

Space-Time Resolving Quantum Electrodynamic States of Two Charges

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A senior thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Bachelor of Science

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April 2022

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ABSTRACT

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The degree to which predictions made by Quantum Field Theory (QFT) agree with experiment make it one of the most accurate physical theories ever constructed. Quantum Electrodynamics (QED) is the branch of QFT that deals with charged particles and their interactions. Generally, one approaches problems in QED using scattering theory, and while this one-time "snapshot" of the input and output states is useful in many applications, it does not tell the whole story of the interaction. In this thesis, a non-perturbative approach of finding space-time resolved dynamics of a system of two spin-up electrons is shown. The techniques applied as well as some of the resulting equations can easily be extended to states with electrons, positrons, or photons of any spin.

Keywords: Quantum Electrodynamics, QED, Quantum Field Theory, QFT, scattering, dynamics, multi-particle states, space-time resolve

ACKNOWLEDGMENTS

I would like to thank the BYU Mathematics Department, who funded this research. I would also like to thank Dr. Scott Glasgow who constantly pushed me to do better and waited patiently as I struggled to do so.

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Chapter 1

Introduction

In this chapter, we give a very brief introduction to Quantum Electrodynamics (QED). We also give a brief introduction to both the typical method of approaching problems in QED as well as the method we will be using. We then give the motivation for this research and introduce the state we are interested in examining. Lastly, we define all the terms needed to solve the Schrodinger equation for our system.

1.1 Introduction to Quantum Electrodynamics

Quantum Electrodynamics (QED) is a branch of Quantum Field Theory (QFT) that describes the interactions between charged particles. Although this theory is much more complicated to use than classical electrodynamics, the degree to which it agrees with experiment makes it one of the most accurate theories ever constructed and thus is much more desirable to use than classical electrodynamics. In what follows, we show both the standard method of approaching problems in QED as well as the method we will be using in order to find the space-time-resolved dynamics of a system of two charged particles.

1.1.1 Standard Approach: Scattering Theory

Scattering theory employs perturbation theory to map states from the distant past to states in the distant future, where do not some sort of particle interaction in the middle. In the past, as well as in the future, the particles do not "know about" each other so there is no interaction there. Then the perturbation to this system is that they have some nontrivial interaction in between the two extremes where they do not know about each other. This method of solving problems in QED can be quite effective, but it provides a one-time, static "picture" of the interactions between particles and requires perturbation theory to do so.

1.1.2 Our Approach: Space-Time Resolving QED

Contrary to scattering theory, our method does not require perturbation theory to be solved and can provide a much more detailed view of particle interactions. This is made possible through solving the Schrodinger equation for our system, which allows us to see how the wave function (probability amplitude) evolves over time. This allows us to see more of a "movie" of the interactions between particles instead of just a snapshot.

1.2 Motivation

Space-time resolving QED has historically been avoided, but the reality is that if we want to further our understanding of the fundamental particles and interactions of the universe, there comes a point where we can no longer see the "simplified" universe, we must see it for what it truly is. We draw analogy to our physics education here at BYU; in PHYS121 we learn about Newton's laws of motion and how well they describe our universe, then in PHYS123 we learn how we were actually somewhat lied to in 121 and the universe isn't quite as simple as it was made out to be previously. Each successive course talks about how the previous course was actually an approximation, but

it helped us form the basis we needed to be able to understand the current course. And so it is with QED; scattering theory can help us understand the basics of QED and use it as a tool in our physics tool-belt, but to come to really understand its description of particles, we cannot ignore their space-time resolved interactions. We seek to show how to resolve these interactions in an attempt to further our understanding of the universe and the fundamental charged particles that make it up.

1.3 Introduction to the State We Will be Working With

One of the amazing things about QFT in general (and thus QED), is that it allows for the creation and annihilation of particles, which other theories do not allow for. This means we can mathematically create a universe with only a few particles in and thus more easily see how those particles interact. We will be starting with a universe with just two spin-up electrons, represented by

$$c_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle. \quad (1.1)$$

Where $|\rangle$ represents the bare vacuum state, and thus, $c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle$ creates a spin up electron¹ with momentum $\hbar\mathbf{k}_2$.² We chose to look at this particular state for various reasons. The main reason is that no one has used QED to show dynamics between multiple charged particles before, so solving this would allow us to create the most accurate³ movie of the interaction between two charged particles. Another major reason we chose this state is because we generally ascribe the Coulomb interaction to photons being sent between the charged particles, although we cannot actually detect these photons. If we solve the Schrodinger equation using the QED Hamiltonian in the Coulomb

¹thus dubbed "creation operator" or "creator"

²Although this could be a single value, that would prevent us knowing anything about the particles in real space, due to the Heisenberg uncertainty principle. Thus, when solving with numerics, this would need to be a distribution in momentum space rather than a single value so the Fourier Transform into real space can be taken. This will allow for seeing the dynamics in real space, where we are a lot more used to thinking about events occurring.

³due to the fact that we are using QED, whose predictions agree with experiment to an unprecedented degree

Gauge⁴, which is exactly what we will prepare for in Chapter 2, then the theory would agree with experiment in that we do not countenance any photons that could actually be responsible for the Coulomb interaction. However, the QED Hamiltonian in the Lorenz gauge⁵ countenances "fake" photons that may or may not be responsible for the Coulomb interaction. Thus, if we solve the same Schrodinger equation, but in the Lorenz gauge—which will be done in a different project—we could compare the results of the two and see if these fake photons really are the cause of the Coulomb Force, and what that might mean physically. Although we were not counting on this initially, we will see in Chapter 2 that there is even more we can get out of this research, such as seeing the dynamics of infinite rods or plates of charge.

1.4 Definitions

To obtain the dynamics of our system, we must solve the Schrodinger equation,

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle, \quad (1.2)$$

where $|\Psi(t)\rangle$ represents the acceptable states we can be in⁶.

We start with the Hamiltonian in the Coulomb Gauge given by Cohen-Tannoudji et al. [1]:

$$H = H_D + H_C + H_R + H_I. \quad (1.3)$$

Each term in the Hamiltonian corresponds to a different phenomena affecting the energy— H_D give the bare energy of our particles, H_C gives the Coulomb interaction between the particles, H_R gives the energy operator for photons, and H_I gives the interaction between radiation and matter.

⁴using the standard definition of the Coulomb Gauge, where $\nabla \cdot \mathbf{A} = 0$, where $\mathbf{A}(\mathbf{r}, t)$ is the magnetic vector potential

⁵using the standard definition of the Lorenz gauge, where $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$, where $\mathbf{A}(\mathbf{r}, t)$ is the magnetic vector potential and V is the electric potential.

⁶meaning, the eigenstates of the Hamiltonian H

Rather than working in physical space, we will work in the Fourier transform space (hereafter referred to as momentum-space). We will use the derived momentum-space expressions for each of these terms given by Glasgow and Ware [2]:

$$H_D = \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) \tilde{\Psi}^\dagger(\mathbf{k}) U(\mathbf{k}) \beta U^\dagger(\mathbf{k}) \tilde{\Psi}(\mathbf{k}) \quad (1.4)$$

Where $\int d^3 \mathbf{k} = \int dk_x \int dk_y \int dk_z$ and we write the dk before the expression being integrated so we can easily see what the integral is over.

We also have that $\tilde{\Psi}(\mathbf{k})$ is the inverse Fourier transform of the four-component column vector $\Psi(\mathbf{k})$. Or, more explicitly:

$$\tilde{\Psi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3 \mathbf{r} \Psi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}}, \quad (1.5)$$

and we define the following:

$$\beta := \begin{bmatrix} \mathbb{1}_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & -\mathbb{1}_{2 \times 2} \end{bmatrix} \quad (1.6)$$

Where the notation $\mathbb{1}_{2 \times 2}$ means the 2×2 identity, etc.

We also use:

$$\varepsilon(\mathbf{k}) = \sqrt{(mc^2)^2 + (\hbar c \mathbf{k})^2}, \quad (1.7)$$

$$U(\mathbf{k}) = \cos\left(\frac{\theta_{\mathbf{k}}}{2}\right) \mathbb{1}_{4 \times 4} - \sin\left(\frac{\theta_{\mathbf{k}}}{2}\right) \beta \boldsymbol{\alpha} \cdot \hat{\mathbf{k}} \quad (1.8)$$

where

$$\theta_{\mathbf{k}} = \arctan\left(\frac{\hbar \|\mathbf{k}\|}{mc}\right), \hat{\mathbf{k}} = \frac{\mathbf{k}}{\|\mathbf{k}\|} \quad (1.9)$$

and where $\boldsymbol{\alpha}$ is the three component vector containing the α_i matrices, which are defined as

$$\alpha_i := \begin{bmatrix} 0_{2 \times 2} & \sigma_i \\ \sigma_i & 0_{2 \times 2} \end{bmatrix} \quad (1.10)$$

with σ_i representing the familiar Pauli Matrices:

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.11)$$

To see the quantum field nature of $\tilde{\Psi}$, we write it in terms of creation and annihilation operators, first letting:

$$\chi(\mathbf{k}) = \begin{bmatrix} C_{\uparrow}(\mathbf{k}) \\ C_{\downarrow}(\mathbf{k}) \\ D_{\uparrow}^{\dagger}(\mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}) \end{bmatrix} \quad (1.12)$$

where C_{\uparrow}^{\dagger} creates a spin-up (using the standard basis (i.e. z)) electron, while C_{\downarrow} annihilates a spin-down electron, etc. The D 's do the same, but for positrons.⁷ Then

$$U^{\dagger}(\mathbf{k})\tilde{\Psi} = \chi(\mathbf{k}) \iff \tilde{\Psi} = U(\mathbf{k})\chi(\mathbf{k}) \quad (1.13)$$

If we use this in (2.3), while enforcing normal ordering, we get:

$$H_D = \int d^3\mathbf{k} \boldsymbol{\varepsilon}(\mathbf{k}) [C_{\uparrow}^{\dagger}(\mathbf{k})C_{\uparrow}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k})C_{\downarrow}(\mathbf{k}) + D_{\uparrow}^{\dagger}(\mathbf{k})D_{\uparrow}(\mathbf{k}) + D_{\downarrow}^{\dagger}(\mathbf{k})D_{\downarrow}(\mathbf{k})]. \quad (1.14)$$

Next, we use the momentum space definition of the radiation term of the Hamiltonian given by Glasgow and Ware [2]:

$$H_R = \int d^3\mathbf{k} \hbar\omega_{\mathbf{k}} [\mathbf{a}_1^{\dagger}(\mathbf{k}) \mathbf{a}_1(\mathbf{k}) + \mathbf{a}_2^{\dagger}(\mathbf{k}) \mathbf{a}_2(\mathbf{k})] \quad (1.15)$$

where $\omega_{\mathbf{k}} = c\|\mathbf{k}\| = ck$ and $\mathbf{a}_j^{\dagger}(\mathbf{k})$ creates a photon with linear polarization $\boldsymbol{\varepsilon}_j\mathbf{k}$ and definite momentum $\hbar\mathbf{k}$ while $\mathbf{a}_j(\mathbf{k})$ annihilates one of those same photons.

we adapt the definition of the interaction term of the Hamiltonian from Glasgow et al. [3] as follows:

$$H_I = b_I \int d^3\mathbf{k} \int d^3\mathbf{k}' \frac{O(\mathbf{k}', \mathbf{k})}{\sqrt{\hbar\|(\mathbf{k}' - \mathbf{k})\|}} \quad (1.16)$$

⁷You'll notice that we have written the creation and annihilation operators using capital letters (and we will use large typeset for photonic operators later on), which is not in accordance with convention. We will use our notation for all definitions that are part of the QED Hamiltonian for reasons discussed in section 2.1.1

where:

$$b_I = -q \sqrt{\frac{c}{2\epsilon_0(2\pi)^3}} \quad (1.17)$$

and

$$O(\mathbf{k}', \mathbf{k}) =: \chi^\dagger(\mathbf{k}') \left(U^\dagger(\mathbf{k}') \boldsymbol{\alpha} \cdot \mathbf{a}_\perp((\mathbf{k}' - \mathbf{k})) U(\mathbf{k}) \right) \chi(\mathbf{k}) : \quad (1.18)$$

where the colons indicate normal ordering⁸ and we have

$$\mathbf{a}_\perp(\mathbf{n}) = \mathbf{a}_1(\mathbf{n}) \boldsymbol{\epsilon}_1(\mathbf{n}) + \mathbf{a}_2(\mathbf{n}) \boldsymbol{\epsilon}_2(\mathbf{n}) + \mathbf{a}_1^\dagger(-\mathbf{n}) \boldsymbol{\epsilon}_1(-\mathbf{n}) + \mathbf{a}_2^\dagger(-\mathbf{n}) \boldsymbol{\epsilon}_2(-\mathbf{n}) \quad (1.19)$$

Lastly, we write⁹ the Coulomb term of the Hamiltonian, using the given definitions by Glasgow et al. [3]

$$H_C = b_C \int d^3\mathbf{k} \frac{1}{\|\mathbf{k}\|^2} \int d^3\mathbf{k}' \int d^3\mathbf{k}'' \tilde{\Psi}^\dagger(\mathbf{k}'') \tilde{\Psi}^\dagger(\mathbf{k}'' - \mathbf{k}) \tilde{\Psi}^\dagger(\mathbf{k}') \tilde{\Psi}^\dagger(\mathbf{k}' + \mathbf{k}) \quad (1.20)$$

where

$$b_C = \frac{q^2}{2\epsilon_0(2\pi)^3}. \quad (1.21)$$

With this last definition, we have not completely defined the QED Hamiltonian in momentum space, which will allow us to prepare the Schrodinger equation to be solved. We do exactly that in the next chapter.

