Designing a Map for Quantum Teleportation Protocols

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ABSTRACT

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Many different quantum teleportation protocols have been proposed since the idea was first introduced in the 1990s. The variety of teleportation protocols can be described by identifying relevant degrees of freedom (DOF). I create a map that organizes teleportation protocols according to the following five DOF: total qubits teleported, minimum qubits needed, type and number of quantum resources used, and the number of classical bits of information required. This map is created to achieve two goals, firstly as a means of organizing the current state of teleportation research, and secondly to search for additional teleportation protocols. By considering unfilled spaces on the map, one new teleportation protocol—the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios—is identified. I design its corresponding quantum circuit diagram and verify its validity. This discovery validates the usefulness of such a map in extending the scope of quantum teleportation.

Keywords: teleportation, quantum information, quantum circuitry, quantum computing, Bell state, map, degrees of freedom, two-qubit state, three-qubit state, quantum resources

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this Chapter, necessary introductory material is provided that will make understanding the methods(Chapter 2) and results(Chapter 3) more graspable. Section 1.1 introduces the basic laws of quantum mechanics and quantum information that enable the existence and properties of qubits. Section 1.2 introduces the quantum circuitry description of quantum computing. Entanglement resources are defined in Section 1.3. In Section 1.4, quantum teleportation is given a qualitative description, to be followed up in Section 2.1 by a quantitative description. Lastly, an overview of the content of Chapters 2 and 3 is provided in Section 1.5.

1.1 Quantum Information

Quantum information is the information that is contained within a quantum state (Nielsen & Chuang 2011). Quantum information science seeks to understand, analyze, and process this information using the laws of quantum mechanics. The theory and techniques differ substantially from how everyday computers process information. Computers use laws of classical physics—especially electricity and magnetism—to analyze and process information. The use of the laws of quantum mechanics in quantum information science allows some unique problems to be solved that everyday computers cannot do efficiently. This is done by using specifically designed quantum algorithms, or algorithms that can be run on quantum computers. Quantum computers are those that can manipulate simple quantum states known as qubits . A notable example of a quantum algorithm that can be run on a quantum computer but can't be run on a classical computer is Shor's algorithm, which can find the prime factors of large integers efficiently (Shor 1997).

Quantum states follow different physical laws than those predicted by Newtonian physics. The Schrödinger equation describes quantum systems. The wavefunction that solves the Schrödinger equation for a particular system can only reveal information about the probabilities that a particular result will occur as the outcome of a measurement. This is different from Newtonian mechanics, where one can know exactly what the state of a system will be if one knows its current momentum and position. A quantum system exists within a superposition of states until a measurement outcome is observed, at which point its wavefunction collapses onto that particular outcome. Thus by knowing the wavefunction, one can only know that probabilities of a measurement outcome.

There are also laws of uncertainty in quantum mechanics that say one can only know information for related characteristics within a quantum system such as position and momentum or energy and time to a certain extent. Additionally, the quantum phenomena of entanglement—to which there is no true classical analog—is possible between quantum systems. This allows for some possibilities in information processing that are not possible for classical computers. In Section 1.3, I describe various ways in which particles can be entangled with each other.

Where classical computers use bits as the basic unit of information processing, quantum computers use quantum bits, or qubits. Qubits differ from classical bits in some important ways. Classical bits can have one of two states: 0 or 1. Operations can be performed on them to change from one state to the other, but they only ever exist as either 0 or 1. Laws of uncertainty and superposition in quantum mechanics cause the qubit to behave differently. One can only know the probability that measuring a qubit will result in a 0 or a 1. These two outcomes are called the computational basis states for qubits. A qubit $|\varphi\rangle$ in a linear vector space V can be expressed mathematically as $|\varphi\rangle = a|0\rangle + b|1\rangle$, where the complex numbers a and b satisfy $|a|^2 + |b|^2 = 1$.

Another important difference is that, unlike qubits, classical bits can be copied. This is experienced on a daily basis when a file is copied into another location on your computer. On the other hand, the "No Cloning Theorem" in quantum mechanics forbids the creation of an exact copy of a quantum state, or less generally a qubit, without destroying the original state. This motivates an important question: how can quantum information be shared or transferred without having to move the physical particle to which the qubit is attached if information can't be copied? This is accomplished by quantum teleportation, which is explained in Section 1.4.

1.2 Quantum Circuitry

Qubits are the building pieces of quantum computers. Within a quantum computer, changes to the state of a qubit can be made. Qubits can also be made to change conditionally on the current state of another qubit. The most widely used model for the representation of the operations of a quantum computer is the quantum circuit model (Nielsen & Chuang 2011).

