

Numerical renormalization of divergences for pair-production in QED

Ethan Gibson

A senior thesis submitted to the faculty of

Brigham Young University

In partial fulfillment of the requirements for the degree of

Bachelor of Science

Scott Glasgow and Benjamin Frandsen, Advisors

Department of Physics and Astronomy

Brigham Young University

February 2021

Copyright © 2021 Ethan Gibson

## ABSTRACT

Numerical renormalization of divergences for pair-production in QED

Ethan Gibson

Department of Physics and Astronomy

Bachelor of Science

Quantum electrodynamics (QED) is a quantum field theory for electromagnetic interactions. By quantizing the electric field, we can analyze the properties of the electromagnetic interactions between quantum systems. Historically, QED has been able to predict the behavior of quantum particles with unprecedented success. However, the success of QED has relied upon perturbation theory to approximate results to a given order of precision. Physicists have largely abandoned analytical, closed form solutions to QED due to the problematic mathematical divergences in attempts to resolve systems non-perturbatively. By using the dressed vacuum as a basis, we may be able to renormalize systems which possess Hamiltonians of similar attributes. We use analytical tactics to avoid the divergence, and form a basis upon which the Hamiltonian can be normalized for the dressed vacuum, giving hope that we can obtain an analytical, closed-form solution for some systems in QED. By doing so, we accomplish an analytical, closed-form type of solution to problems within QED.

Keywords: quantum electrodynamics; renormalization; divergence

# 1 TABLE OF CONTENTS

---

1	Introduction.....	4
1.1	Introduction to Quantum Electrodynamics .....	4
1.1.1	Motivation for quantum field theories .....	4
1.1.2	Quantum Electrodynamics.....	5
1.2	Problems with (And Solutions to) QED .....	5
1.2.1	Infrared Divergence .....	5
1.2.2	QED Dressed Vacuum.....	6
1.2.3	Numerical Renormalization .....	6
2	Theoretical Methods .....	8
2.1	Mathematical Background.....	8
2.1.1	The Hamiltonian Operator .....	8
2.1.2	The QED Hamiltonian .....	9
2.1.3	The Projection method.....	10
2.2	Analytic Mathematical Methods.....	11
2.2.1	Projection onto one spatial dimension .....	11
2.2.2	The Dirac Term.....	12
2.2.3	The Radiation Term .....	14
2.2.4	The Interaction Term .....	15
2.2.5	Photon “Mass” .....	20
2.3	Numerical Mathematical Methods.....	21
2.3.1	Transformation to Unitless Variables .....	21
2.3.2	Solving for the Ground-State Energy.....	23
3	Results and Conclusion.....	27
3.1	Python Methods .....	27
3.1.1	Setting up Momentum Grids.....	27
3.1.2	Solving the Transcendental Equation.....	28
3.2	Discussion of Results .....	30
3.2.1	Theoretical Assumptions.....	30
3.2.2	Potential for Future Non-Perturbative QED .....	31
3.3	Conclusion .....	32
	Bibliography.....	33
	List of Figures.....	34

# 1 INTRODUCTION

---

In this chapter, we will provide background to quantum field theory, and more specifically quantum electrodynamics (QED). We will discuss its successes and its failures, particularly its theoretical shortcomings. We will discuss our attempts to overcome these shortcomings with both analytical and numerical methods. We will introduce the concept of the dressed vacuum, and how the analysis of it can lead us to resolve infrared divergences in QED.

## 1.1 INTRODUCTION TO QUANTUM ELECTRODYNAMICS

### 1.1.1 Motivation for quantum field theories

Quantum field theories (QFTs) developed as a relativistic interpretation to quantum mechanics. In essence, it attempted to apply the principles of early quantum mechanics to particles traveling at speeds in which special relativity applied. The earliest attempts at a quantum field theory resulted in the Klein-Gordon equation, which is given by:

$$\frac{1}{c^2} \frac{\partial^2}{\partial u^2} \psi - \nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0 \quad (0.0.1)$$

The idea is basically a “second quantization.” The first quantization is that of quantum mechanics, where particles become wave functions, whose modulus square describes the probability of locating the particle at a given point upon collapse of the wave function. The second quantization involves representing physical observables (position, momentum, etc.) as variables, and fields as fundamental operators<sup>1</sup>. Particles can be thought of as excitations of these quantum fields.

### 1.1.2 Quantum Electrodynamics

Quantum electrodynamics (QED) is a quantum field theory revolving around the electric field, and the fundamental electromagnetic interactions of quantum particles. QED describes the energy of quantum field excitations by assuming a setting where electromagnetism is the only dominant force. As such, we will not be considering strong and weak interactions.

## 1.2 PROBLEMS WITH (AND SOLUTIONS TO) QED

### 1.2.1 Infrared Divergence

QED has historically seen much success. Its predictions are some of the most precise in the history of physics. However, QED has some theoretical shortcomings.

The problems with QED have been well-documented since its beginnings. As will be seen hereafter, the nature of the QED Hamiltonian leads it to yield divergences, thus disallowing theorists from finding a mathematical solution to QED systems using mathematical analysis.

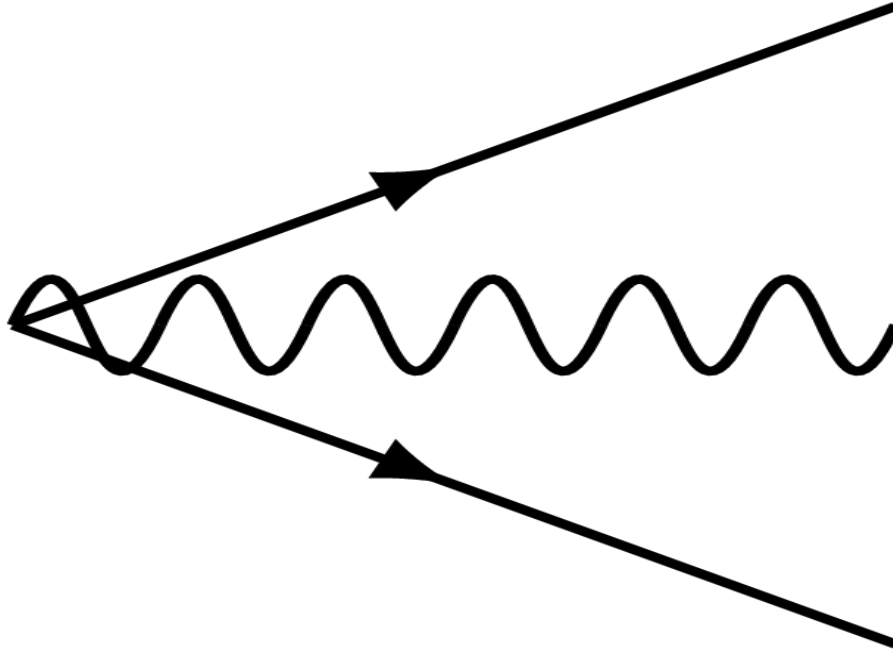
In order to solve problems in QED, one is forced to resort to perturbation theory, a series of approximations to the QED Hamiltonian, in order to find solutions. Despite the aforementioned success of these techniques, the absence of an analytical, closed-form solution is a glaring flaw in the theory that leads many physicists to believe that QED may be incomplete.

The type of divergence which we will encounter in this work is known as infrared divergence, or in other words, divergence in the low-energy limit. It is characterized by the presence of an unresolvable singularity in the denominator, which blows up the value of the integrand as it passes through zero. We postulate that through numerical methods, and a number of constraints that will be discussed hereafter, we can find a way around these problematic divergences.

### 1.2.2 QED Dressed Vacuum

The basic idea is to extensively and mathematically analyze what we will call the dressed vacuum.

Consider the following Feynman diagram:



*Figure 1.1 The Feynman diagram of the dressed vacuum system*

This diagram represents the dressed vacuum system. The curved line in the middle represents a photon, and the straight lines represent an electron-positron pair. All particles are produced as an excitation of the electric field, and are thus produced from the vacuum. We will be using the projector method (to be described in Chapter Two) to analyze this system, so we will be taking different time slices of this Feynman diagram.

### 1.2.3 Numerical Renormalization

The idea is to relate the dressed vacuum to other, similar types of excitations. Even though the dressed vacuum cannot be normalized due to the divergences that arise in it, if we adjust the parameters, we may

just have a chance. We will see that by introducing a new parameter, we can beat the infrared divergence, allowing us to normalize the dressed vacuum. We call this process renormalization. The hope is to use the normalized dressed vacuum state as a basis where other systems of similar particle excitations can be treated as perturbations of the dressed vacuum, and thus renormalized using the basis we will have found. With this background in mind, we can begin our analysis of the dressed vacuum.