⁸this idea will be discussed in more detail in section 2.1.1

⁹to get from the definitions given by Glasgow et al. [3] to equation (1.20), one needs to remember the convolution theorem since arriving here requires taking the Fourier Transform of a product

Chapter 2

Methods

In this chapter, we prepare the Schrodinger equation for our system to be solved using numerical techniques. We do this by: first, allowing the Hamiltonian to act on our state and examining the resulting particle spray¹, second, deriving the Hamiltonian projected onto the state we care about solving for (a state with two spin up electrons), and third, simplifying the equation to a point where the numerics become possible using standard computation techniques.

This approach is summarized in figure 2.1 below. The red arrows indicate the steps we will take in this chapter, while the black and blue arrows represent steps we will not take. More specifically, the black arrows show steps that, if taken, prevent us from finding the dynamics we are seeking. On the contrary, the blue arrows indicate steps that are necessary to take in order to find the dynamics of our system.

¹a general term, which in this case describes the particle content output by the Hamiltonian when it acts on the given state

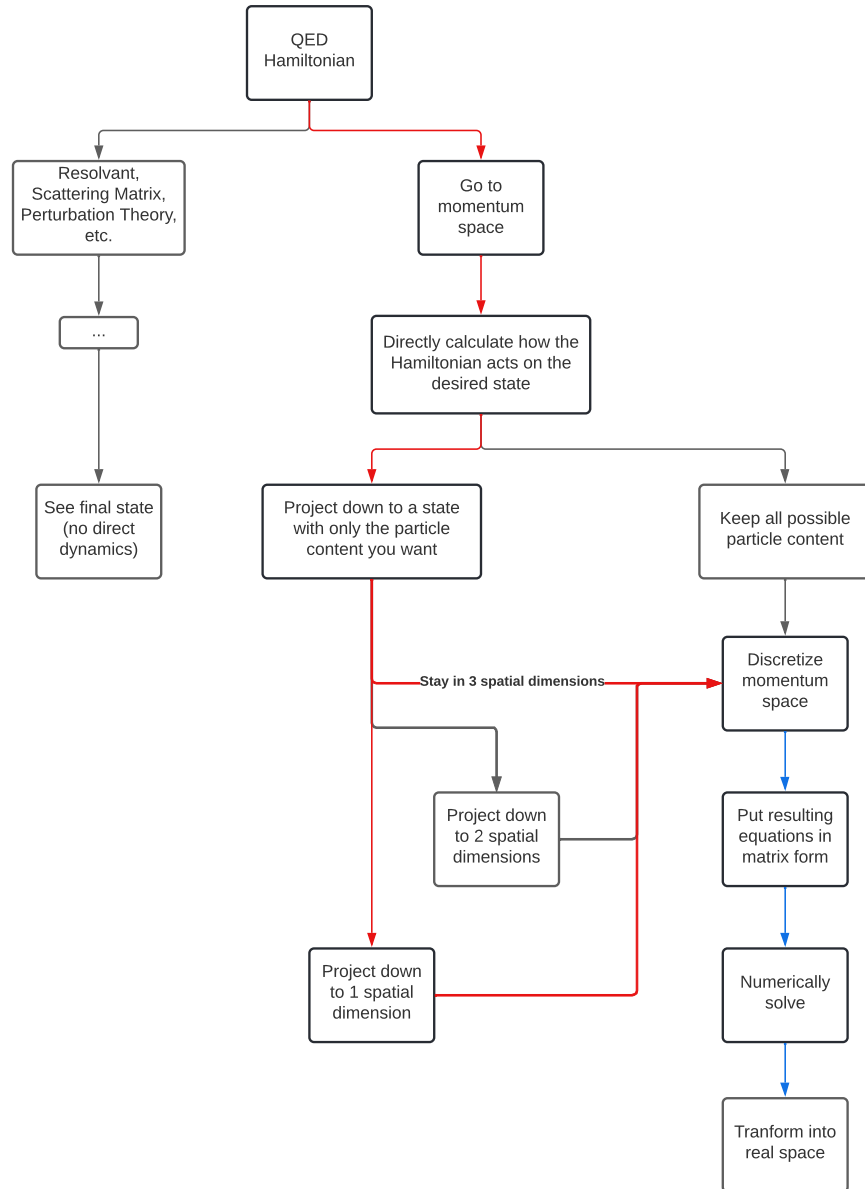


Figure 2.1 Diagram showing various ways to approach a problem in QED. The topic of the left-hand side is scattering theory, which is not the focus of this paper, so we have not included any of the intermediate steps taken in that approach. The red arrows indicate steps we will take in this paper, while the black arrows indicate steps we will not take. The blue arrows indicate steps that must be taken in order to see the physical dynamics.

2.1 Derivation of the Reduced Hamiltonian

With the given definitions and background, we are now ready to apply this Hamiltonian onto the state of two spin up electrons. This state has the form:

$$|\Psi(t)\rangle = \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.1)$$

Which, in shorthand², is also written:

$$\int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.2)$$

Then we need to solve:

$$i\hbar \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \frac{\partial}{\partial t} \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}_1, \mathbf{k}_2\rangle =: H : \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.3)$$

And since H is not a function of the variables in the integrals, and commutes with Ψ we can bring the Hamiltonian into the integrals and to the other side of Ψ , giving us:

$$i\hbar \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \frac{\partial}{\partial t} \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}_1, \mathbf{k}_2\rangle = \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) : H : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.4)$$

Where the colons around H indicate normal-ordering.

2.1.1 Introduction to Normal Ordering

The notation $: H :$ enforces normal-ordering of the creation and annihilation operators inside H , which for fermions, has the familiar relations:

$$: c^{\dagger} c := c^{\dagger} c, \quad : c c^{\dagger} := -c^{\dagger} c \quad (2.5)$$

and for bosons:

$$: a^{\dagger} a := a^{\dagger} a, \quad : a a^{\dagger} := a^{\dagger} a \quad (2.6)$$

²In the rest of the paper, we will refrain from using the shorthand for anything besides electron creation or annihilation operators acting on empty kets or bras, respectively.

Enforcing normal ordering basically takes care of re-normalization for us, while not changing the dynamics we are seeking, as explained by Glasgow [2].

Until this point, we have ignored the fact that we have not been writing the creation and annihilation operators completely in accordance with convention. Normally, they are written using all lower-case letters, as seen in (2.25). One may notice that in our definitions above of the various terms of the Hamiltonian, we have used all big letters (whether upper-case or just bigger font). The reason we have done so is related to normal ordering. In a way, in the process of normal ordering, we have allowed the creation and annihilation operators to interact with each other. This means we cannot have them interact with each other again, or we will get nonsense out when our Hamiltonian acts on our state. So we have used the big letters for creation³ and annihilation⁴ operators solely in the Hamiltonian to remind ourselves that they must be normal ordered, and thus can pass through each other without creating delta-functions. Other than that, they are exactly the same as all other creation and annihilation operators.

2.1.2 Derivation of the Reduced Interaction Hamiltonian Term

We can now move on to letting the Hamiltonian act on our interesting state. From above, we see we need to first calculate:

$$\begin{aligned} & : H : |\mathbf{k}_1, \mathbf{k}_2\rangle \\ & =: H_D + H_I + H_R + H_C : |\mathbf{k}_1, \mathbf{k}_2\rangle \end{aligned} \tag{2.7}$$

We will calculate each term separately, starting with the Interaction term:

$$\begin{aligned} & : H_I : |\mathbf{k}_1, \mathbf{k}_2\rangle =: H_I : c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \\ & = b_I \int d^3 \mathbf{k} \int d^3 \mathbf{k}' \frac{O(\mathbf{k}', \mathbf{k})}{\sqrt{\hbar ||(\mathbf{k}' - \mathbf{k})||}} c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \end{aligned} \tag{2.8}$$

³such as $C^{\dagger}(\mathbf{k}), D^{\dagger}(\mathbf{k})$, etc.

⁴such as $C(\mathbf{k}), C(\mathbf{k})$, etc.

$O(\mathbf{k}', \mathbf{k})$ is made up of electron, positron, and photon creation and annihilation operators, along with some values from u^\dagger, U matrices. To calculate (2.8), we will need to use these operators along with the fact that bosonic and fermionic creation and annihilation operators commute.

We see that the u^\dagger, U matrices in $O(\mathbf{k}', \mathbf{k})$ will allow any (every) operator in χ^\dagger, χ , to interact with any (every) operator in a_\perp , so we see that the interaction Hamiltonian acts on this state proportional to

$$\begin{aligned}
& : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \left[\mathbf{a}_1((\mathbf{k}' - \mathbf{k})) + \mathbf{a}_2((\mathbf{k}' - \mathbf{k})) + \mathbf{a}_1^\dagger(\mathbf{k} - \mathbf{k}') + \mathbf{a}_2^\dagger(\mathbf{k} - \mathbf{k}') \right] \\
& \quad \times \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& =: \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \mathbf{a}_1((\mathbf{k}' - \mathbf{k})) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \mathbf{a}_2((\mathbf{k}' - \mathbf{k})) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \mathbf{a}_1^\dagger(\mathbf{k} - \mathbf{k}') \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \mathbf{a}_2^\dagger(\mathbf{k} - \mathbf{k}') \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& \hspace{15em} (2.9)
\end{aligned}$$

Due to commutation of fermionic and bosonic creation and annihilation operators, this is equal to

$$\begin{aligned}
& : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : \mathbf{a}_1(\mathbf{k}' - \mathbf{k}) |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : \mathbf{a}_2(\mathbf{k}' - \mathbf{k}) |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : \mathbf{a}_1^\dagger(\mathbf{k} - \mathbf{k}') |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& + : \left(C_\uparrow^\dagger(\mathbf{k}') + C_\downarrow^\dagger(\mathbf{k}') + D_\uparrow(\mathbf{k}') + D_\downarrow(\mathbf{k}') \right) \left(C_\uparrow(\mathbf{k}) + C_\downarrow(\mathbf{k}) + D_\uparrow^\dagger(\mathbf{k}) + D_\downarrow^\dagger(\mathbf{k}) \right) : \mathbf{a}_2^\dagger(\mathbf{k} - \mathbf{k}') |\mathbf{k}_1, \mathbf{k}_2\rangle \\
& \hspace{15em} (2.10)
\end{aligned}$$

And we know that an annihilation operator acting on an empty (vacuum) ket immediately gives zero, making the first two terms zero. This also makes it so when we expand out the remaining terms above, any of the expanded terms with any positron annihilator or spin down electron annihilator

will also be zero. This simplifies (2.10) to

$$\begin{aligned}
& [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& + [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}') | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& \quad + \delta(\mathbf{k}_1 - \mathbf{k}') [\mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] | \mathbf{k}_2 \rangle \\
& \quad - \delta(\mathbf{k}_2 - \mathbf{k}') [\mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] | \mathbf{k}_1 \rangle
\end{aligned} \tag{2.11}$$

Which we are saying : $H_I : | \mathbf{k}_1, \mathbf{k}_2 \rangle$ is proportional to.

