseif arx(i)<nmb && arx(i)>nmb
        cn=cn+0;

    end
end
gaussx(nmb)=cn;
end



for nmb=1:100
cn=0;
for i=1:length(ary);
    if ary(i)<=nmb && ary(i)>=nmb
        cn=cn+1;
    elseif ary(i)<nmb && ary(i)>nmb
        cn=cn+0;

    end
end
gaussy(nmb)=cn;
end

figure
bar(gaussx,'r');
ylabel('Probability (not normalized)','fontsize',12)
```

```
xlabel('amplitude','fontsize',12)
title(['for x: r=4 and a= ',num2str(a)])
hold on;

figure
bar(gaussy,'b')
ylabel('Probability (not normalized)','fontsize',12)
xlabel('Amplitude','fontsize',12)
title(['for y: r=4 and a= ',num2str(a)])
```

Index