## 2 THEORETICAL METHODS

---

In this chapter, we will walk through the analytical methods used to renormalize the dressed vacuum. We will introduce definitions of mathematical constructs relevant to the problem at hand. We will address the steps taken to solve the problem, then introduce a unitless set of variables. We will introduce the mollifier, which will aide us in resolving the infrared divergence. We will arrive at a normalizable expression for the QED Hamiltonian, which we may use to numerically renormalize the dressed vacuum.

### 2.1 MATHEMATICAL BACKGROUND

#### 2.1.1 The Hamiltonian Operator

Classically, Hamiltonian mechanics is a framework in which we can mathematically derive the equations of motion for a system using what we call the Hamiltonian of the system. The Hamiltonian is a *function* used to introduce a system of potentially coupled differential equations. In cases where the behavior of the pertinent variables of the system is time-independent with respect to the Cartesian plane<sup>1</sup>, the Hamiltonian becomes the total energy of the system as a function of the pertinent variables.

In quantum mechanics, the Hamiltonian becomes an *operator* in abstract vector space. The Hamiltonian is known as a linear operator, meaning that it can be separated into a sum of its constituent parts. In quantum mechanics, the Hamiltonian is the potential energy operator plus the kinetic energy operator given by:

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\mathbf{r}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \quad (0.0.2)$$



Where the potential is a function of the spatial coordinates. The Hamiltonian operator plays a critical role in the study of quantum systems, as it is the key to understanding the energy of quantum systems. The Hamiltonian is related to energy in the following way:

$$\hat{H}\psi = E_n\psi \quad (0.0.3)$$

This is an eigenvalue equation, as the Hamiltonian is operating in Hilbert space on eigenvectors known as eigenstates, and the energy is a scalar to be multiplied into the same eigenstates. The energy is therefore an eigenvalue of the Hamiltonian. If we can find the eigenvalues of a given Hamiltonian, we may figure out the specific “allowed energy” of any quantum system.

### 2.1.2 The QED Hamiltonian

In quantum electrodynamics (QED), the Hamiltonian plays as important a role in understanding a system as in elementary quantum mechanics. Recall that QED is a quantum field theory, and that a quantum field theory describes particles as excitations of a quantum field. The Hamiltonian operator in quantum field theory is therefore a tool to understand the excitation of the relevant quantum field. In QED, we quantize the electric field, leading to a fully-fledged quantum field theory for the electric field. Much of the work of this quantization has previously been investigated in depth. For our purposes, we will draw from *Photons and Atoms* by Claude Cohen-Tannoudji<sup>2</sup> for pertinent mathematical results regarding the QED Hamiltonian. We will split the linear operator into four separate operators, each of which will represent a property of the system. We define the Hamiltonian operator as:

$$H = H_D + H_R + H_I + H_C \quad (0.0.4)$$

We will call these terms, respectively: the Dirac term, the radiation term, the interaction term, and the Coulomb term. The Dirac term describes the free energy of the massive particles of the system. The radiation term describes the free photonic radiation of the system. The interaction term describes the

coupling between massive particles and photons in the electric field. Finally, the Coulomb term describes the electromagnetic interaction of the particles of the system. The definitions of the respective terms of the Hamiltonian are:

$$\begin{aligned}
H_D &= \int d^3\mathbf{r} \Psi^\dagger(\mathbf{r}) [\beta mc^2 - i\hbar c \boldsymbol{\alpha} \cdot \nabla] \Psi(\mathbf{r}) \\
H_R &= \int d^3\mathbf{k} \hbar \omega_{\mathbf{k}} (a_1^\dagger(\mathbf{k}) a_1(\mathbf{k}) + a_2^\dagger(\mathbf{k}) a_2(\mathbf{k})) \\
H_I &= -qc \int d^3\mathbf{r} \Psi^\dagger(\mathbf{r}) [\boldsymbol{\alpha} \cdot \mathbf{A}_\perp(\mathbf{r})] \Psi(\mathbf{r}) \\
H_C &= \int d^3\mathbf{r} \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}) \rho(\mathbf{r}')}{8\pi\epsilon_0 \|\mathbf{r} - \mathbf{r}'\|}
\end{aligned} \tag{0.0.5}$$

A full explanation of the symbols in this equation will be given hereafter. By solving the eigenvalue equation using this definition for the Hamiltonian operator, we can find the allowed energies for our quantum system. As discussed in the previous chapter, finding the ground state energy for the dressed vacuum will be crucial in our attempts to create a basis for renormalization upon which we can construct solutions for other quantum states.

### 2.1.3 The Projection method

One last point of mathematical background is the concept of the projector. This technique will allow us to extract the information we need out of the Hamiltonian with relative ease. In quantum mechanics, we can solve for all potential eigen-states of the Hamiltonian. Much of the time, most of the possible states will disappear due to the nature of the system. By using what is known as a projector, we can single out only the states that will survive, allowing us to single in on the energies which we wish to find.

We will define the projector for the dressed vacuum state to be:

$$P_{\uparrow\downarrow} := | \rangle \langle | + \int d\mathbf{k}_\gamma \int d\mathbf{k}_{e^-} \int d\mathbf{k}_{e^+} a_p^\dagger(\mathbf{k}_\gamma) c_\uparrow^\dagger(\mathbf{k}_{e^-}) d_\downarrow^\dagger(\mathbf{k}_{e^+}) | \rangle \langle | d_\downarrow(\mathbf{k}_{e^+}) c_\uparrow(\mathbf{k}_{e^-}) a_p(\mathbf{k}_\gamma) \tag{0.0.6}$$

The reader will note that we have included a bare vacuum term and an “excited” state. This indicates that by taking time slices of the Feynman diagram (Fig. 1.1) for the dressed vacuum, we have a state of bare vacuum added with a state involving the production of an electron-positron pair and a photon.

The main idea is to “sandwich” the Hamiltonian with the projector, which will “spit out” the desired energies for the system. This will look like:

$$\begin{aligned} P_{\uparrow\downarrow} H P_{\uparrow\downarrow} &= P_{\uparrow\downarrow} (H_D + H_R + H_I + H_C) P_{\uparrow\downarrow} \\ &= P_{\uparrow\downarrow} H_D P_{\uparrow\downarrow} + P_{\uparrow\downarrow} H_R P_{\uparrow\downarrow} + P_{\uparrow\downarrow} H_I P_{\uparrow\downarrow} + P_{\uparrow\downarrow} H_C P_{\uparrow\downarrow} \end{aligned} \quad (0.0.7)$$

We can thus project onto each individual component of the Hamiltonian. We will now go into a detailed overview of the mathematical methods used in resolving the Hamiltonian.

## 2.2 ANALYTIC MATHEMATICAL METHODS

### 2.2.1 Projection onto one spatial dimension

An important feature of our theoretical investigation is projecting our system down to one spatial dimension. Recall that our Hamiltonian involves mostly integrals over all space in three spatial dimensions. For computational ease, we wish to push this down to be a one-dimensional problem. The mechanism which we will use to project down is to constrain our system to a “tube” of length L, and assume a periodicity in the Hamiltonian in two of the arbitrary orthogonal spatial coordinates (reminiscent of a quantum standing wave). For the Dirac term this looks like:

$$H_D = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk_z \sqrt{(mc^2)^2 + (\hbar ck_z)^2} (c_{\uparrow}^{\dagger}(k_z)c_{\uparrow}(k_z) + c_{\downarrow}^{\dagger}(k_z)c_{\downarrow}(k_z) + d_{\uparrow}^{\dagger}(k_z)d_{\uparrow}(k_z) + d_{\downarrow}^{\dagger}(k_z)d_{\downarrow}(k_z)) \quad (0.1.1)$$

Where two of our coordinates have been transformed using a Fourier-like double sum to represent the periodicity. We will see soon that these sums will disappear due to the commutation relations introduced in the simplification.