Equation (2.11) means that when the Interaction Hamiltonian acts on our state, it can pair produce⁵ as well as produce a photon of opposite momentum (thus conserving total momentum).

2.1.3 Derivation of the Reduced Radiation Hamiltonian Term

We now move to the Radiation Hamiltonian:

$$\begin{aligned}
& : H_R : | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& = \int d^3 \mathbf{k} \hbar \omega_{\mathbf{k}} [\mathbf{a}_1^{\dagger}(\mathbf{k}) \mathbf{a}_1(\mathbf{k}) + \mathbf{a}_2^{\dagger}(\mathbf{k}) \mathbf{a}_2(\mathbf{k})] c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle
\end{aligned} \tag{2.12}$$

Due to the commutation of bosonic with fermionic creation and annihilation operators, we see this is equal to the following:

$$\begin{aligned}
& \int d^3 \mathbf{k} \hbar \omega_{\mathbf{k}} \left(\mathbf{a}_1^{\dagger}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) \mathbf{a}_1(\mathbf{k}) | \rangle + \mathbf{a}_2^{\dagger}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) \mathbf{a}_2(\mathbf{k}) | \rangle \right) \\
& = 0.
\end{aligned} \tag{2.13}$$

So we see : $H_R : | \mathbf{k}_1, \mathbf{k}_2 \rangle = 0$. In other words, the radiation Hamiltonian does not contribute to our system. This matches our intuition that since our state does not have photons, their energy must be zero.

⁵production of an electron and positron, which conserves baryon number

2.1.4 Derivation of the Reduced Dirac Hamiltonian Term

Next, we look at the Dirac Hamiltonian:

$$\begin{aligned} & : H_D : | \mathbf{k}_1, \mathbf{k}_2 \rangle \\ & = \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) [C_{\uparrow}^{\dagger}(\mathbf{k}) C_{\uparrow}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}) C_{\downarrow}(\mathbf{k}) + D_{\uparrow}^{\dagger}(\mathbf{k}) D_{\uparrow}(\mathbf{k}) + D_{\downarrow}^{\dagger}(\mathbf{k}) D_{\downarrow}(\mathbf{k})] c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \end{aligned} \quad (2.14)$$

Since the D's (creation and annihilation operators of positrons) commute with C's, we see that similar to the Interaction Hamiltonian derivation above, the last two terms with D's in them will be zero since the annihilation operator will act on the empty ket. Similarly, the spin down electron operators commute with the spin up ones, leaving us with just:

$$\int d^3 \mathbf{k} \varepsilon(\mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) C_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.15)$$

we will now need to employ the anti-commutation of fermions:

$$\{c_{\uparrow}(\mathbf{k}), c_{\uparrow}^{\dagger}(\mathbf{k}')\} := c_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}') + c_{\uparrow}^{\dagger}(\mathbf{k}') c_{\uparrow}(\mathbf{k}). \quad (2.16)$$

Which means

$$c_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}') = \delta(\mathbf{k}' - \mathbf{k}) - c_{\uparrow}^{\dagger}(\mathbf{k}') c_{\uparrow}(\mathbf{k}). \quad (2.17)$$

Applying this to what is left of the the Dirac Hamiltonian term (eq 2.35), we see

$$\begin{aligned} & \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) C_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \\ & = \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) (\delta(\mathbf{k}_1 - \mathbf{k}) - c_{\uparrow}^{\dagger}(\mathbf{k}_1) C_{\uparrow}(\mathbf{k})) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \\ & = \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) \delta(\mathbf{k}_1 - \mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle - \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) C_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \end{aligned} \quad (2.18)$$

Remembering that the Dirac delta function of a vector is really three delta functions:

$$\delta(\mathbf{k}_1 - \mathbf{k}) := \delta(k_{1x} - k_x) \delta(k_{1y} - k_y) \delta(k_{1z} - k_z) \quad (2.19)$$

we see that the term with the delta function(s) will be able to turn the integral(s) into the integrand evaluated at the point where the argument of the delta function(s) goes to zero, making (2.18):

$$= \varepsilon(\mathbf{k}_1) C_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle - \int d^3 \mathbf{k} \varepsilon(\mathbf{k}) C_{\uparrow}^{\dagger}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_1) C_{\uparrow}(\mathbf{k}) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.20)$$

We repeat the process on the second term in (2.18) to get:

$$\begin{aligned}
& \varepsilon(\mathbf{k}_1)C_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle - \int d^3\mathbf{k}\varepsilon(\mathbf{k})C_{\uparrow}^{\dagger}(\mathbf{k})c_{\uparrow}^{\dagger}(\mathbf{k}_1)C_{\uparrow}(\mathbf{k})c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle \\
&= \varepsilon(\mathbf{k}_1)C_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle - \varepsilon(\mathbf{k}_2)C_{\uparrow}^{\dagger}(\mathbf{k}_2)c_{\uparrow}^{\dagger}(\mathbf{k}_1)|\rangle + \int d^3\mathbf{k}\varepsilon(\mathbf{k})C_{\uparrow}^{\dagger}(\mathbf{k})c_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)C_{\uparrow}(\mathbf{k})|\rangle \quad (2.21) \\
&= \varepsilon(\mathbf{k}_1)C_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle - \varepsilon(\mathbf{k}_2)C_{\uparrow}^{\dagger}(\mathbf{k}_2)c_{\uparrow}^{\dagger}(\mathbf{k}_1)|\rangle.
\end{aligned}$$

So, in the end, we have for the Dirac Hamiltonian acting on our state:

$$:H_D:|\mathbf{k}_1,\mathbf{k}_2\rangle = \varepsilon(\mathbf{k}_1)|\mathbf{k}_1,\mathbf{k}_2\rangle - \varepsilon(\mathbf{k}_2)|\mathbf{k}_2,\mathbf{k}_1\rangle. \quad (2.22)$$

Which we could interpret as giving us the bare energies⁶ of the present particles.

2.1.5 Derivation of the Reduced Coulomb Hamiltonian Term

Lastly, we calculate how the Coulomb Hamiltonian acts on our state.

$$\begin{aligned}
& :H_C:|\mathbf{k}_1,\mathbf{k}_2\rangle \\
&= b_C \int d^3\mathbf{k} \frac{1}{\|\mathbf{k}\|^2} \int d^3\mathbf{k}' \int d^3\mathbf{k}'' \tilde{\Psi}^{\dagger}(\mathbf{k}'')\tilde{\Psi}^{\dagger}(\mathbf{k}''-\mathbf{k})\tilde{\Psi}^{\dagger}(\mathbf{k}')\tilde{\Psi}^{\dagger}(\mathbf{k}'+\mathbf{k})|\mathbf{k}_1,\mathbf{k}_2\rangle \quad (2.23)
\end{aligned}$$

If we ignore the Coefficients that will come from the $U^{\dagger}U$ matrices for now, and just focus on how the creation and annihilation operators will interact with our state, we can see

$$\begin{aligned}
& :H_C:|\mathbf{k}_1,\mathbf{k}_2\rangle \\
& \propto: \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ C_{\downarrow}^{\dagger}(\mathbf{k}'') \\ D_{\uparrow}(\mathbf{k}'') \\ D_{\downarrow}(\mathbf{k}'') \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}''-\mathbf{k}) \\ C_{\downarrow}(\mathbf{k}''-\mathbf{k}) \\ D_{\uparrow}^{\dagger}(\mathbf{k}''-\mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}''-\mathbf{k}) \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ C_{\downarrow}^{\dagger}(\mathbf{k}') \\ D_{\uparrow}(\mathbf{k}') \\ D_{\downarrow}(\mathbf{k}') \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'+\mathbf{k}) \\ C_{\downarrow}(\mathbf{k}'+\mathbf{k}) \\ D_{\uparrow}^{\dagger}(\mathbf{k}'+\mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}'+\mathbf{k}) \end{bmatrix} :c_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2)|\rangle \quad (2.24)
\end{aligned}$$

⁶Meaning, the energy of the bare particle. The bare particle being one that has not been "dressed" by a pair of virtual particles. The exact nature and details of virtual and bare particles are beyond the scope of this paper, but can be easily researched for a better understanding.

This gives us a 1×1 matrix made of a summation of $4^4 = 256$ terms of one electron or positron creation or annihilation operator from each of the matrices shown above. To know how H_C affects our state, we have to figure out what each of those terms is. To do so, we need to find a way to simplify this expression.

Again due to our choice of state, we can do as we have done with the other terms of the Hamiltonian. We see here that any matrix element with a positron annihilator or spin down electron annihilator will result in that element going to zero since an annihilator will act on an empty ket. This allows us to simplify (2.24) to be effectively:

$$: \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ C_{\downarrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ C_{\downarrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \end{bmatrix} : c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.25)$$

We interpret this new equation as showing the fact that a particle that was never there cannot be annihilated.

2.1.6 The Full, Reduced Hamiltonian

We do not have exactly what the the Hamiltonian acting on our state is, but combining (2.11), (2.13), (2.22), (2.25), we have:

$$:H : |\mathbf{k}_1, \mathbf{k}_2\rangle \propto \quad (2.26)$$

$$\begin{aligned} & [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathfrak{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') |\mathbf{k}_1, \mathbf{k}_2\rangle \\ & + [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathfrak{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}') |\mathbf{k}_1, \mathbf{k}_2\rangle \\ & + \delta(\mathbf{k}_1 - \mathbf{k}') [\mathfrak{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathfrak{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] |\mathbf{k}_2\rangle - \delta(\mathbf{k}_2 - \mathbf{k}') [\mathfrak{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathfrak{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] |\mathbf{k}_1\rangle \\ & + \varepsilon(\mathbf{k}_1) |\mathbf{k}_1, \mathbf{k}_2\rangle - \varepsilon(\mathbf{k}_2) |\mathbf{k}_2, \mathbf{k}_1\rangle \\ & + : \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ C_{\downarrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ C_{\downarrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \end{bmatrix} : |\mathbf{k}_1, \mathbf{k}_2\rangle \end{aligned}$$

Or, in words, letting our Hamiltonian act on our system of two spin up electrons, it will keep the electrons we started with (whose bare energies are given by the Dirac Hamiltonian), and do not some probability it produces electrons, positrons, and/or photons of various spins.

As we will eventually be looking at the dynamics using numerics, this is much too complicated to try to compute. Even if we could put this in the form that numerics could be used, and had a super computer to run them on, it would take far too long to calculate.

Although we know do not a lot of interesting physics in (2.26), such as production of an electron-positron pair and a photon, our original interest was to see how QED shows two electrons interacting. Thus, we will restrict the particle spray of the Hamiltonian to solely two spin up electrons. This way, there is much less to compute but we do not lose the physics we were interested in seeing. We will accomplish this through a projection; we will project (2.26) down to the state with just two spin

up electrons. We will use the projector⁷, P , defined as:

$$P := \frac{1}{2} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| \quad (2.27)$$

So we now need to calculate⁸

$$P : H : |\Psi(t)\rangle \propto P : H : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.28)$$

to see only the particle content we want and to have a system that is computationally possible.

2.2 Derivation of the Projected Hamiltonian

As above, we will work through this derivation one term of the Hamiltonian at a time, although now we only have three terms to calculate since we found $H_R |\Psi\rangle = 0$.

2.2.1 Derivation of the Projected Interaction Hamiltonian Term

We again start with the Interaction Hamiltonian:

$$P : H_I : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.29)$$

With the projector acting on the left of the Hamiltonian, the important piece to calculate is the inner product between the right side of the projector and what is left of the Hamiltonian acting on our state:

$$\langle \mathbf{k}'_2, \mathbf{k}'_1 | : H_I : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.30)$$

⁷see section A.1 for a simple physical description of projectors, as well as a simple proof showing our projector really is a projector

⁸since we are going to use the projection operator on both sides of the equation, we immediately drop the factor of $\frac{1}{2}$

Using (2.11), we see $\langle \mathbf{k}'_2, \mathbf{k}'_1 | : H_I : | \mathbf{k}_1, \mathbf{k}_2 \rangle$ is proportional to

$$\begin{aligned}
& \langle \mathbf{k}'_2, \mathbf{k}'_1 | [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& + \langle \mathbf{k}'_2, \mathbf{k}'_1 | [C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\uparrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\uparrow}^{\dagger}(\mathbf{k}) + C_{\downarrow}^{\dagger}(\mathbf{k}')D_{\downarrow}^{\dagger}(\mathbf{k})] \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}') | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& \quad + \langle \mathbf{k}'_2, \mathbf{k}'_1 | \delta(\mathbf{k}_1 - \mathbf{k}') [\mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] | \mathbf{k}_2 \rangle \\
& \quad - \langle \mathbf{k}'_2, \mathbf{k}'_1 | \delta(\mathbf{k}_2 - \mathbf{k}') [\mathbf{a}_1^{\dagger}(\mathbf{k} - \mathbf{k}') + \mathbf{a}_2^{\dagger}(\mathbf{k} - \mathbf{k}')] | \mathbf{k}_1 \rangle
\end{aligned} \tag{2.31}$$

due to the commutation of bosonic and fermionic creators and annihilators.