Ultimately, the other terms in our Hamiltonian become:

$$\begin{aligned}
H_R &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk_z \hbar ck_z \left( a_1^\dagger(k_z) a_1(k_z) + a_2^\dagger(k_z) a_2(k_z) \right) \\
\frac{H_I}{-cq} &= \int_0^L dx \int_0^L dy \int_{-\infty}^{+\infty} dz \Psi^\dagger(\vec{r}) (\vec{\alpha} \cdot A_\perp(\vec{r})) \Psi(\vec{r}) \\
&= \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \sum_{\tilde{n}=-\infty}^{+\infty} \sum_{\tilde{m}=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} \int d\tilde{k}_z \int dk'_z \cdot \\
&\begin{pmatrix} C_\uparrow^\dagger(\tilde{k}) \\ C_\downarrow^\dagger(\tilde{k}) \\ D_\uparrow(\tilde{k}) \\ D_\downarrow(\tilde{k}) \end{pmatrix}^T \sum_{j=1}^2 \frac{a_j(\tilde{k}-k') U^\dagger(\tilde{k}) \vec{\alpha} \cdot \vec{\varepsilon}_j(\tilde{k}-k') U(k') + a_j^\dagger(k'-\tilde{k}) U^\dagger(\tilde{k}) \vec{\alpha} \cdot \vec{\varepsilon}_j(k'-\tilde{k}) U(k')}{\sqrt{\|\tilde{k}-k'\|}} \begin{pmatrix} C_\uparrow(k') \\ C_\downarrow(k') \\ D_\uparrow^\dagger(k') \\ D_\downarrow^\dagger(k') \end{pmatrix}
\end{aligned} \tag{0.1.2}$$

We will also be neglecting the Coulomb term in our renormalization procedure, for the sake of simplicity.

It is an effort that will be worthwhile in the long-term, but unnecessary for our current purposes. With these definitions set forth, we are ready to project onto the Hamiltonian.

## 2.2.2 The Dirac Term

Let us start with the Dirac term. Recall that the process of projecting onto this term will yield the free energy for our massive particles. We begin by left and right-multiplying our projector onto the Hamiltonian:

$$\begin{aligned}
h_D = & \left[ \left| \right\rangle \left\langle \right| + \int d\mathbf{k}_\gamma \int d\mathbf{k}_{e^-} \int d\mathbf{k}_{e^+} a_p^\dagger(\mathbf{k}_\gamma) c_\uparrow^\dagger(\mathbf{k}_{e^-}) d_\downarrow^\dagger(\mathbf{k}_{e^+}) \right] \left\langle \right| \left| d_\downarrow(\mathbf{k}_{e^+}) c_\uparrow(\mathbf{k}_{e^-}) a_p(\mathbf{k}_\gamma) \right\rangle \\
& \cdot \left[ \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk_z \sqrt{(mc^2)^2 + (\hbar ck_z)^2} \left( c_\uparrow^\dagger(k_z) c_\uparrow(k_z) + c_\downarrow^\dagger(k_z) c_\downarrow(k_z) + d_\uparrow^\dagger(k_z) d_\uparrow(k_z) + d_\downarrow^\dagger(k_z) d_\downarrow(k_z) \right) \right] \\
& \cdot \left[ \left| \right\rangle \left\langle \right| + \int d\mathbf{k}_\gamma \int d\mathbf{k}_{e^-} \int d\mathbf{k}_{e^+} a_p^\dagger(\mathbf{k}_\gamma) c_\uparrow^\dagger(\mathbf{k}_{e^-}) d_\downarrow^\dagger(\mathbf{k}_{e^+}) \right] \left\langle \right| \left| d_\downarrow(\mathbf{k}_{e^+}) c_\uparrow(\mathbf{k}_{e^-}) a_p(\mathbf{k}_\gamma) \right\rangle
\end{aligned}
\tag{0.1.3}$$

The reader will note that we will refer to the projected Hamiltonian using the lowercase  $h$ . We will spare the reader the trouble of wading through the minutia, and ultimately, we arrive at:

$$\begin{aligned}
= & \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \int dk'_\gamma \int dk'_{e^-} \int dk'_{e^+} \int dk \sqrt{(mc^2)^2 + (\hbar ck)^2} \\
& \cdot \left( \delta^3(k'_{e^+} - k_{e^+}) \delta^3(k'_\gamma - k_\gamma) \delta^3(k - k_{e^-}) \delta^3(k'_{e^-} - k) + \delta^3(k'_{e^-} - k_{e^-}) \delta^3(k - k_{e^+}) \delta^3(k'_{e^+} - k) \delta^3(k'_\gamma - k_\gamma) \right) \\
& \cdot a_p^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) \left| \right\rangle \left\langle \right| d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^-}) a_p(k'_\gamma)
\end{aligned}
\tag{0.1.4}$$

Where we have obtained delta functions from the *canonical commutation relations*. In quantum field theories, two creation and/or annihilation operators follow the following rules for commutation:

For bosons:

$$\left[ a_i, a_j^\dagger \right] = a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}
\tag{0.1.5}$$

And for fermions:

$$\left\{ a_i, a_j^\dagger \right\} = a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}
\tag{0.1.6}$$

We will see soon that under a continuous index, the commutation relations produce a Dirac delta function, while discrete indices produce the Kronecker delta function. In our case, we have three-dimensional delta

functions corresponding to our 3 indices or dimensions of concern. These delta functions are actually the mechanism that allows us to constrain the system to one spatial dimension. The three-dimensional delta functions above are defined by the following:

$$\delta^3(k_i - k_j) \equiv \delta_{1,m} \delta_{1,n} \delta(k_i - k_j) \quad (0.1.7)$$

We find that only the  $m=1$  and  $n=1$  terms will survive, and the summations disappear leaving the terms inside unaffected. This gives us:

$$\begin{aligned} h_D = & \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \int dk'_\gamma \int dk'_{e^-} \int dk'_{e^+} \int dk \sqrt{(mc^2)^2 + (\hbar ck)^2} \\ & \cdot (\delta(k'_{e^+} - k_{e^+}) \delta(k'_\gamma - k_\gamma) \delta(k - k_{e^-}) \delta(k'_{e^-} - k) + \delta(k'_{e^-} - k_{e^-}) \delta(k - k_{e^+}) \delta(k'_{e^+} - k) \delta(k'_\gamma - k_\gamma)) \\ & \cdot a_p^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) | \rangle \langle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^-}) a_p(k'_\gamma) \end{aligned} \quad (0.1.8)$$

The Dirac delta functions collapse the majority of these integrals, leaving us with:

$$\begin{aligned} h_D = & \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \left( \sqrt{(mc^2)^2 + (\hbar ck_{e^+})^2} + \sqrt{(mc^2)^2 + (\hbar ck_{e^-})^2} \right) \\ & \cdot a_p^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) | \rangle \langle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^-}) a_p(k'_\gamma) \end{aligned} \quad (0.1.9)$$

One may note that this expression makes sense, as we have recovered inside the integrals a relativistic expression for free energy corresponding to one electron and one positron.

### 2.2.3 The Radiation Term

We will now perform the same procedure for the radiation term. Recall that we expect to recover photon energies from this procedure. We have:

$$\begin{aligned}
h_R = & \left( | \rangle \langle | + \int d\mathbf{k}_\gamma \int d\mathbf{k}_{e^-} \int d\mathbf{k}_{e^+} a_p^\dagger(\mathbf{k}_\gamma) c_\uparrow^\dagger(\mathbf{k}_{e^-}) d_\downarrow^\dagger(\mathbf{k}_{e^+}) | \rangle \langle | d_\downarrow(\mathbf{k}_{e^+}) c_\uparrow(\mathbf{k}_{e^-}) a_p(\mathbf{k}_\gamma) \right) \\
& \cdot \left( \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk_z \hbar ck_z \left( a_1^\dagger(k_z) a_1(k_z) + a_2^\dagger(k_z) a_2(k_z) \right) \right) \\
& \cdot \left( | \rangle \langle | + \int d\mathbf{k}_\gamma \int d\mathbf{k}_{e^-} \int d\mathbf{k}_{e^+} a_p^\dagger(\mathbf{k}_\gamma) c_\uparrow^\dagger(\mathbf{k}_{e^-}) d_\downarrow^\dagger(\mathbf{k}_{e^+}) | \rangle \langle | d_\downarrow(\mathbf{k}_{e^+}) c_\uparrow(\mathbf{k}_{e^-}) a_p(\mathbf{k}_\gamma) \right)
\end{aligned} \tag{0.1.10}$$

The reader will note that the creation and annihilation operators that represent the photons in this equation contain an arbitrary polarization state  $p$ . This polarization was not of importance in the Dirac term, as all photonic terms went to zero. In the radiation term, however, this comes into importance for the commutation relations. We will select the polarization of our photonic terms in the projector to be 1 for the remainder of our problem. With this in mind, eq. 2.2.10 simplifies to:

$$\begin{aligned}
h_R = & \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int dk \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \int dk'_\gamma \int dk'_{e^-} \int dk'_{e^+} \hbar ck \\
& \delta^3(k'_{e^+} - k_{e^+}) \delta^3(k'_{e^-} - k_{e^-}) \delta^3(k - k_\gamma) \delta^3(k'_\gamma - k) \\
& \cdot a_p^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) | \rangle \langle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^-}) a_p(k'_\gamma)
\end{aligned} \tag{0.1.11}$$

As before, the delta functions will collapse the summations and some of the integrals. Thus, the radiation term reduces to:

$$h_R = \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \hbar ck_\gamma \cdot a_p^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) | \rangle \langle | d_\downarrow(k_{e^+}) c_\uparrow(k_{e^-}) a_p(k_\gamma) \tag{0.1.12}$$

#### 2.2.4 The Interaction Term

Now we move onto the interaction term of the QED Hamiltonian. Recall that the interaction term describes the coupling between massive particles and photons in the electric field. The interaction term will essentially “perturb” the energy of the system (we use the word independently of perturbation theory, which is a whole branch of study within quantum mechanics). Ultimately, the interaction term has a tendency to lower the energy of the system by an amount determined by the interaction term itself. In

their work on dressed electron eigen-states<sup>3</sup>, Scott Glasgow et. al produced the following figure displaying how the interaction term pulls down the energy of electron eigen-states:

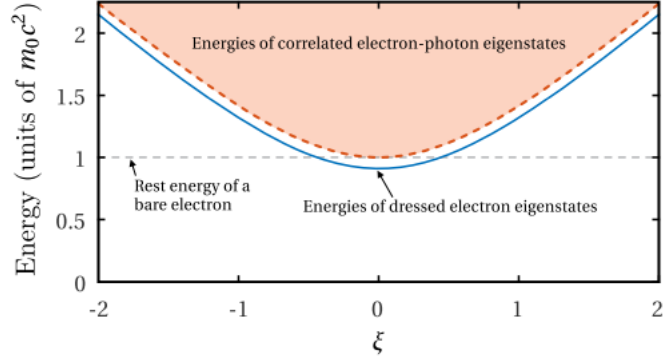


Figure 1.1: Illustration of effect of the interaction term on dressed electron eigen-states, very much analogous to the problem at hand. The red curve illustrates the free energy of the eigen-states independent of the interaction term, while the blue curve includes the effect of the interaction term

This effect of the interaction term on the dressed electron eigenstates is analogous to our investigation on the dressed vacuum. The Dirac and radiation terms will pull out eigen-energies of the particles, and the interaction term will perturb that total energy downward. We will see this reflected in the sign of the interaction term of the Hamiltonian.

Now that we have more thoroughly established the effect of the interaction term, we are prepared to apply our projector. As mentioned in Eq. (2.2.2), the interaction term as constrained to one spatial dimension becomes:

$$\frac{H_I}{-cq} = \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \sum_{\tilde{n}=-\infty}^{+\infty} \sum_{\tilde{m}=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} \int d\tilde{k}_z \int dk'_z \cdot \quad (0.1.13)$$

$$\begin{pmatrix} C_{\uparrow}^{\dagger}(\tilde{k}) \\ C_{\downarrow}^{\dagger}(\tilde{k}) \\ D_{\uparrow}(\tilde{k}) \\ D_{\downarrow}(\tilde{k}) \end{pmatrix}^T \sum_{j=1}^2 \frac{a_j(\tilde{k}-k')U^{\dagger}(\tilde{k})\vec{\alpha} \cdot \vec{\varepsilon}_j(\tilde{k}-k')U(k') + a_j^{\dagger}(k'-\tilde{k})U^{\dagger}(\tilde{k})\vec{\alpha} \cdot \vec{\varepsilon}_j(k'-\tilde{k})U(k')}{\sqrt{\|\tilde{k}-k'\|}} \begin{pmatrix} C_{\uparrow}(k') \\ C_{\downarrow}(k') \\ D_{\uparrow}^{\dagger}(k') \\ D_{\downarrow}^{\dagger}(k') \end{pmatrix}$$



It is important at this point to introduce some relevant definitions for several of the terms involved in the interaction term.

The  $\vec{\alpha}$  in Eq. (2.2.13) is related to the famous Pauli spin matrices by the following rule:

$$\vec{\alpha} \equiv (\gamma^1 \quad \gamma^2 \quad \gamma^3), \gamma^\mu \equiv \begin{pmatrix} 0_{2 \times 2} & \sigma_\mu \\ \sigma_\mu & 0_{2 \times 2} \end{pmatrix}, \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (0.1.14)$$

The dot product  $\vec{\alpha} \cdot \vec{\varepsilon}_j$  in the sum essentially pulls out the relevant spin matrix in the form  $\gamma^\mu$ . For example:

$$\vec{\alpha} \cdot \vec{\varepsilon}_1 (\vec{k} - \vec{k}') = \text{sign}(\mathbf{k}'_\gamma) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (0.1.15)$$

And likewise for the higher-order j terms. The U-matrices are defined as follows:

$$U(\vec{k}) = \cos(\theta_{\vec{k}} / 2) I - \sin(\theta_{\vec{k}} / 2) \beta \vec{\alpha} \cdot \hat{k} \quad (0.1.16)$$

Where

$$\theta_{\vec{k}} = \arctan\left(\frac{\hbar \|\vec{k}\|}{mc}\right), \beta \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (0.1.17)$$

Recall that we are constraining this system to one spatial dimension, and our momentum state will be constrained to the z-axis. Therefore, we will pass in  $(0, 0, k_z)$  into our U matrix:

$$\begin{aligned}
U(0,0,k_z) &= \cos(\theta_{(0,0,k_z)}/2)I - \sin(\theta_{(0,0,k_z)}/2)\beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \cos\left(\frac{\theta_{k_z}}{2}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \sin\left(\frac{\theta_{k_z}}{2}\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(\frac{\theta_{k_z}}{2}\right) & 0 & -\sin\left(\frac{\theta_{k_z}}{2}\right) & 0 \\ 0 & \cos\left(\frac{\theta_{k_z}}{2}\right) & 0 & \sin\left(\frac{\theta_{k_z}}{2}\right) \\ -\sin\left(\frac{\theta_{k_z}}{2}\right) & 0 & \cos\left(\frac{\theta_{k_z}}{2}\right) & 0 \\ 0 & \sin\left(\frac{\theta_{k_z}}{2}\right) & 0 & \cos\left(\frac{\theta_{k_z}}{2}\right) \end{pmatrix} \tag{0.1.18}
\end{aligned}$$

With these definitions in mind, we may apply the projector to the interaction term. We find:

$$\begin{aligned}
h_I &= \left(\frac{\hbar}{2\varepsilon_0 c 2\pi L^2}\right)^{\frac{1}{2}} \int d\tilde{k}_z \int dk'_z \int dk'_\gamma \int dk'_{e^-} \int dk'_{e^+} \delta(k'_\gamma - (\tilde{k}_z - k'_z)) \delta(k'_{e^+} - \tilde{k}_z) \delta(k'_{e^-} - k'_z) \\
&\quad | \rangle \langle | \mathbf{e}_4^T \frac{U^\dagger(\tilde{k}_z) \vec{\alpha} \cdot \vec{\varepsilon}_1(\tilde{k}_z - k'_z) U(k'_z)}{\sqrt{|\tilde{k}_z - k'_z|}} \mathbf{e}_1 | \rangle \langle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^-}) a_1(k'_\gamma) \\
&\quad + \left(\frac{\hbar}{2\varepsilon_0 c 2\pi L^2}\right)^{\frac{1}{2}} \int d\tilde{k}_z \int dk'_z \int dk_\gamma \int dk_{e^-} \int dk_{e^+} \delta(\tilde{k}_z - k_{e^-}) \delta((k'_z - \tilde{k}_z) - k_\gamma) \delta(k'_z - k_{e^+}) \\
&\quad a_1^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^-}) d_\downarrow^\dagger(k_{e^+}) | \rangle \langle | \mathbf{e}_1^T \frac{U^\dagger(\tilde{k}_z) \vec{\alpha} \cdot \vec{\varepsilon}_1(k'_z - \tilde{k}_z) U(k'_z)}{\sqrt{|k'_z - \tilde{k}_z|}} \mathbf{e}_4 | \rangle \langle | \\
\end{aligned} \tag{0.1.19}$$