Now that we have projected down to certain particle content (that of two spin-up electrons), the Interaction Hamiltonian adds nothing to our system. This confirms that the projector is doing what we hoped it would: restricting what we have to calculate and the particle spray we face. We will continue to see the projector do this as we derive the next projected terms below.

2.2.2 Derivation of the Projected Dirac Hamiltonian Term

Now we derive the projected Dirac Hamiltonian term:

$$\begin{aligned}
P : H_D : | \mathbf{k}_1, \mathbf{k}_2 \rangle & \propto \\
& \langle \mathbf{k}'_2, \mathbf{k}'_1 | : H_D : | \mathbf{k}_1, \mathbf{k}_2 \rangle \\
& = \langle \mathbf{k}'_2, \mathbf{k}'_1 | \left(\varepsilon(\mathbf{k}_1) | \mathbf{k}_1, \mathbf{k}_2 \rangle - \varepsilon(\mathbf{k}_2) | \mathbf{k}_2, \mathbf{k}_1 \rangle \right) \\
& = \varepsilon(\mathbf{k}_1) \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}(\mathbf{k}'_1) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle - \varepsilon(\mathbf{k}_2) \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}(\mathbf{k}'_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) c_{\uparrow}^{\dagger}(\mathbf{k}_1) | \rangle
\end{aligned} \tag{2.32}$$

Applying the same technique that led us to $: H_D : | \mathbf{k}_1, \mathbf{k}_2 \rangle$ in the first place⁹, this becomes:

$$\begin{aligned}
& = \varepsilon(\mathbf{k}_1) (\delta(\mathbf{k}_1 - \mathbf{k}'_1) \delta(\mathbf{k}_2 - \mathbf{k}'_2) - \delta(\mathbf{k}_1 - \mathbf{k}'_2) \delta(\mathbf{k}_2 - \mathbf{k}'_1)) \\
& \quad - \varepsilon(\mathbf{k}_2) (\delta(\mathbf{k}_1 - \mathbf{k}'_2) \delta(\mathbf{k}_2 - \mathbf{k}'_1) - \delta(\mathbf{k}_1 - \mathbf{k}'_1) \delta(\mathbf{k}_2 - \mathbf{k}'_2)) \\
& = (\varepsilon(\mathbf{k}_1) + \varepsilon(\mathbf{k}_2)) (\delta(\mathbf{k}_1 - \mathbf{k}'_1) \delta(\mathbf{k}_2 - \mathbf{k}'_2) - \delta(\mathbf{k}_1 - \mathbf{k}'_2) \delta(\mathbf{k}_2 - \mathbf{k}'_1))
\end{aligned} \tag{2.33}$$

⁹the projector proof in section A.1 includes a derivation very similar to this

We use this to find $P : H_D : |\Psi(t)\rangle$:

$$P : H_D : |\Psi(t)\rangle = \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| : H_D : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.34)$$

We have already done a significant portion of this work, so plugging (2.33) into (2.34) and using the delta functions to evaluate as many integrals as we can, we see

$$P : H_D : |\Psi(t)\rangle = \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 (\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2)) (\Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) - \Psi(\mathbf{k}'_2, \mathbf{k}'_1, t)) |\mathbf{k}'_1, \mathbf{k}'_2\rangle \quad (2.35)$$

Taking advantage of the antisymmetry of fermions, namely that their wave functions must be antisymmetric¹⁰, we see that (2.35) can be simplified:

$$P : H_D : |\Psi(t)\rangle = \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 (2\Psi(\mathbf{k}'_1, \mathbf{k}'_2, t)) (\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2)) |\mathbf{k}'_1, \mathbf{k}'_2\rangle \quad (2.36)$$

2.2.3 Derivation of the Projected Coulomb Hamiltonian Term

Now we move to the Coulomb term, using (2.25) we see

$$P : H_C : |\mathbf{k}_1, \mathbf{k}_2\rangle \propto \quad (2.37)$$

$$\begin{aligned} & \langle \mathbf{k}'_2, \mathbf{k}'_1| : \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ C_{\downarrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ C_{\downarrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \end{bmatrix} : |\mathbf{k}_1, \mathbf{k}_2\rangle \\ & =: \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}(\mathbf{k}'_1) \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ C_{\downarrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ C_{\downarrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ D_{\uparrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \\ D_{\downarrow}^{\dagger}(\mathbf{k}' + \mathbf{k}) \end{bmatrix} : c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle. \end{aligned}$$

¹⁰i.e. $\Psi(\mathbf{k}_1, \mathbf{k}_2, t) = -\Psi(\mathbf{k}_2, \mathbf{k}_1, t)$

We immediately see that any matrix element with at least one positron creator (spin up or spin down) will be zero since that creator will act on the empty bra. We also see that when a spin down electron creator is chosen, the same thing happens.

This effectively leaves us with:

$$\langle |c_{\uparrow}(\mathbf{k}'_2)c_{\uparrow}(\mathbf{k}'_1) \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ 0 \\ 0 \end{bmatrix} : c_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.38)$$

This means do not only one nonzero matrix element, and that occurs when all spin up electron operators are chosen.

Enforcing normal ordering, we see this element must be:

$$- \langle |c_{\uparrow}(\mathbf{k}'_2)c_{\uparrow}(\mathbf{k}'_1)C_{\uparrow}^{\dagger}(\mathbf{k}'')C_{\uparrow}^{\dagger}(\mathbf{k}')C_{\uparrow}(\mathbf{k}'' - \mathbf{k})C_{\uparrow}(\mathbf{k}' + \mathbf{k})c_{\uparrow}^{\dagger}(\mathbf{k}_1)c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \quad (2.39)$$

We again employ the technique from above for determining what this inner product is. By doing that, we get:

$$\begin{aligned} & \delta(\mathbf{k}'' - \mathbf{k}'_1)\delta(\mathbf{k}' - \mathbf{k}'_2)\delta(\mathbf{k}_1 - (\mathbf{k}'' - \mathbf{k}))\delta(\mathbf{k}_2 - (\mathbf{k}' + \mathbf{k})) \\ & - \delta(\mathbf{k}'' - \mathbf{k}'_1)\delta(\mathbf{k}' - \mathbf{k}'_2)\delta(\mathbf{k}_1 - (\mathbf{k}' + \mathbf{k}))\delta(\mathbf{k}_2 - (\mathbf{k}'' - \mathbf{k})) \\ & - \delta(\mathbf{k}'' - \mathbf{k}'_2)\delta(\mathbf{k}' - \mathbf{k}'_1)\delta(\mathbf{k}_1 - (\mathbf{k}'' - \mathbf{k}))\delta(\mathbf{k}_2 - (\mathbf{k}' + \mathbf{k})) \\ & + \delta(\mathbf{k}'' - \mathbf{k}'_2)\delta(\mathbf{k}' - \mathbf{k}'_1)\delta(\mathbf{k}_1 - (\mathbf{k}' + \mathbf{k}))\delta(\mathbf{k}_2 - (\mathbf{k}'' - \mathbf{k})) \end{aligned} \quad (2.40)$$

The actual projected Coulomb term is only proportional to this though, so now we work on calculating the whole term.

We start by finding the $U^{\dagger}U$ coefficients we have been ignoring. Adding those in to (2.57), we have

that the full matrix in the coulomb term will effectively be

$$: \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}'') \\ 0 \\ 0 \\ 0 \end{bmatrix}^T U^{\dagger}(\mathbf{k}'')U(\mathbf{k}'' - \mathbf{k}) \begin{bmatrix} C_{\uparrow}(\mathbf{k}'' - \mathbf{k}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C_{\uparrow}^{\dagger}(\mathbf{k}') \\ 0 \\ 0 \\ 0 \end{bmatrix}^T U^{\dagger}(\mathbf{k}')U(\mathbf{k}' + \mathbf{k}) \begin{bmatrix} C_{\uparrow}(\mathbf{k}' + \mathbf{k}) \\ 0 \\ 0 \\ 0 \end{bmatrix} : \quad (2.41)$$

With the zeros where they are, the first entry (the entry in column, 1 row 1) of each $U^{\dagger}U$ matrix will be selected out. Multiplying $U^{\dagger}U$ and taking the first element, we get:

$$\begin{aligned} & \left(U^{\dagger}(\mathbf{k}'')U(\mathbf{k}'' - \mathbf{k}) \right)_{11} \left(U^{\dagger}(\mathbf{k}')U(\mathbf{k}' + \mathbf{k}) \right)_{11} \\ &= \left(\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\cos\left(\frac{\theta_{(\mathbf{k}'' - \mathbf{k})}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}'' - \mathbf{k})}}{2}\right) \left(\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k}) + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \right) \\ & \quad \times \left(\cos\left(\frac{\theta_{\mathbf{k}'}}{2}\right)\cos\left(\frac{\theta_{(\mathbf{k}' + \mathbf{k})}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}' + \mathbf{k})}}{2}\right) \left(\hat{\mathbf{k}}' \cdot (\mathbf{k}' + \mathbf{k}) + i(\hat{\mathbf{k}}' \times (\mathbf{k}' + \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \right) \end{aligned} \quad (2.42)$$

So we have for the projected Coulomb term:

$$\begin{aligned} & \langle \mathbf{k}'_2, \mathbf{k}'_1 | : H_C : | \mathbf{k}_1, \mathbf{k}_2 \rangle \\ &= \langle \mathbf{k}'_2, \mathbf{k}'_1 | b_C \int d^3 \mathbf{k} \frac{1}{\|\mathbf{k}\|^2} \int d^3 \mathbf{k}' \int d^3 \mathbf{k}'' \tilde{\Psi}^{\dagger}(\mathbf{k}'') \tilde{\Psi}^{\dagger}(\mathbf{k}'' - \mathbf{k}) \tilde{\Psi}^{\dagger}(\mathbf{k}') \tilde{\Psi}^{\dagger}(\mathbf{k}' + \mathbf{k}) | \mathbf{k}_1, \mathbf{k}_2 \rangle \\ & \quad = b_C \int d^3 \mathbf{k} \frac{1}{\|\mathbf{k}\|^2} \int d^3 \mathbf{k}' \int d^3 \mathbf{k}'' \left(\right. \\ & \quad \left(\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\cos\left(\frac{\theta_{(\mathbf{k}'' - \mathbf{k})}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}'' - \mathbf{k})}}{2}\right) \left(\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k}) + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \right) \\ & \quad \times \left(\cos\left(\frac{\theta_{\mathbf{k}'}}{2}\right)\cos\left(\frac{\theta_{(\mathbf{k}' + \mathbf{k})}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}' + \mathbf{k})}}{2}\right) \left(\hat{\mathbf{k}}' \cdot (\mathbf{k}' + \mathbf{k}) + i(\hat{\mathbf{k}}' \times (\mathbf{k}' + \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \right) \\ & \quad \times \left(\delta(\mathbf{k}'' - \mathbf{k}'_1) \delta(\mathbf{k}' - \mathbf{k}'_2) \delta(\mathbf{k}_1 - (\mathbf{k}'' - \mathbf{k})) \delta(\mathbf{k}_2 - (\mathbf{k}' + \mathbf{k})) \right. \\ & \quad - \delta(\mathbf{k}'' - \mathbf{k}'_1) \delta(\mathbf{k}' - \mathbf{k}'_2) \delta(\mathbf{k}_1 - (\mathbf{k}' + \mathbf{k})) \delta(\mathbf{k}_2 - (\mathbf{k}'' - \mathbf{k})) \\ & \quad - \delta(\mathbf{k}'' - \mathbf{k}'_2) \delta(\mathbf{k}' - \mathbf{k}'_1) \delta(\mathbf{k}_1 - (\mathbf{k}'' - \mathbf{k})) \delta(\mathbf{k}_2 - (\mathbf{k}' + \mathbf{k})) \\ & \quad \left. \left. + \delta(\mathbf{k}'' - \mathbf{k}'_2) \delta(\mathbf{k}' - \mathbf{k}'_1) \delta(\mathbf{k}_1 - (\mathbf{k}' + \mathbf{k})) \delta(\mathbf{k}_2 - (\mathbf{k}'' - \mathbf{k})) \right) \right) \end{aligned} \quad (2.43)$$

With each set of four delta functions multiplied together, the three integrals will each be evaluated slightly differently due to the arguments of the delta functions varying slightly.