Where again, we obtain delta functions from the commutation relations for the creation and annihilation operators. Using our definitions from before, we can multiply the matrices together with relatively little difficulty. This yields:

$$\begin{aligned}
\frac{h_l}{-qc} &= \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \int dk'_\gamma \int dk'_{e^+} \\
& \frac{\text{sign}(k'_\gamma) \left( C \left( \frac{\hbar(k'_{e^+} - k'_\gamma)}{mc} \right) C \left( \frac{\hbar k'_{e^+}}{mc} \right) - S \left( \frac{\hbar k'_{e^+}}{mc} \right) S \left( \frac{\hbar(k'_{e^+} - k'_\gamma)}{mc} \right) \right)}{\sqrt{|k'_\gamma|}} \langle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^+} - k'_\gamma) a_1(k'_\gamma) \\
& + \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \int dk_\gamma \int dk_{e^+} \\
& \frac{\text{sign}(k_\gamma) \left( C \left( \frac{\hbar(k_{e^+} - k_\gamma)}{mc} \right) C \left( \frac{\hbar k_{e^+}}{mc} \right) + S \left( \frac{\hbar k_{e^+}}{mc} \right) S \left( \frac{\hbar(k_{e^+} - k_\gamma)}{mc} \right) \right)}{\sqrt{|k_\gamma|}} \langle | a_1^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^+} - k_\gamma) d_\downarrow^\dagger(k_{e^+}) | \rangle
\end{aligned}
\tag{0.1.20}$$

The S and C functions in eq. 2.2.20 are likely unfamiliar notation. They are derived from the cosine and sine functions introduced in eq. 2.2.16 and are defined as follows:

$$\begin{aligned}
\sin(\theta_{k_z} / 2) &= \sqrt{\frac{1}{2}(1 - \cos \theta_{k_z})} = \sqrt{\frac{1}{2} \left( 1 - \cos \left( \arctan \left( \frac{\hbar k_z}{mc} \right) \right) \right)} \\
&= \sqrt{\frac{1}{2} \left( 1 - \frac{mc}{\sqrt{((mc)^2 + \hbar^2 k_z^2)}} \right)} = \sqrt{\frac{1}{2} \left( 1 - \frac{1}{\sqrt{\left( 1 + \frac{\hbar k_z}{mc} \right)}} \right)} \\
&\equiv S \left( \frac{\hbar k_z}{mc} \right)
\end{aligned}
\tag{0.1.21}$$

$$\begin{aligned}
\cos(\theta_{k_z} / 2) &= \sqrt{\frac{1}{2}(1 + \cos(\theta_{k_z}))} = \sqrt{\frac{1}{2}\left(1 + \cos\left(\arctan\left(\frac{\hbar k_z}{mc}\right)\right)\right)} \\
&= \sqrt{\frac{1}{2}\left(1 + \frac{mc}{\sqrt{(mc)^2 + \hbar^2 k_z^2}}\right)} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \frac{\hbar k_z}{mc}}}\right)} \\
&\equiv C\left(\frac{\hbar k_z}{mc}\right)
\end{aligned} \tag{0.1.22}$$

This notation offers us conciseness, and more importantly, a way to create a function whose argument is

$\frac{\hbar k}{mc}$ . We will later define unitless variables in terms of this quantity.

### 2.2.5 Photon “Mass”

We have now found a complete simplified term for the QED Hamiltonian for the dressed vacuum system.

There is one glaring issue with this. The reader may notice that the interaction term goes as  $\frac{1}{\sqrt{k_\gamma}}$ . This is

very problematic, as this integral clearly diverges as the photon momentum approaches zero. Here, quantum field theorists historically have turned to perturbation theory to solve the remainder of the problem. We, however, seek a more analytical method of solving for these energy states. We will therefore use a mathematical trick known as a mollifier.

The mollifier will be introduced in the denominator of the interaction term, in order to allow for a normalizable function. Our addition of the mollifier can be thought of as a photon “mass” term, which we will allow to approach zero. Numerically, the mollifier allows us to obtain non-infinite values for the interaction term, which we can use as a basis for renormalization. Our transformation will be as follows:

$$\sqrt{k_\gamma} \rightarrow (k_\gamma^2 + \varepsilon^2)^{1/4} \tag{0.1.23}$$

Where epsilon acts as our mollifier. We have encoded a virtual “mass” inside epsilon, and in the limit that epsilon goes to zero, we have our initial denominator. Our interaction term then becomes:

$$\begin{aligned}
h_I = & -cq \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \int dk'_\gamma \int dk'_{e^+} \\
& | \rangle \rangle \left| \frac{\text{sign}(k'_\gamma) \left( C \left( \frac{\hbar(k'_{e^+} - k'_\gamma)}{mc} \right) C \left( \frac{\hbar k'_{e^+}}{mc} \right) - S \left( \frac{\hbar k'_{e^+}}{mc} \right) S \left( \frac{\hbar(k'_{e^+} - k'_\gamma)}{mc} \right) \right)}{(k'^2_\gamma + \varepsilon^2)^{1/4}} \right| \rangle \rangle | d_\downarrow(k'_{e^+}) c_\uparrow(k'_{e^+} - k'_\gamma) a_1(k'_\gamma) \\
& -cq \left( \frac{\hbar}{2\varepsilon_0 c 2\pi L^2} \right)^{\frac{1}{2}} \int dk_\gamma \int dk_{e^+} \\
& a_1^\dagger(k_\gamma) c_\uparrow^\dagger(k_{e^+} - k_\gamma) d_\downarrow^\dagger(k_{e^+}) | \rangle \rangle \left| \frac{\text{sign}(k_\gamma) \left( C \left( \frac{\hbar(k_{e^+} - k_\gamma)}{mc} \right) C \left( \frac{\hbar k_{e^+}}{mc} \right) + S \left( \frac{\hbar k_{e^+}}{mc} \right) S \left( \frac{\hbar(k_{e^+} - k_\gamma)}{mc} \right) \right)}{(k_\gamma^2 + \varepsilon^2)^{1/4}} \right| \rangle \rangle |
\end{aligned}$$

(0.1.24)

With this in mind, we are now ready to numerically solve the eigenvalue problem for the ground state energy of the dressed vacuum.

## 2.3 NUMERICAL MATHEMATICAL METHODS

We have found analytical expressions for the QED Hamiltonian in our dressed vacuum system. To be able to properly solve it for the allowed energy states, we need to turn to numerical methods to solve for the energies. We will ultimately find that the preciseness of our ability to find energies is constrained by our insertion of the mollifier.

### 2.3.1 Transformation to Unitless Variables

We wish to work in purely unitless quantities. Thus, we define the transformations,

$$p_i = \frac{\hbar k_i}{mc}, k_i = \frac{p_i \cdot mc}{\hbar}, dk_i = \frac{mc}{\hbar} dp_i. \text{ We additionally allow } \alpha = \frac{q^2}{4\pi\epsilon_0\hbar c}, \text{ which is a unitless quantity.}$$

We begin with the interaction term of the QED Hamiltonian. When applying the variable transformations,

we obtain:

$$\begin{aligned} h_I = & -\frac{\hbar c}{L} \sqrt{\alpha} \left( \frac{mc}{\hbar} \right)^2 \int dp'_\gamma \int dp'_{e^+} \\ & \frac{\text{sign}(p'_\gamma) \left( C(p'_{e^+} - p'_\gamma) C(p'_{e^+}) - S(p'_{e^+}) S(p'_{e^+} - p'_\gamma) \right)}{\sqrt{\frac{mc}{\hbar} (p'^2_\gamma + \epsilon^2)^{1/4}}} | \rangle \langle | d_\downarrow \left( \frac{mc}{\hbar} p'_{e^+} \right) c_\uparrow \left( \frac{mc}{\hbar} (p'_{e^+} - p'_\gamma) \right) a_1 \left( \frac{mc}{\hbar} p'_\gamma \right) \\ & -\frac{\hbar c}{L} \sqrt{\alpha} \left( \frac{mc}{\hbar} \right)^2 \int dp_\gamma \int dp_{e^+} \\ & a_1^\dagger \left( \frac{mc}{\hbar} p_\gamma \right) c_\uparrow^\dagger \left( \frac{mc}{\hbar} (p_{e^+} - p_\gamma) \right) d_\downarrow^\dagger \left( \frac{mc}{\hbar} p_{e^+} \right) | \rangle \langle | \frac{\text{sign}(p_\gamma) \left( C(p_{e^+} - p_\gamma) C(p_{e^+}) + S(p_{e^+}) S(p_{e^+} - p_\gamma) \right)}{\sqrt{\frac{mc}{\hbar} (p^2_\gamma + \epsilon^2)^{1/4}}} \end{aligned} \quad (0.2.1)$$