After the integral evaluations, we get that (2.43) equals

$$\begin{aligned}
& \frac{-b_C}{\|(\mathbf{k}_1 - \mathbf{k}'_2)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2)) \\
& + \frac{b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2)) \\
& + \frac{b_C}{\|(\mathbf{k}_1 - \mathbf{k}'_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2)) \\
& + \frac{-b_C}{\|(\mathbf{k}'_2 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2))
\end{aligned} \tag{2.44}$$

Which we can easily see equals

$$\begin{aligned}
& \frac{-2b_C}{\|(\mathbf{k}_1 - \mathbf{k}'_2)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2)) \\
& + \frac{2b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_2}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot \hat{\mathbf{k}}_2 + i(\hat{\mathbf{k}}'_2 \times \hat{\mathbf{k}}_2) \cdot \hat{\mathbf{z}} \right) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}'_1 + \mathbf{k}'_2)) \\
& = \langle \mathbf{k}'_2, \mathbf{k}'_1 | : H_C : | \mathbf{k}_1, \mathbf{k}_2 \rangle
\end{aligned} \tag{2.45}$$

Using the same process as for the Dirac Hamiltonian term, we can show that for

$$P : H_C : |\Psi(t)\rangle = \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 \Psi(\mathbf{k}_1, \mathbf{k}_2, t) |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| : H_C : |\mathbf{k}_1, \mathbf{k}_2\rangle \quad (2.46)$$

the following is true:

$$\begin{aligned} P : H_C : |\Psi(t)\rangle = & \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 2\Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \left(\right. \\ & \frac{2bc}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\ & \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \right. \\ & \left. \left. + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1) + i(\hat{\mathbf{k}}'_2 \times (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1)) \cdot \hat{\mathbf{z}} \right) \right) \right) |\mathbf{k}'_1, \mathbf{k}'_2\rangle. \end{aligned} \quad (2.47)$$

This gives us the final piece we need to be able to assemble the full, projected Hamiltonian.

2.2.4 The Full, Projected Hamiltonian

Thus, combining (2.13), (2.31), (2.36) and (2.47), we see

$$\begin{aligned} P : H : |\Psi(t)\rangle = & \\ & 0 + 0 + 2 \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \left[\right. \\ & \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) \left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) + \int d^3 \mathbf{k}_1 \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \left(\right. \\ & \frac{2bc}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\ & \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \right. \\ & \left. \left. + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1) + i(\hat{\mathbf{k}}'_2 \times (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1)) \cdot \hat{\mathbf{z}} \right) \right) \right) \left. \right] |\mathbf{k}'_1, \mathbf{k}'_2\rangle. \end{aligned} \quad (2.48)$$

Since we projected the right hand side of the Schrodinger equation, we know that (2.48) equals the projected left hand side of the Schrodinger equation:

$$i\hbar P \frac{d}{dt} |\Psi(t)\rangle = 2i\hbar \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \frac{d}{dt} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) \quad (2.49)$$

Canceling the factors of two, we have so far:

$$\begin{aligned} i\hbar \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \frac{d}{dt} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) = \\ \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \left[\left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) \right. \\ \left. + \int d^3 \mathbf{k}_1 \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \left(\frac{2bc}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \right. \right. \\ \left. \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \right. \right. \\ \left. \left. + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1) + i(\hat{\mathbf{k}}'_2 \times (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1)) \cdot \hat{\mathbf{z}} \right) \right) \right] \end{aligned} \quad (2.50)$$

Since the integrals are over the exact same volume, we know the integrands must be equal.¹¹ So given any state with fixed $\mathbf{k}'_1, \mathbf{k}'_2$, we know

¹¹this can also be derived if we left multiply both sides of the equation by $\langle \mathbf{k}_2, \mathbf{k}_1 |$. In this case, antisymmetry of fermionic wave functions must be taken of advantage, as well as the fact that in the Coulomb term, one can switch the arbitrary indices, which will then allow for the previous fact to be taken advantage of—this is much easier to see if you do this step before using the delta function that shows conservation of momentum to get rid of one of the two directions

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) = & \\
& \left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) + \int d^3 \mathbf{k}_1 \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \left[\right. \\
& \frac{2b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\
& \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \right. \\
& \left. \left. + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1) + i(\hat{\mathbf{k}}'_2 \times (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1)) \cdot \hat{\mathbf{z}} \right) \right) \right]
\end{aligned} \tag{2.51}$$

Which is the full, projected Schrodinger equation.

2.3 Simplifications Made to be Able to Run Numerics Without a Supercomputer

We can use numerics to solve our infinite system of differential equations to find the wave function, Ψ , but unfortunately the number of particles and dimensions we still have would require the use of a very powerful supercomputer.¹² In this section, we will attempt to simplify the system enough such that numerics can be successfully run on a regular computer, while still keeping the interesting physics present.

¹²We estimate the numerical complexity of matrix multiplication for the cases of 1-3 spatial dimensions in subsections 3.2.1, 3.2.2, 3.2.3 respectively

2.3.1 Simplification #1: non-relativistic particles

The first simplification we will use is to make our particles non-relativistic. When this is true, the arguments of the sine and cosine functions in (2.47) get really small.

Using the small angle approximations for sine and cosine, we get, for the first term of sines and cosines of (2.47):

$$1 - \frac{\theta_{\mathbf{k}'_1}^2}{8} - \frac{\theta_{\mathbf{k}_1}^2}{8} + \frac{\theta_{\mathbf{k}'_1}^2 \theta_{\mathbf{k}_1}^2}{64} + \frac{\theta_{\mathbf{k}'_1} \theta_{\mathbf{k}_1}}{4} \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \quad (2.52)$$

Which, if we ignore all terms with order θ^2 , we get exactly 1.

This gives us

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) = \left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) + \int d^3 \mathbf{k}_1 \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \frac{2b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \quad (2.53)$$

This form speeds up the numerics by making the computer calculate up to 16 less vector norms, 8 cosine or sine terms, 4 dot products, and 2 cross products at *every combination of $\mathbf{k}_1, \mathbf{k}'_1, \mathbf{k}'_2$* ,¹³ which have 3 dimensions each.

This simplification also gives with a previous requirement we enforced to be able to use the projection operator: the particles must be low enough energy that they do not pair produce—so this simplification will not get rid of any more interesting physics.

2.3.2 Simplification #2: Constriction of two dimensions

Reducing the number of spatial dimensions obviously reduces computational complexity significantly,¹⁴ but it can also ruin inherently three dimensional physics, such as spin, $\frac{1}{r^2}$ force dependence, etc., if we are not careful. Thus, besides messing with units, if we just "erase" two of our integrals

¹³while conserving momentum, of course

¹⁴As we will see in section 3.2

we lose the ability to see the force between the particles fall off like $\frac{1}{r^2}$, which is one of the physical principles we are interested in seeing with our system.

We reduce our computation requirements by projecting down to quasi one spatial dimension. To do this, we enforce spatial periodicity in the x,y directions (hence the phrase "quasi one spatial dimension") with arbitrary lateral dimensions $\ell \times \ell$. This will give us, for (2.53):

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) = \left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) + \sum_{n,m} \int dk_{1z} \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \frac{2b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \quad (2.54)$$

where

$$\mathbf{k} = \mathbf{k}_{(nm)} = \left(\frac{2\pi n}{\ell}, \frac{2\pi m}{\ell}, k_z \right) \quad (2.55)$$

and n, m are integers that range from negative infinity to positive infinity in the sums.

To make this computationally effective, we keep only the lowest mode – so we set $n = m = 0$ to get

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}'_{1(00)}, \mathbf{k}'_{2(00)}, t) = \left(\varepsilon(\mathbf{k}'_{1(00)}) + \varepsilon(\mathbf{k}'_{2(00)}) \right) \Psi(\mathbf{k}'_{1(00)}, \mathbf{k}'_{2(00)}, t) + \int dk_{1z} \Psi(\mathbf{k}_{1(00)}, (\mathbf{k}'_{1(00)} + \mathbf{k}'_{2(00)} - \mathbf{k}_{1(00)}), t) \frac{2b_C}{\|(\mathbf{k}'_{1(00)} - \mathbf{k}_{1(00)})\|^2} \quad (2.56)$$

Or, to simplify the notation, we let $\mathbf{k}'_1 \rightarrow \mathbf{k}, \mathbf{k}'_2 \rightarrow \mathbf{k}', \mathbf{k}_1 \rightarrow \mathbf{k}''$:

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}_{00}, \mathbf{k}'_{00}, t) = \left(\varepsilon(\mathbf{k}_{00}) + \varepsilon(\mathbf{k}'_{00}) \right) \Psi(\mathbf{k}_{00}, \mathbf{k}'_{00}, t) + \int dk''_z \Psi(\mathbf{k}''_{00}, (\mathbf{k}_{00} + \mathbf{k}'_{00} - \mathbf{k}''_{00}), t) \frac{2b_C}{\|\mathbf{k}_{00} - \mathbf{k}''_{00}\|^2} \quad (2.57)$$

With (2.57), we try to preserve the three dimensionality of the physics, while decreasing the computational complexity significantly. We discuss whether we think we accomplished both of those things in chapter 3.

2.3.3 Simplification #3: discretization of momentum space

Now that we have our system projected down to quasi one spatial dimension, we discretize momentum space as the final step in preparing our system for numerics on a regular computer. If we think of an integral as a sum of a bunch of smaller integrals, we can see why the following is true in the limit as $\Delta x \rightarrow 0$:

$$\int dx f(x) = \sum_i \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} dx f(x) \quad (2.58)$$

Where Δx is the distance between discrete points in space.

Normally, for a single function value, $f(x_i)$, and small Δx , we would approximate the integral in (2.58) according to:

$$f(x_i)\Delta x = \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} dx f(x) \quad (2.59)$$

Applying this concept to our state, $|\mathbf{k}_{00}, \mathbf{k}'_{00}\rangle$, we would thus have

$$\Delta k \Delta k' |\mathbf{k}_{00}, \mathbf{k}'_{00}\rangle = \int_{k_i - \frac{\Delta k}{2}}^{k_i + \frac{\Delta k}{2}} dk_z \int_{k_j - \frac{\Delta k'}{2}}^{k_j + \frac{\Delta k'}{2}} dk'_z |\mathbf{k}_{00}, \mathbf{k}'_{00}\rangle \quad (2.60)$$

The problem is that we are dealing with a state which needs to stay normalized.¹⁵ Normalizing is easily done¹⁶, and gives a slightly different version of (2.60):

$$\sqrt{\Delta k} \sqrt{\Delta k'} |\mathbf{k}_{00}, \mathbf{k}'_{00}\rangle = \int_{k_{zi} - \frac{\Delta k}{2}}^{k_{zi} + \frac{\Delta k}{2}} dk_z \int_{k_{zj} - \frac{\Delta k'}{2}}^{k_{zj} + \frac{\Delta k'}{2}} dk'_z |\mathbf{k}_{00}, \mathbf{k}'_{00}\rangle \quad (2.61)$$

we now drop the subscript 0's from our \mathbf{k} 's (remembering that we have done so) in (2.57), so we can index the \mathbf{k} 's a different way. we also apply the process of discretization to our system and get:

$$\begin{aligned} \sum_{i,j} \int_{k_{zi} - \frac{\Delta k}{2}}^{k_{zi} + \frac{\Delta k}{2}} dk_z \int_{k'_{zj} - \frac{\Delta k'}{2}}^{k'_{zj} + \frac{\Delta k'}{2}} dk'_z i\hbar \frac{d}{dt} \Psi(\mathbf{k}, \mathbf{k}', t) |\mathbf{k}, \mathbf{k}'\rangle = \\ \sum_{i,j} \int_{k_{zi} - \frac{\Delta k}{2}}^{k_{zi} + \frac{\Delta k}{2}} dk_z \int_{k'_{zj} - \frac{\Delta k'}{2}}^{k'_{zj} + \frac{\Delta k'}{2}} dk'_z \left((\varepsilon(\mathbf{k}) + \varepsilon(\mathbf{k}')) \Psi(\mathbf{k}, \mathbf{k}', t) \right. \\ \left. + \sum_f \int_{k''_{zf} - \frac{\Delta k''}{2}}^{k''_{zf} + \frac{\Delta k''}{2}} dk''_z \Psi(\mathbf{k}'', (\mathbf{k} + \mathbf{k}' - \mathbf{k}''), t) \frac{2b_C}{\|\mathbf{k} - \mathbf{k}''\|^2} \right) |\mathbf{k}, \mathbf{k}'\rangle \quad (2.62) \end{aligned}$$

¹⁵meaning we need $\langle \mathbf{k}'_{00}, \mathbf{k}_{00} | \mathbf{k}_{00}, \mathbf{k}'_{00} \rangle = 1$

¹⁶see section A.3 for a derivation

we choose the same discretization value for each term and call it Δk , which gives us:

$$\begin{aligned}
\Delta k \sum_{i,j} i\hbar \frac{d}{dt} \Psi(\mathbf{k}_i, \mathbf{k}'_j, t) |\mathbf{k}_i, \mathbf{k}'_j\rangle &= \\
\Delta k \sum_{i,j} \left(\left(\varepsilon(\mathbf{k}_i) + \varepsilon(\mathbf{k}'_j) \right) \Psi(\mathbf{k}_i, \mathbf{k}'_j, t) + \Delta k \sum_f \Psi(\mathbf{k}'_f, (\mathbf{k}_i + \mathbf{k}'_j - \mathbf{k}''_f), t) \frac{2b_C}{\|\mathbf{k}_i - \mathbf{k}''_f\|^2} \right) |\mathbf{k}_i, \mathbf{k}'_j\rangle & \\
\iff & \\
i\hbar \frac{d}{dt} \Psi(\mathbf{k}_i, \mathbf{k}'_j, t) &= \\
\left(\varepsilon(\mathbf{k}_i) + \varepsilon(\mathbf{k}'_j) \right) \Psi(\mathbf{k}_i, \mathbf{k}'_j, t) + \Delta k \sum_f \Psi(\mathbf{k}'_f, (\mathbf{k}_i + \mathbf{k}'_j - \mathbf{k}''_f), t) \frac{2b_C}{\|\mathbf{k}_i - \mathbf{k}''_f\|^2}. & \quad (2.63)
\end{aligned}$$

Where, of course, in $i\hbar\dots$, we have $i = \sqrt{-1}$ (meaning, it is not an index).¹⁷

The superscripts above were to distinguish the variables, but since we now have indices now on all of our \mathbf{k} 's—placing them inside the discretization of the whole momentum space— we can drop any superscripts we have. This gives us

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}_i, \mathbf{k}_j, t) = \left(\varepsilon(\mathbf{k}_i) + \varepsilon(\mathbf{k}_j) \right) \Psi(\mathbf{k}_i, \mathbf{k}_j, t) + \Delta k \sum_f \Psi(\mathbf{k}_f, (\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_f), t) \frac{2b_C}{\|\mathbf{k}_i - \mathbf{k}_f\|^2}. \quad (2.64)$$

With that, we summarize what we have done so far: we defined the QED Hamiltonian, which we then let act on our state of two spin up electrons. Due to the particle spray present when that happens, we projected our particle content back down to what we originally had. We then enforced

¹⁷The astute reader may notice we seem to have ended up with the wrong coefficient on the sum over f in the last term. Before seeing the derivation in section A.3, one would've thought this should be as it is due to discretizing the integral (which is exactly right, it turns out), but after normalizing, one might think it requires only a factor of $\sqrt{\Delta k}$ instead of $\sqrt{\Delta k^2}$. But indeed our initial intuition would have been right here. This is because the state we normalized for the last term was the same state as the other terms in (2.63), which for the *two*-label state gave us a factor of $\sqrt{\Delta k^2}$ (multiplied in front of the RHS in that equation). The last integral we discretized was done in the normal way, leaving us with a full factor of Δk .

periodic boundary conditions in two dimensions and discretized momentum space, which allowed us to find an equation, namely (2.64), that could be solved using a standard computer. Numerically solving for eigenvalues and eigenstates of $\Psi(\mathbf{k}_i, \mathbf{k}_j, t)$ would allow us to then transform into real space, thus finally obtaining the dynamics we have been seeking.

Chapter 3

Results and Discussion

In this chapter, we briefly go over the equations we worked so hard to obtain. We also discuss what it would take to solve these equations numerically and why it has not been done up to this point. Lastly, we discuss the next steps to continue this research.

3.1 Results

We again show figure 2.1 below, where what we have accomplished so far is shown by the red lines. We started with the QED Hamiltonian in momentum space, which we allowed to act on a state with two spin up electrons. We then projected the spray down to just the two spin up electrons we originally wanted to see. One reason we were justified in doing this is because the initial electrons would have to have extremely high energies in order to pair produce. Making our electrons have a sufficiently low energy ensures we only see the particle content we projected onto. Our path then went two ways: first, we discretized momentum space of our three spatial dimensional Schrodinger equation, and second, we projected down to one spatial dimension and then discretized momentum space.

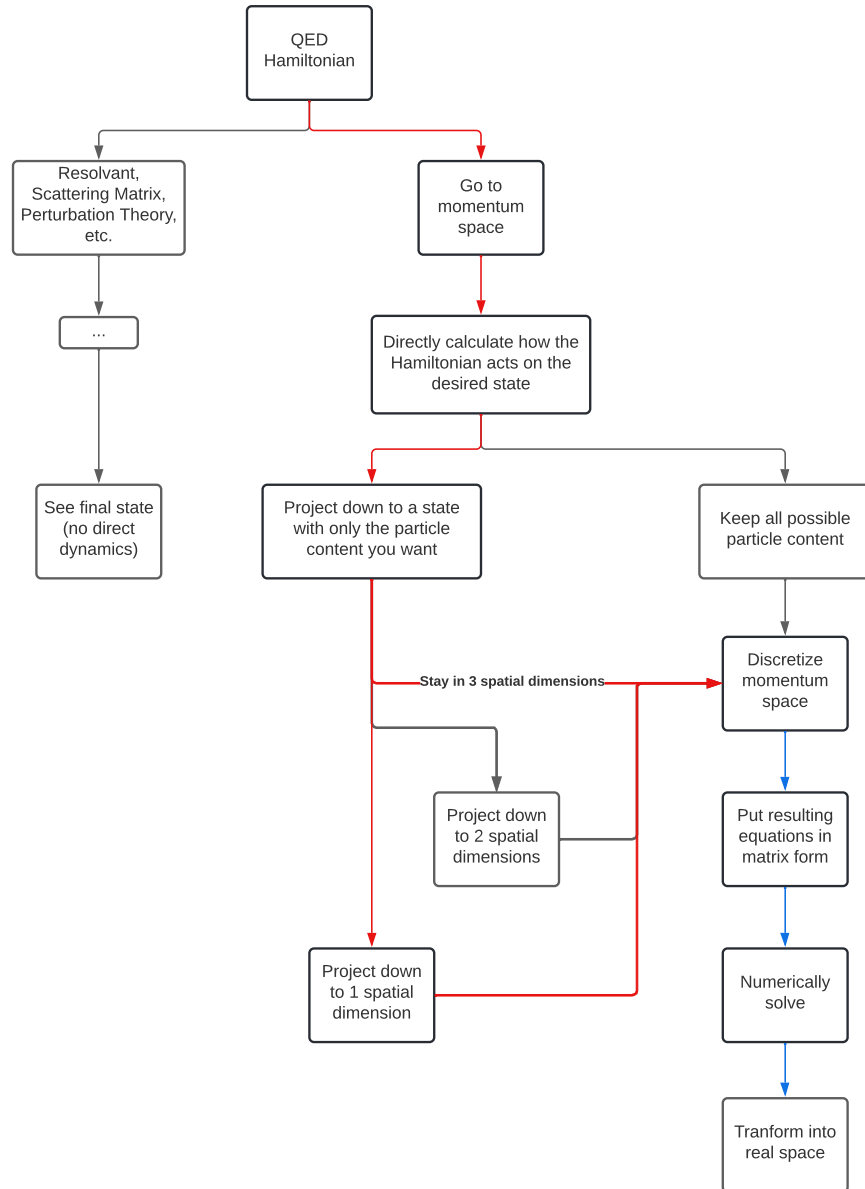


Figure 3.1 Diagram showing various ways to approach a problem in QED. The topic of the left-hand side is scattering theory, which is not the focus of this paper, so we have not included any of the intermediate steps taken in that approach. The red arrows indicate steps we will take in this paper, while the black arrows indicate steps we will not take. The blue arrows indicate steps that must be taken in order to see the physical dynamics.

3.1.1 Result #1: Numerically Solvable Schrodinger equation in 3 Spatial Dimension For System of 2 Spin-Up Electrons

Given a state of two spin-up electrons, we were able to use the Hamiltonian in the Coulomb Gauge given by references [1–3] to find an expression, namely equation (2.51), that can be solved using a supercomputer¹. For the ease of the reader, we include that equation here:

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) = & \\
 & \left(\varepsilon(\mathbf{k}'_1) + \varepsilon(\mathbf{k}'_2) \right) \Psi(\mathbf{k}'_1, \mathbf{k}'_2, t) + \int d^3 \mathbf{k}_1 \Psi(\mathbf{k}_1, (\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1), t) \left[\right. \\
 & \frac{2b_C}{\|(\mathbf{k}'_1 - \mathbf{k}_1)\|^2} \left(\cos\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}'_1}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}_1}}{2}\right) \left(\hat{\mathbf{k}}'_1 \cdot \hat{\mathbf{k}}_1 + i(\hat{\mathbf{k}}'_1 \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{z}} \right) \right) \\
 & \times \left(\cos\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \cos\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \right. \\
 & \left. \left. + \sin\left(\frac{\theta_{\mathbf{k}'_2}}{2}\right) \sin\left(\frac{\theta_{(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1)}}{2}\right) \left(\hat{\mathbf{k}}'_2 \cdot (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1) + i(\hat{\mathbf{k}}'_2 \times (\mathbf{k}'_1 + \hat{\mathbf{k}}'_2 - \mathbf{k}_1)) \cdot \hat{\mathbf{z}} \right) \right) \right] ,
 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
 \varepsilon(\mathbf{k}) &= \sqrt{(mc^2)^2 + (\hbar c \mathbf{k})^2}, \\
 b_C &= \frac{q^2}{2\varepsilon_0(2\pi)^3}, \\
 \theta_{\mathbf{k}} &= \arctan\left(\frac{\hbar \|\mathbf{k}\|}{mc}\right).
 \end{aligned} \tag{3.2}$$

Which is the biggest result from our work since this equation can be solved numerically, giving the dynamics we've been seeking.²

¹an estimate of the numerical complexity of solving this equation is given in section 3.2.3

²although doing so requires the use of a powerful supercomputer with powerful GPU's

3.1.2 Result #2: Numerically Solvable Schrodinger equation in 1 Spatial Dimension For System of 2 Spin-Up Electrons

Given the solution in (2.51), we then reduced our system to one spatial dimension and discretized momentum space, giving us (2.64), which we again include below for convenience:

$$i\hbar \frac{d}{dt} \Psi(\mathbf{k}_i, \mathbf{k}_j, t) = \left(\varepsilon(\mathbf{k}_i) + \varepsilon(\mathbf{k}_j) \right) \Psi(\mathbf{k}_i, \mathbf{k}_j, t) + \Delta k \sum_f \Psi(\mathbf{k}_f, (\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_f), t) \frac{2b_C}{\|\mathbf{k}_i - \mathbf{k}_f\|^2}. \quad (3.3)$$

This is another huge result since this equation can be solved numerically without requiring the use of a supercomputer.

3.2 Discussion on Why Numerics Have not Been Used Yet

Although we have equations that can be solved numerically, giving us the dynamics we have been seeking, we still have not actually run the numerics, as discussed below (covering the case of each number of spatial dimensions). We start each of the following subsections by estimating the approximate numerical complexity of multiplying the matrices in that equation.