To streamline our notation, we introduce the functions:

$$\begin{aligned} g'(p'_{e^+}, p'_\gamma) & := -\frac{\hbar\sqrt{\alpha}}{mcL} \frac{\text{sign}(p'_\gamma) \left( C(p'_{e^+} - p'_\gamma) C(p'_{e^+}) - S(p'_{e^+}) S(p'_{e^+} - p'_\gamma) \right)}{(p'^2_\gamma + \epsilon^2)^{1/4}} \\ g(p_{e^+}, p_\gamma) & := -\frac{\hbar\sqrt{\alpha}}{mcL} \frac{\text{sign}(p_\gamma) \left( C(p_{e^+} - p_\gamma) C(p_{e^+}) + S(p_{e^+}) S(p_{e^+} - p_\gamma) \right)}{(p^2_\gamma + \epsilon^2)^{1/4}} \end{aligned} \quad (0.2.2)$$

We additionally define unitless basis vectors:

$$\langle p'_{e^+}, p'_{e^+} - p'_\gamma, p'_\gamma | = \langle \left| \left( \frac{mc}{\hbar} \right)^{3/2} d_\downarrow \left( \frac{mc}{\hbar} p'_{e^+} \right) c_\uparrow \left( \frac{mc}{\hbar} (p'_{e^+} - p'_\gamma) \right) a_1 \left( \frac{mc}{\hbar} p'_\gamma \right) \right. \rangle \quad (0.2.3)$$

And likewise for the kets. This leaves our simplified interaction term as:

$$\begin{aligned} \frac{h_I}{mc^2} = & \int dp'_\gamma \int dp'_{e^+} g'(p'_{e^+}, p'_\gamma) | \rangle \langle p'_{e^+}, p'_{e^+} - p'_\gamma, p'_\gamma | \\ & + \int dp_\gamma \int dp_{e^+} g(p_{e^+}, p_\gamma) | p_\gamma, p_{e^+} - p_\gamma, p_{e^+} \rangle \langle | \end{aligned} \quad (0.2.4)$$

For the Dirac and interaction terms, we define our unitless basis vectors as:

$$\begin{aligned} | p_\gamma, p_{e^-}, p_{e^+} \rangle \langle p_\gamma, p_{e^-}, p_{e^+} | = & \left( \frac{mc}{\hbar} \right)^3 \cdot a_1^\dagger \left( \frac{mc}{\hbar} p_\gamma \right) c_1^\dagger \left( \frac{mc}{\hbar} p_{e^-} \right) d_1^\dagger \left( \frac{mc}{\hbar} p_{e^+} \right) | \rangle \langle | \\ & | d_1 \left( \frac{mc}{\hbar} p_{e^+} \right) c_1 \left( \frac{mc}{\hbar} p_{e^-} \right) a_1 \left( \frac{mc}{\hbar} p_\gamma \right) \end{aligned} \quad (0.2.5)$$

After the variable transformations, this gives us a fully simplified Hamiltonian (in units of  $mc^2$ ):

$$\begin{aligned} H = & \frac{h_D}{mc^2} + \frac{h_R}{mc^2} + \frac{h_I}{mc^2} \\ = & \int dp_\gamma \int dp_{e^-} \int dp_{e^+} \left[ \left( \sqrt{1+p_{e^-}^2} + \sqrt{1+p_{e^+}^2} + p_\gamma \right) \cdot | p_\gamma, p_{e^-}, p_{e^+} \rangle \langle p_\gamma, p_{e^-}, p_{e^+} | \right. \\ & \left. + g(p_{e^+}, p_\gamma) | p_\gamma, p_{e^+} - p_\gamma, p_{e^+} \rangle \langle | \right. \\ & \left. + \int dp'_\gamma \int dp'_{e^+} g'(p'_{e^+}, p'_\gamma) | \rangle \langle p'_{e^+}, p'_{e^+} - p'_\gamma, p'_\gamma | \right] \end{aligned} \quad (0.2.6)$$

### 2.3.2 Solving for the Ground-State Energy

We can now analytically solve the eigenvalue equation where the energies will be the eigenvalues of the Hamiltonian. For renormalization, we are only concerned with the lowest or ground-state eigenvalue of the Hamiltonian.

We have to solve the eigenvalue problem:

$$H\left|\phi(p_{e^-}, p_{e^+}, p_\gamma, E)\right\rangle = E\left|\phi(p_{e^-}, p_{e^+}, p_\gamma, E)\right\rangle \quad (0.2.7)$$

We assume the eigenstates to have the general form:

$$\left|\phi(p_{e^-}, p_{e^+}, p_\gamma, E)\right\rangle := c_1(E)\left|\right\rangle + \int dp_{e^-} \int dp_{e^+} \int dp_\gamma c_2(p_{e^-}, p_{e^+}, p_\gamma, E) \left|a_1^\dagger(p_\gamma) c_{\uparrow}^\dagger(p_{e^-}) d_{\downarrow}^\dagger(p_{e^+})\right\rangle \quad (0.2.8)$$

With two terms. One corresponds to the bare vacuum, and the other corresponds to the state involving and electron-positron pair. In order to solve for the ground state energy, we have to apply this function to the unitless Hamiltonian, and solve for the energy. This results in:

$$E\left|\phi(p_{e^-}, p_{e^+}, p_\gamma, E)\right\rangle = \left( \int dp_\gamma \int dp_{e^-} \int dp_{e^+} \left[ \left( \sqrt{1+p_{e^-}^2} + \sqrt{1+p_{e^+}^2} + p_\gamma \right) \left|p_\gamma, p_{e^-}, p_{e^+}\right\rangle \left\langle p_\gamma, p_{e^-}, p_{e^+} \right| \right. \right. \\ \left. \left. + g(p_{e^+}, p_\gamma) \left|p_\gamma, p_{e^+} - p_\gamma, p_{e^+}\right\rangle \left\langle \right| \right. \right. \\ \left. \left. + \int dp'_\gamma \int dp'_{e^+} g'(p'_{e^+}, p'_\gamma) \left| \right\rangle \left\langle p'_{e^+}, p'_{e^+} - p'_\gamma, p'_\gamma \right| \right. \right. \\ \left. \left. \cdot \left( c_1(E) \left|\right\rangle + \int d\tilde{p}_{e^-} \int d\tilde{p}_{e^+} \int d\tilde{p}_\gamma c_2(\tilde{p}_{e^-}, \tilde{p}_{e^+}, \tilde{p}_\gamma, E) \left|a_1^\dagger(\tilde{p}_\gamma) c_{\uparrow}^\dagger(\tilde{p}_{e^-}) d_{\downarrow}^\dagger(\tilde{p}_{e^+})\right\rangle \right) \right] \right) \quad (0.2.9)$$

Again, we find commutation relations that give us delta functions that remove the tilde variables of integration, and leave us with:

$$E\left|\phi(p_{e^-}, p_{e^+}, p_\gamma, E)\right\rangle = \int dp_\gamma \int dp_{e^-} \int dp_{e^+} c_1(E) g(p_{e^+}, p_\gamma) \left|p_\gamma, p_{e^+} - p_\gamma, p_{e^+}\right\rangle \\ + \left(\frac{mc}{\hbar}\right)^{3/2} \int dp_\gamma \int dp_{e^-} \int dp_{e^+} c_2(p_{e^-}, p_{e^+}, p_\gamma, E) \left( \sqrt{1+p_{e^-}^2} + \sqrt{1+p_{e^+}^2} + p_\gamma \right) \left|p_\gamma, p_{e^-}, p_{e^+}\right\rangle \\ + \left(\frac{mc}{\hbar}\right)^{3/2} \int dp'_\gamma \int dp'_{e^+} c_2(p'_{e^+} - p'_\gamma, p'_{e^+}, p'_\gamma, E) g'(p'_{e^+}, p'_\gamma) \left| \right\rangle \quad (0.2.10)$$



From here, we can obtain two equations for two unknowns. One is found by right-multiplying both sides by  $\langle \text{---} |$ , which gives us the equation:

$$\begin{aligned} & \left(\frac{mc}{\hbar}\right)^{3/2} \int dp'_\gamma \int dp'_{e^+} c_2(p'_{e^+} - p'_\gamma, p'_{e^+}, p'_\gamma, E) g'(p'_{e^+}, p'_\gamma) \\ & = Ec_1(E) \langle \text{---} | \rangle + E \int dp_{e^-} \int dp_{e^+} \int dp_\gamma \left( \begin{aligned} & c_2(p_{e^-}, p_{e^+}, p_\gamma, E) \\ & \langle \text{---} | a_1^\dagger(p_\gamma) c_1^\dagger(p_{e^-}) d_1^\dagger(p_{e^+}) \rangle \end{aligned} \right) \end{aligned} \quad (0.2.11)$$