Figure 3.2 shows the important pieces of information needed to estimate the numerical complexity. The process for doing that will be as follows: first, find the length of the vector $\Psi(\mathbf{k}_1, \mathbf{k}_2, t)$, which is $N^{(IV)}$ in the figure. Second, determine the numerical complexity of the matrix multiplication $P : H : \Psi(\mathbf{k}_1, \mathbf{k}_2, t)$.

The first step is straightforward. Since the vector is one dimensional, the length will simply be the number of ways the wave function could be called. We find this number by raising the number of points in the discretization, N , to the number of independent spatial variables able to be called in $\Psi(\mathbf{k}_1, \mathbf{k}_2, t)$. For three spatial dimensions, the maximum number the exponent can be is 6, since each of the two particles have momenta in 3 dimensions.³

³meaning each \mathbf{k} in the argument of the wave function has three components: $\Psi(\mathbf{k}_1(k_{1x}, k_{1y}, k_{1z}), \mathbf{k}_2(k_{2x}, k_{2y}, k_{2z}), t)$

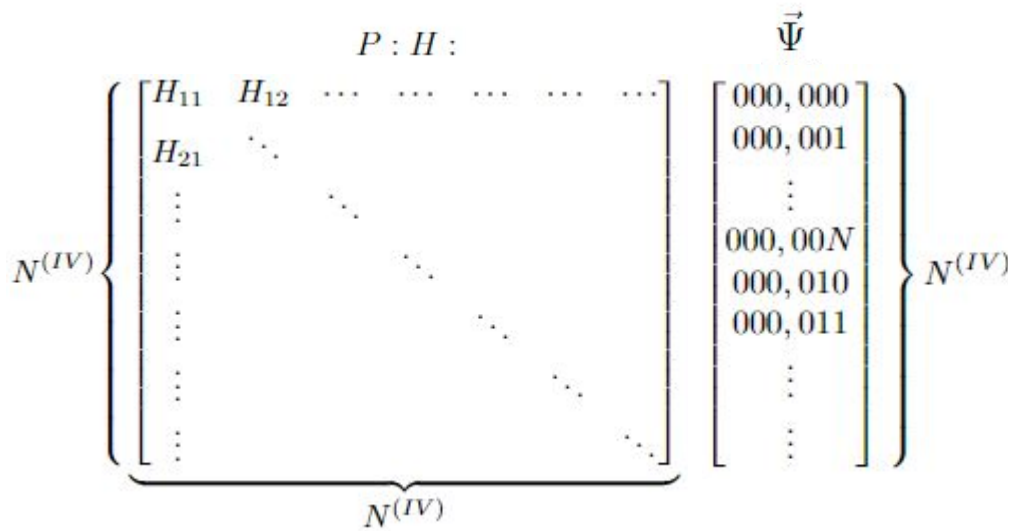


Figure 3.2 A representation of matrix multiplication numerical complexity. The $N^{(IV)} \times N^{(IV)}$ matrix on the left is $P : H :$, while the $N^{(IV)} \times 1$ matrix on the right is $\vec{\Psi}$, where IV is the number of independent variables in $\Psi(\mathbf{k}_1, \mathbf{k}_2, t)$ (besides t), which changes depending on how many dimensions are included. $\vec{\Psi}$ is indexed using up to six labels (one label for each dimension for each particle); the initial structure of one such indexing method is shown.

For the second step, we use the simplest approach to matrix multiplying and the resulting computational complexity.⁴ The simplest approach is to count the number of individual multiplications that have to happen in order to multiply the matrices. We do this by simply multiplying the dimension of $P : H :$ by the dimension of $\vec{\Psi}$. With the given $P : H :$, $\vec{\Psi}$, this would give us $(N^{(IV)} \times N^{(IV)})$, since each element of $P : H :$ needs to be multiplied exactly once with an element of $\vec{\Psi}$ (exactly once since $\vec{\Psi}$ is $1 \times N^{(IV)}$). Unfortunately, it is not just this number of calculations done at each point in time; to actually see the dynamics, $N^{(IV)}$ points must be plotted at each point in time, which requires a lot of GPU⁵ power in addition to the computing power already needed to do the calculations.

3.2.1 Regarding the Case of 1 Spatial Dimension

The vector form of Ψ in equation (2.64) is indexed according to $\Psi(\mathbf{k}(0, 0, k_{iz}), \mathbf{k}(0, 0, k_{jz}), t)$, and thus has 2 independent spatial variables. This makes the length of $\vec{\Psi}$ equal to N^2 . Thus, for each row of $P : H :$, there are N^2 entries being multiplied with $\vec{\Psi}$. We do this multiplication N^2 times, resulting in a maximum computational complexity of $O(N^2 * N^2) = O(N^4)$, where the O stands for "order" in the Big O notation. Given that N is the number of points in one entire dimension, we have to have an N of at least 1000, which makes even this relatively small—when compared with that of higher dimensions—matrix multiplication quite numerically expensive. Additionally, this is the numerical complexity of just one matrix multiplication; we have not taken into account any additional steps actually solving the differential equation or plotting the results.

Although (2.64) could be solved using a standard matrix approach in many computers, we realized that the periodic boundary conditions we enforced in 2.3.2 would essentially look like two infinite planes of charges. This is extremely similar to infinite capacitor plates (differing by just the

⁴due to conservation of momentum, the most of the off-diagonal terms in $P : H :$ are 0—so $P : H :$ ends up being a block diagonal matrix. This means the numerical complexity is not quite as much as $N^{(IV)} \times N^{(IV)}$, but is somewhere between that and $N^{(IV)-\# \text{ spatial dimensions}} \times N^{(IV)}$ if the correct numerical techniques are employed.

⁵graphics processing unit

sign of the charge on one of the plates), so we know the electric field, and thus the force, between the plates would be constant.

At first glance, one might think that a constant force, no matter what the distance between the objects is, would break energy conservation laws. On the contrary, a standard calculation in classical electrodynamics books shows that the energy stored in a field is equal to the work done to assemble the charges (or currents) that make up the field. This means that although allowing the plates of charges to move freely would allow them to gain infinite energy, that same amount of energy was needed to place the plates there in the first place. As the plates move, they convert that potential energy (energy stored in the field) to kinetic energy, thus conserving energy.

Despite the fact that our reduced system seems to make physical sense, we still decided not to run the numerics because approximating infinite plates of charge can easily be done through classical electrodynamics and thus does not need to be done using QED.

3.2.2 Regarding the Case of 2 Spatial Dimensions

Here, $\vec{\Psi}$ would be indexed according to $\Psi(\mathbf{k}_1(0, k_{1y}, k_{1z}), \mathbf{k}_2(0, k_{2y}, k_{2z}), t)$, resulting in a length of $\vec{\Psi}$ of N^4 . The resulting matrix multiplication would have a complexity of up to $O(N^{4+4} = O(N^8))$. We never solved for this case, but using the techniques in 2.3.2, one could easily project down to two spatial dimensions instead of one. With the periodic boundary conditions we enforced, we would be simulating infinite rods of charge. This, too, could be easily solved using classical electrodynamics (Although QED would be more accurate?), but it is complicated enough computationally that it would need a supercomputer (it would be approximately the square of the computational complexity of the one spatial dimension case—assuming the same discretization). In addition to super-computing power, we would need to have excellent GPU's to be able to plot so many spatial points at many locations through time.

3.2.3 Regarding the Case of 3 Spatial Dimensions

In this case, $\vec{\Psi}$ would be indexed according to $\Psi(\mathbf{k}_1(k_{1x}, k_{1y}, k_{1z}), \mathbf{k}_2(k_{2x}, k_{2y}, k_{2z}), t)$, resulting in a length of $\vec{\Psi}$ of N^6 . The resulting matrix multiplication would have maximum matrix multiplication complexity of $O(N^{6+6} = O(N^{12}))$. This case would clearly also require a super computer, albeit a better one than the two dimensional case, in addition to requiring even better GPU's to help with the plotting. But in solving this case, we have the full power of QED describing single particles interacting in every dimension we know of. For this reason, and the fact that it is only relatively more complex than the two spatial dimension case while not requiring simplifications, we decided to run numerics for this case.

We aren't including these numerics in this thesis because it is still going to be a lot of work to get to the point where we can run numerics on it; after discretizing (2.51), one would have to put the equation in matrix form, choose an indexing method to let the state with two labels, $\mathbf{k}'_1, \mathbf{k}'_2$ be an array instead of a matrix, as well as include an integral over all \mathbf{k}_1 (multiplying specific states in $\vec{\Psi}$ by some constants in each case) within each matrix entry of a given $\mathbf{k}'_1, \mathbf{k}'_2$.

3.3 Next Steps

In the following sections, we describe what the next steps of this research could be. First, we describe how to solve the system of equations numerically. Second, how to find the dynamics numerically. And third, we describe further research after finding the dynamics.

3.3.1 Finding the Allowed Energies of Our Hamiltonian

Earlier, we decided to use a projection operator to restrict our particle content. This actually served us in another way, namely, it guaranteed us a solution to the Schrodinger equation. We explain:

Let's say we are trying to solve the matrix equation

$$Ax = b \quad (3.4)$$

Where A is a matrix and x, b are vectors.

This equation does not always have a solution, such as when b is not in the column space of A . The way to approach this is to left-multiply by A^T on both sides. This guarantees that b is in the column space of $A^T A$ and is the least-squares solution of $Ax = b$.

So if there isn't a solution to the matrix equation

$$H |\Psi\rangle = i\hbar \frac{d}{dt} |\Psi\rangle \quad (3.5)$$

Then applying the projector to each side is like applying A^T and thus guarantees us a solution, and if it is not a solution to the original equation (3.5) then it is as close as possible to one. With that in mind, our goal is to make (2.64) look like a matrix equation, which makes it easy to use Matlab to solve for the eigenvalues of the Hamiltonian acting on our state of two spin-up electrons:

$$H |\Psi\rangle = E |\Psi\rangle \quad (3.6)$$

Where E is a constant, and where solving for all possible E values gives all the allowed energies of the system.

Doing this requires input of some initial conditions. To choose these, we remember simplification #1 (from section 2.3.1), where we required that the particles be non-relativistic, and the Heisenberg uncertainty principle which dictates how precisely the momentum and position of a particle can be known at the same time.

If we want to know anything about the particles' positions, we cannot have the initial momenta be numbers. Instead we use Gaussian distributions, which are easy to work with and still one to know about the particles' respective momenta as and positions. Furthermore, we have to make these distributions peak in lower velocities so as to stay non-relativistic.

3.3.2 Finding the Dynamics

Once the energies (eigenvalues of the Hamiltonian) and eigenstates in momentum space are obtained, we take the inverse Fourier transform to get into real space.

Using the typical solution to the Schrodinger equation :

$$\Psi(x, t) = e^{-iH(x)t/\hbar} \quad (3.7)$$

we then propagate our solution in time. Thus finally obtaining the dynamics we have been seeking.

3.3.3 Steps After Finding Dynamics

Part of the interesting physics presented by the QED Hamiltonian in the Coulomb Gauge is that it does not countenance any photons, while still allowing for a Coulomb interaction. Because of this, another important step after this is to do the analogous calculation in the Lorenz gauge, which does countenance photons, and see how the two results differ. This could show us if the "fake" longitudinal and scalar photons⁶ in the Lorenz gauge are actually responsible for the Coulombic Interaction. From there, delving into the reason for the differences between the solutions in each gauge could lead to huge insights on the nature of the interaction of charged quantum particles and how "fake" photons interact with those particles.

Due to the nature of each of the terms in the QED Hamiltonian, any choice of state with exactly two charged particles (electrons, positrons, or one of each) will result in almost identical equations as shown above. The Radiation and Interaction terms will remain zero as long as the projection does not include photons, while the Dirac term could have slightly different form if a state of one positron and one electron is chosen or identical particles with different spins are chosen. The Coulomb term

⁶Longitudinal photons propagate in the same direction as the oscillating field that creates the photons, while scalar photons propagate in the direction of time. Neither of these states is recognized as possible in physical space.

would change the most significantly, but would still be easy to calculate.⁷ One could use these facts to apply this work to others, such as allowing pair-produced particles to have interaction where it has been ignored previously.⁸ This would allow the pair to annihilate after their creation, which is a much more physical view of the situation.

Additionally, one could use the techniques presented in this document to more easily find the dynamics of states with any spin electrons, any spin positrons, real photons, and any combination thereof. Each of these situations could lead to additional insights on the quantum nature of these particles.