The second equation is obtained by right-multiplying both sides by  $\langle \tilde{p}_\gamma, \tilde{p}_{e^-}, \tilde{p}_{e^+} |$ . Again, delta functions will remove the tilde variables, giving us:

$$c_1(E) g(p_{e^+}, p_\gamma) + c_2(p_{e^-}, p_{e^+}, p_\gamma, E) \left( \sqrt{1 + p_{e^-}^2} + \sqrt{1 + p_{e^+}^2} + p_\gamma \right) = Ec_2(p_{e^-}, p_{e^+}, p_\gamma, E) \quad (0.2.12)$$

And thus, our system of equations is:

$$\begin{aligned} & \int dp'_\gamma \int dp'_{e^+} c_2(p'_{e^+} - p'_\gamma, p'_{e^+}, p'_\gamma, E) g'(p'_{e^+}, p'_\gamma) = Ec_1(E) \\ & c_1(E) g(p_{e^+}, p_\gamma) + c_2(p_{e^-}, p_{e^+}, p_\gamma, E) \left( \sqrt{1 + p_{e^-}^2} + \sqrt{1 + p_{e^+}^2} + p_\gamma \right) = Ec_2(p_{e^-}, p_{e^+}, p_\gamma, E) \end{aligned} \quad (0.2.13)$$

From here, we simply solve for  $c_2$  using the second equation, and insert it back into the first equation. We find that  $c_1$  cancels out, leaving us with:

$$E = - \int dp_\gamma \int dp_{e^+} \frac{g(p_{e^+}, p_\gamma) g'(p_{e^+}, p_\gamma)}{-E + \left( \sqrt{1 + (p_{e^+} - p_\gamma)^2} + \sqrt{1 + p_{e^+}^2} + p_\gamma \right)} \quad (0.2.14)$$

We have a transcendental equation for E! Luckily, we can solve transcendental equations without too much issue using computer programming. The programming method we will be investigating here is

using python code to solve this equation for various values of E and plotting the solutions against the values for E.

In order to numerically compute these integrals, we will begin by discretizing our equation in order to code it and solve it. We will create a 2x2 grid, with grid points of size  $\Delta p$  in each direction of p. We have to be careful with our choice of grid point size. We will choose our grid points to be smaller than our value for epsilon. The discretized numerical version of our transcendental equation for ground state energy becomes:

$$E = - \sum_{m,n=-\infty}^{\infty} \frac{g(p_m, p_n) g'(p_m, p_n)}{-E + \left( \sqrt{1 + (p_m - p_n)^2} + \sqrt{1 + p_m^2 + p_n^2} \right)} (\Delta p_m) (\Delta p_n) \quad (0.2.15)$$

The results of the numerical calculations will be discussed hereafter in chapter 3.

## 3 RESULTS AND CONCLUSION

---

As discussed in chapter 2, our ultimate goal was to use our analytical techniques to open the door to finding a numerical basis using the dressed state of the vacuum. This involves numerically solving the Hamiltonian introduced in subsection 2.3.2. This chapter will provide the results of this numerical analysis, including a numerical value for the ground state energy of the dressed vacuum with given parameters. Implications of these results for projects involving similar particle conditions will additionally be discussed.

### 3.1 PYTHON METHODS

#### 3.1.1 Setting up Momentum Grids

As introduced in section 2.3.2, the Hamiltonian gave us a transcendental equation for the energies of the dressed vacuum system under given parameters. We have:

$$E = - \sum_{m,n=-\infty}^{\infty} \frac{g(p_m, p_n) g'(p_m, p_n)}{-E + \left( \sqrt{1 + (p_m - p_n)^2} + \sqrt{1 + p_m^2} + p_n \right)} (\Delta p_m) (\Delta p_n) \quad (1.1.1)$$

Where the definitions of the  $g$ -functions are given in chapter 2 by Eq. (2.3.2). If it is helpful, the reader can imagine that the dressed vacuum system that we are describing is one that has a finite “Planck length,” where the Planck length is the value which we choose for the grid spacing. We are therefore abandoning our continuous labels in favor of a numerically palatable discretized grid in momentum space.

The discrete method to solving an integral equation, however, comes with the caveat that we have to be conscious of what we choose to be our bounds for integration. If we choose them to be too small, we risk missing out on points of the integration, leaving our ultimate calculations vastly imprecise. To get an idea of the grid that we will be working with, consider the following plot:

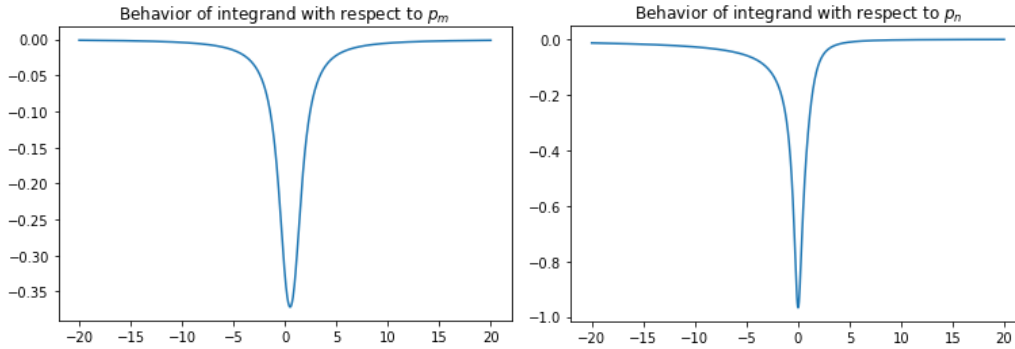


Figure 2.1 Behavior of the integrands in the momentum coordinate for the photon and positron directions in the dressed vacuum system. The reader will note that the functions quickly move to zero as momentum moves away from zero.

The figure shows the behavior of the integrand in the  $p_m$  direction. We fix both  $p_n$  and  $E$  to be one, and examine the behavior of the function with respect to  $p_m$ . We likewise do the same process for  $p_n$ . The behavior of the functions indicates fairly clearly that the functions essentially approach zero by the time we reach  $\pm 20$  on the momentum axis. This tells us that  $\pm 20$  is a good limit to use in both momentum coordinates. We therefore assign a  $2 \times 2$  grid with those limits and a fixed number of grid points in each direction.

### 3.1.2 Solving the Transcendental Equation

Now that we have a grid over which we may sum, we can proceed to set up our transcendental equation. The standard way to graphically solve a transcendental equation is to graphically compare the left-hand side of the equation to the right-hand side, and find their intersections. In our case, we are purely interested in the lowest (ground-state) energy, so we are looking for the intersection that is closest to zero. In Python, we create an energy grid. We constrain the grid to contain only values of energy between 0 and 2. The energy grid will have a fixed number grid points over which we can evaluate the integral for the given value of  $E$ . By performing the integration process described in subsection 3.1.1, and looping through each point on the energy grid, we can obtain an expression for the right-hand side as a function of  $E$ . We may perform this with varying parameters; however, we will fix epsilon to be .2, the number of

momentum grid points to be 2000, and the number of energy grid points to be 20. When we plot the right-hand side function superimposed on a plot of the line  $f(x) = x$  we obtain the following plot:

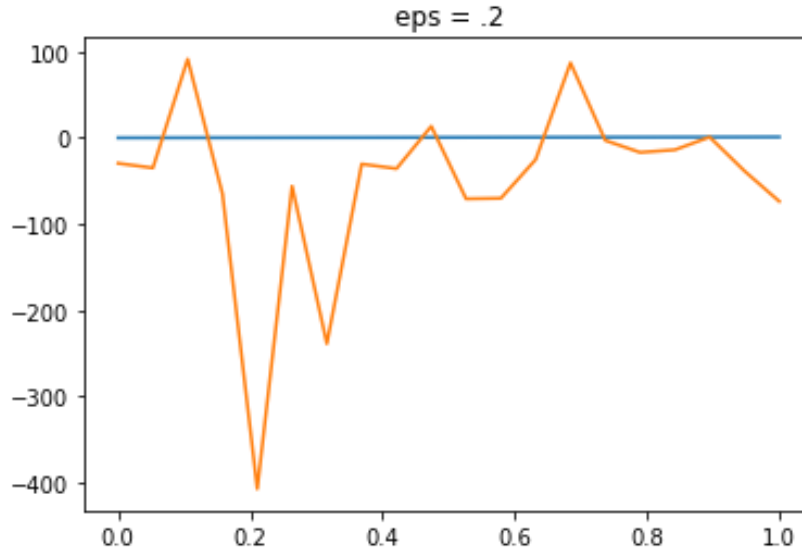


Figure 3.2 Plot of the right-hand side of Eq. (3.1.1) vs. the line  $f(x) = x$

Much of the right-hand side function is out of the window, as we have focused in on the relevant window, where we can obtain intersections with the left-hand side function. Furthermore, when we zoom into the first (lowest energy) intersection, we obtain the following plot:

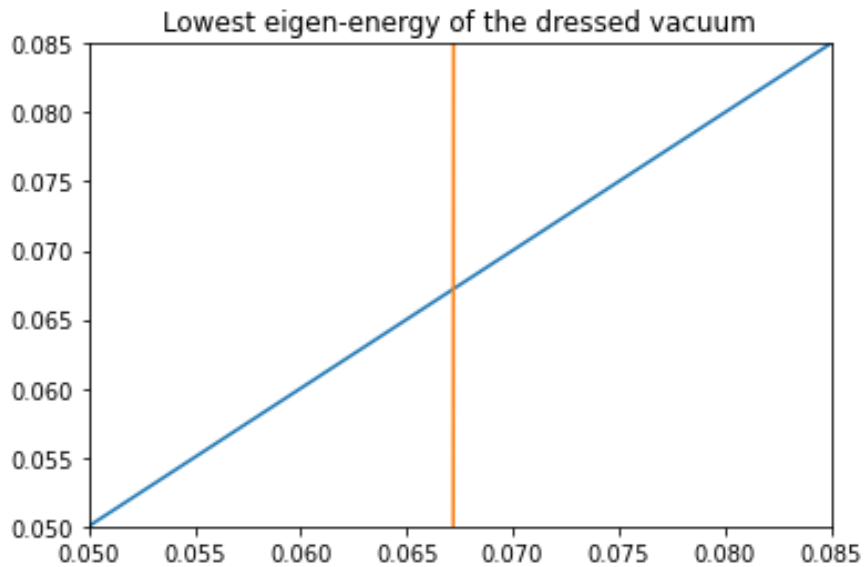


Figure 3.3 The zoomed-in look at the lowest-energy solution to the transcendental equation. We have additionally added a horizontal line to highlight the value of the intersection.

From the plot, we ascertain that the value of the ground-state energy for the dressed vacuum is about  $E = 0.067mc^2$ .

## 3.2 DISCUSSION OF RESULTS

### 3.2.1 Theoretical Assumptions

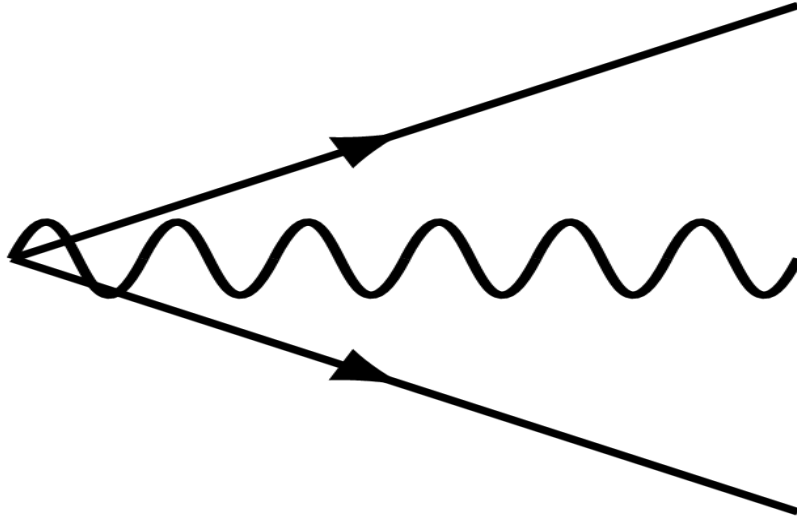
Many of the theoretical methods we have used in our analysis of the QED dressed vacuum may make the reader somewhat uncomfortable, and rightfully so. We have made many wild, sweeping assumptions about the universe that don't exist in reality. Throughout this experiment, we have almost been "playing God" by our choice of conditions that brought us to an analytical solution.

For starters, we have made the assumption that in our fictional universe, the only fundamental force is the electromagnetic interaction. Strong and weak interactions do not affect our fundamental particles. We have chosen our fundamental constants arbitrarily. We have assigned a non-zero massive property to the photon. We have constrained our universe to possess only one spatial dimension, already rendering it unphysical.

With all of these modifications to the assumptions of what reality is in this experiment, one may justly ask, what is the point? Aside from the pure fun of producing dozens of pages of mathematical documents, we have done something that hasn't really been done before: we have solved QED analytically! We can furthermore discover in the future what happens when we let the massive property of our photon get closer and closer to zero. Barring our ability to find a limit to our integrals as the photon mass goes to zero, we can choose a non-zero mass, and use the  $E = 0.171815mc^2$  from section 3.1, or a result found using a different mass, and form a basis for a similar Hamiltonian.

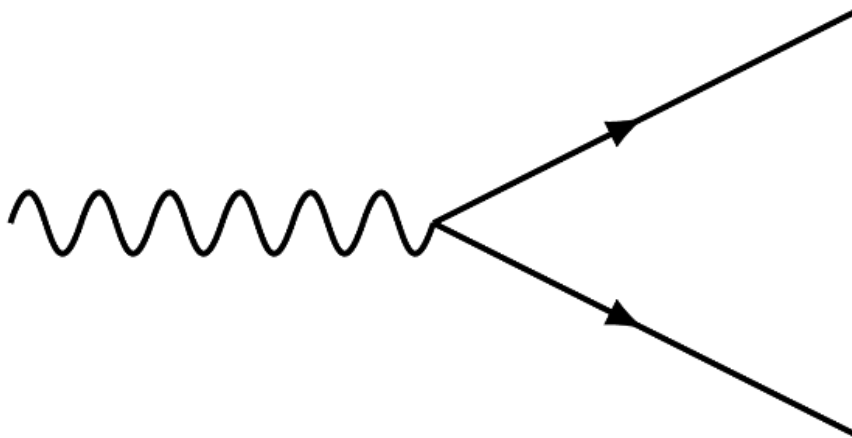
### 3.2.2 Potential for Future Non-Perturbative QED

Consider the Feynman diagram for the dressed vacuum which we have been examining:



*Figure 3.4 Feynman diagram for dressed vacuum system*

As we have discussed extensively, this diagram represents a photon and an electron-positron pair spontaneously being emitted. Consider also the following Feynman diagram:



*Figure 3.5 Feynman diagram of a photon decaying into an electron-positron pair*

Here, we have a system in which one photon decays into an electron-positron pair. It turns out that the Hamiltonian for this system is actually fairly similar to that of the dressed vacuum! Due to the similarity between these systems, we can think of the dressed vacuum as the stage in which this pair-production

system plays its role. We can potentially use the results we found in section 3.1 to renormalize the system, making a “zero” state for the pair production system to exist in.

### **3.3 CONCLUSION**

While this analytical method to QED has its limitations, renormalization allows us to obtain eigenvalues to the Hamiltonian, something that is unavailable to us using perturbation theory. The results of this project ultimately provide a window for us to solve QED without the limitations of perturbation theory. Further investigation, however, must be performed to give us an idea of whether or not this is worth pursuing. It may be possible, in fact, that QED is fundamentally incomplete. While its predictions have been met with unprecedented experimental success, the mathematical issues that arise could very well be insurmountable, particularly if we are to move outside of the one-dimensional world we have been occupying in this experiment. Nonetheless, the success we have obtained in forming a basis for renormalization shows much promise to yield further fruitful investigations into analytical, closed-form QED.



## BIBLIOGRAPHY

- [1] Hatfield, Brian. *Quantum Field Theory of Point Particles and Strings*. CRC Press, Taylor & Francis Group, 2018.
  
- [2] Cohen-Tannoudji, Claude, et al. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 2007.
  
- [3] Taylor, John R. *Classical Mechanics*. University Science Books, 2005.
  
- [4] Glasgow, Scott, et al. "Space-time-resolved quantum electrodynamics: A (1+1) dimensional model" *Physical Review A*. 93, 062106. 2016.

## LIST OF FIGURES

---

<b>Figure 1.1</b> Depiction of Feynman diagram for dressed vacuum.....	6
<b>Figure 2.1</b> Effect of interaction term on the bare energy for electron eigenstates [Ref 1] .....	16
<b>Figure 3.1.</b> Integrand of QED Hamiltonian in one direction.....	28
<b>Figure 3.2.</b> Results of numerical integration.....	29
<b>Figure 3.3.</b> Location of ground-state eigenvalue .....	29
<b>Figure 3.4</b> Depiction of Feynman diagram for dressed vacuum.....	31
<b>Figure 3.5</b> Depiction of Feynman diagram for pair-production.....	31