⁷The $U^\dagger(\mathbf{k}'')U(\mathbf{k}'' - \mathbf{k})$ matrix is shown in section A.2, and one could use that to easily determine $U^\dagger(\mathbf{k}')U(\mathbf{k}' + \mathbf{k})$. Given any selection of particles, one would need to calculate the inner product of the projection with the valid combinations of creators and annihilators given the choice of state. One would use that to then determine the relevant elements of the given matrices to serve as replacements for the sine and cosine terms derived in this document.

⁸A physics senior thesis was completed in 2021 by Ethan Gibson where the Coulomb interaction between a pair-produced electron and positron was ignored. This problem is also currently being worked on by Joshua Newey, who has also ignored the Coulomb interaction

Appendix A

Miscellaneous Derivations, Explanations, etc.

A.1 Projectors (relating to footnotes 7, 9)

When we project down to two spin up electrons, what we're effectively saying is that physically, we're going to measure the momentum¹ of two spin up electrons. So although the particle spray in (2.26) shows us there's potentially a lot more in our universe than just two spin up electrons, we're not going to worry about any of it besides those two electrons we started with.

Mathematically, this is accomplished through a projection operator, which we define in the following way:

$$P := \frac{1}{2} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| \quad (\text{A.1})$$

For this to be a projector, we need to show that the projection of the projector is the projector itself, i.e.

$$P^2 = P. \quad (\text{A.2})$$

¹since we've done all our calculations so far in momentum space

We show this now:

$$\begin{aligned}
P^2 &= \\
& \frac{1}{4} \left(\int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| \right) \left(\int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}_1, \mathbf{k}_2\rangle \langle \mathbf{k}_2, \mathbf{k}_1| \right) \\
&= \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1| \mathbf{k}_1, \mathbf{k}_2\rangle \langle \mathbf{k}_2, \mathbf{k}_1| \\
&= \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}(\mathbf{k}'_1) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \langle \mathbf{k}_2, \mathbf{k}_1|
\end{aligned} \tag{A.3}$$

We can now use the anti-commutation relations of fermions given in (2.16), (2.17) to see that this makes

$$\begin{aligned}
P^2 &= \\
& \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle | c_{\uparrow}(\mathbf{k}'_2) (\delta(\mathbf{k}_1 - \mathbf{k}'_1) - c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}(\mathbf{k}'_1)) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \langle \mathbf{k}_2, \mathbf{k}_1| \\
&= \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \delta(\mathbf{k}_1 - \mathbf{k}'_1) \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \\
&\quad - \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle | c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}(\mathbf{k}'_1) c_{\uparrow}^{\dagger}(\mathbf{k}_2) | \rangle \langle \mathbf{k}_2, \mathbf{k}_1| \\
&= \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \delta(\mathbf{k}_1 - \mathbf{k}'_1) \delta(\mathbf{k}_2 - \mathbf{k}'_2) \langle \mathbf{k}_2, \mathbf{k}_1| \\
&\quad - \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \delta(\mathbf{k}_1 - \mathbf{k}'_2) \delta(\mathbf{k}_2 - \mathbf{k}'_1) \langle \mathbf{k}_2, \mathbf{k}_1| \\
&\quad + \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle | c_{\uparrow}^{\dagger}(\mathbf{k}_1) c_{\uparrow}(\mathbf{k}'_2) c_{\uparrow}^{\dagger}(\mathbf{k}_2) c_{\uparrow}(\mathbf{k}'_1) | \rangle \langle \mathbf{k}_2, \mathbf{k}_1|.
\end{aligned} \tag{A.4}$$

Since creators acting on empty bras give zero (as well as annihilators acting on empty kets), the last

term is zero, which means

$$\begin{aligned}
P^2 &= \\
&\frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \delta(\mathbf{k}_1 - \mathbf{k}'_1) \delta(\mathbf{k}_2 - \mathbf{k}'_2) \langle \mathbf{k}_2, \mathbf{k}_1 | \\
&- \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \delta(\mathbf{k}_1 - \mathbf{k}'_2) \delta(\mathbf{k}_2 - \mathbf{k}'_1) \langle \mathbf{k}_2, \mathbf{k}_1 | \\
&= \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1 | \\
&- \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_1, \mathbf{k}'_2 |.
\end{aligned} \tag{A.5}$$

Due to the anti-commutation of fermions, we see this becomes

$$\begin{aligned}
P^2 &= \\
&\frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1 | \\
&+ \frac{1}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1 | \\
&= \frac{2}{4} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1 | \\
&= \frac{1}{2} \int d^3 \mathbf{k}'_1 \int d^3 \mathbf{k}'_2 |\mathbf{k}'_1, \mathbf{k}'_2\rangle \langle \mathbf{k}'_2, \mathbf{k}'_1 | \\
&= P.
\end{aligned} \tag{A.6}$$

And we see our defined projection operator satisfies the necessary condition.

A.2 4×4 $U^\dagger U$ Matrices (relating to footnote 7)

Using the definition in (1.8), we've calculated the matrix $U^\dagger(\mathbf{k}'')U(\mathbf{k}'' - \mathbf{k})$ and found the four columns to be: Column one:

$$\left[\begin{array}{l} \left(\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) + \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left(\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k}) + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \right) \\ \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{x}} - i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{y}} \right) \\ -\hat{\mathbf{k}}''_z \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) + (\mathbf{k}'' - \mathbf{k})_z \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \\ -\cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) (\hat{\mathbf{k}}''_x + i\hat{\mathbf{k}}''_y) + \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\mathbf{k}'' - \mathbf{k})_x + i(\mathbf{k}'' - \mathbf{k})_y \right) \end{array} \right] \quad (\text{A.7})$$

two:

$$\left[\begin{array}{l} \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{y}} + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{x}} \right) \\ \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right) + \sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) \left((\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k})) - i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \\ -\cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) (\hat{\mathbf{k}}''_x - i\hat{\mathbf{k}}''_y) + \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\mathbf{k}'' - \mathbf{k})_x - i(\mathbf{k}'' - \mathbf{k})_y \right) \\ \hat{\mathbf{k}}''_z \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) - (\mathbf{k}'' - \mathbf{k})_z \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \end{array} \right] \quad (\text{A.8})$$

three:

$$\left[\begin{array}{l} \hat{\mathbf{k}}''_z \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) - (\mathbf{k}'' - \mathbf{k})_z \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \\ \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) (\hat{\mathbf{k}}''_x + i\hat{\mathbf{k}}''_y) - \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\mathbf{k}'' - \mathbf{k})_x + i(\mathbf{k}'' - \mathbf{k})_y \right) \\ \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right) + \sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) \left((\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k})) + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \\ \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left(-(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{y}} + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{x}} \right) \end{array} \right] \quad (\text{A.9})$$

four:

$$\left[\begin{array}{l} \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) (\hat{\mathbf{k}}''_x - i\hat{\mathbf{k}}''_y) - \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\mathbf{k}'' - \mathbf{k})_x - i(\mathbf{k}'' - \mathbf{k})_y \right) \\ -\hat{\mathbf{k}}''_z \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) + (\mathbf{k}'' - \mathbf{k})_z \cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \\ \sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right)\sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right) \left((\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{y}} + i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{x}} \right) \\ \cos\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\cos\left(\frac{\theta_{\mathbf{k}''}}{2}\right) + \sin\left(\frac{\theta_{(\mathbf{k}''-\mathbf{k})}}{2}\right)\sin\left(\frac{\theta_{\mathbf{k}''}}{2}\right) \left((\hat{\mathbf{k}}'' \cdot (\mathbf{k}'' - \mathbf{k})) - i(\hat{\mathbf{k}}'' \times (\mathbf{k}'' - \mathbf{k})) \cdot \hat{\mathbf{z}} \right) \end{array} \right] \quad (\text{A.10})$$

The other matrix needed for the Coulomb term would be $U^\dagger(\mathbf{k}')U(\mathbf{k}' + \mathbf{k})$, which we will not put here but could be easily adapted from the previous matrix by switching the arguments ($\mathbf{k}'' \rightarrow$

\mathbf{k}' , $(\mathbf{k}'' - \mathbf{k}) \rightarrow (\mathbf{k}' + \mathbf{k})$). These two matrices can then be used to determine the coefficients in the Coulomb term for states of any choice of charged particles.

A.3 Normalized States (relating to footnote 16)

In a one particle state, normalization is the process of setting the probability that the particle is found *anywhere* at a given time t is equal to one. Which is intuitive; it "normalizes" the numbers we use to describe the probability of something happening (always between 0 and 1). The typical way this is done in quantum mechanics is to multiply the state by some normalization constant, we'll call it "A," and then set the inner product of the state with itself equal to one. From there, one can generally find what A is explicitly for the given state. Although we don't know exactly what a given state's wave function is, we still want to ensure our state is normalized. To do this, we start with a modified version of (2.60):

$$\Delta k_1 \Delta k_2 |\mathbf{k}_1, \mathbf{k}_2\rangle = A \int_{k_i - \frac{\Delta k_1}{2}}^{k_i + \frac{\Delta k_1}{2}} d^3 \mathbf{k}_1 \int_{k_j - \frac{\Delta k_2}{2}}^{k_j + \frac{\Delta k_2}{2}} d^3 \mathbf{k}_2 |\mathbf{k}_1, \mathbf{k}_2\rangle. \quad (\text{A.11})$$

Where we now see the normalization constant A appearing. It's important to note that the entire RHS is a single state while on the LHS, only the ket is a state. Then the next step is to take the inner product of each side with itself:

$$\Delta k_1 \Delta k_2 \langle \mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_1, \mathbf{k}_2 \rangle = A^2 \int_{k_f - \frac{\Delta k'_1}{2}}^{k_f + \frac{\Delta k'_1}{2}} d^3 \mathbf{k}'_1 \int_{k_g - \frac{\Delta k'_2}{2}}^{k_g + \frac{\Delta k'_2}{2}} d^3 \mathbf{k}'_2 \int_{k_i - \frac{\Delta k_1}{2}}^{k_i + \frac{\Delta k_1}{2}} d^3 \mathbf{k}_1 \int_{k_j - \frac{\Delta k_2}{2}}^{k_j + \frac{\Delta k_2}{2}} d^3 \mathbf{k}_2 \langle \mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_1, \mathbf{k}_2 \rangle. \quad (\text{A.12})$$

Since A is supposed to be part of the state, it got squared while the Δk terms weren't squared because they weren't part of the state, they're just constants out front.

If we force the state on each side to be normalized, that leaves us with

$$\begin{aligned}\Delta k_1 \Delta k_2 &= A^2 \\ &\iff \\ A &= \sqrt{\Delta k_1 \Delta k_2}\end{aligned}\tag{A.13}$$

So we see

$$\Delta k_1 \Delta k_2 |\mathbf{k}_1, \mathbf{k}_2\rangle = \sqrt{\Delta k_1 \Delta k_2} \int_{k_i - \frac{\Delta k_1}{2}}^{k_i + \frac{\Delta k_1}{2}} d^3 \mathbf{k}_1 \int_{k_j - \frac{\Delta k_2}{2}}^{k_j + \frac{\Delta k_2}{2}} d^3 \mathbf{k}_2 |\mathbf{k}_1, \mathbf{k}_2\rangle,\tag{A.14}$$

or

$$\sqrt{\Delta k_1 \Delta k_2} |\mathbf{k}_1, \mathbf{k}_2\rangle = \int_{k_i - \frac{\Delta k_1}{2}}^{k_i + \frac{\Delta k_1}{2}} d^3 \mathbf{k}_1 \int_{k_j - \frac{\Delta k_2}{2}}^{k_j + \frac{\Delta k_2}{2}} d^3 \mathbf{k}_2 |\mathbf{k}_1, \mathbf{k}_2\rangle.\tag{A.15}$$

This is exactly what we claimed to get to equation (2.61). We can further this if we choose the same discretization for $\mathbf{k}_1, \mathbf{k}_2$. Doing so gives

$$\Delta k |\mathbf{k}_1, \mathbf{k}_2\rangle = \int_{k_i - \frac{\Delta k_1}{2}}^{k_i + \frac{\Delta k_1}{2}} d^3 \mathbf{k}_1 \int_{k_j - \frac{\Delta k_2}{2}}^{k_j + \frac{\Delta k_2}{2}} d^3 \mathbf{k}_2 |\mathbf{k}_1, \mathbf{k}_2\rangle,\tag{A.16}$$

Which is ultimately what it took to get to equation (2.64).

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