An example of a quantum circuit is shown in Fig. 1.1. Each horizontal line represents a qubit, and time passes as you move from the left to the right. The input states for each qubit are shown on the left of the circuit, and the final output states are sometimes shown at the right. It is standard protocol for the input states to all be the same, and they usually are input in the computational basis state of $|0\rangle$. Boxes that appear on individual lines are quantum gates that cause a change to occur for the qubit whose line it is on. Some of the possible single qubit operations are the Hadamard gate (H), the Pauli-X gate (X), the Pauli-Y gate (Y), and the Pauli-Z gate (Z). The Pauli-X gate has the effect of changing the state of a qubit from $|1\rangle$ to $|0\rangle$, or from $|0\rangle$ to $|1\rangle$, and so it is sometimes called the quantum NOT gate, or just the NOT gate when referring to quantum circuits. The only two-qubit

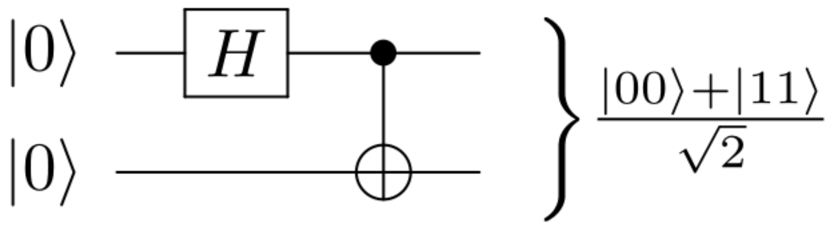


Figure 1.1 A simple quantum circuit.

gate that is needed for any quantum circuit is the C-NOT gate , represented by a horizontal line that connects two qubits. One qubit has a dark circle on it; this is the control qubit. The other qubit has a reticle; this is called the target qubit. The C-NOT gate performs the Pauli-X operation on the target qubit when and only when the control qubit is $|1\rangle$, and leaves it unchanged otherwise. There are other multi qubit quantum gates, but they can all be decomposed using series of single qubit gates and the C-NOT gate. For this reason the C-NOT gate is known as the only *universal* two-qubit gate.

The Hadamard gate H plays an important role in quantum computers. It has the varying effects that

$$H : |0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1.1}$$

and

$$H : |1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) \tag{1.2}$$

.

Notice that the outcome of the gate depends on the state of the qubit. When it is applied to a qubit and followed by a C-NOT gate where the qubit that just had the Hadamard gate applied to it is the control qubit, a state of maximum entanglement is created between the two qubits. This is called a Bell state , and is an important component in many quantum information processes because this entanglement is something classical computers can't efficiently imitate. In Section 1.3, the creation of the four Bell states are described using the quantum circuitry model. Other types of entanglement resources are also introduced.

Measurements can also be represented on a quantum circuit diagram. They appear as a meter with an arrow. Once a measurement is performed, the qubit's superposition of states collapses into a single result. It ends the life of that qubit, outputting the result of the measurement as classical information.

1.3 Entanglement Resources

A single qubit has two computational basis states. In the z-basis for the spin up or down of a particle, these are $|0\rangle$ and $|1\rangle$. These computational basis states are sometimes illustrated with arrows, to better illustrate the physical meaning of the two spin states a quantum particle could have. $|0\rangle$ becomes $|\uparrow\rangle$, and $|1\rangle$ becomes $|\downarrow\rangle$. This thesis makes use of both notations. It is standard for all quantum computer qubits to be initialized in the computational basis state of $|\uparrow\rangle$. Given these initial conditions, the Bell state—the state of maximal entanglement between two qubits—that can be obtained with the fewest number of quantum gates is the $|\Phi^+\rangle$ Bell state. The experimental application of quantum gates introduces noise and come at a monetary cost. This makes the $|\Phi^+\rangle$ Bell state the most efficient and affordable given the initial states. Table 1.1 introduces the quantum circuits that create each Bell state, as well as the bra–ket representation of each state and an accounting of the total number of quantum gates needed to obtain each one.

A two-qubit system has four computational basis states. The Bell basis is one such two-qubit system, and its basis states can be denoted as $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$.

There are more ways for quantum states to be entangled than in the Bell state. The GHZ state generalizes the type of entanglement for a two-qubit entangled state to *n* states. So, the GHZ state

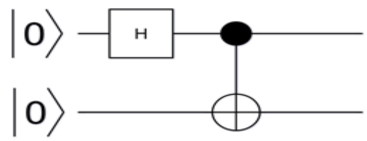
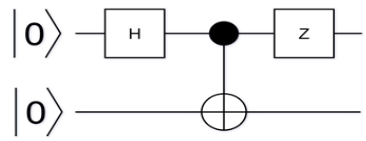
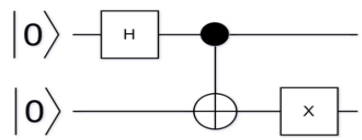
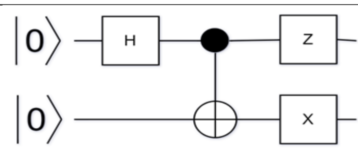
Bell State Creation			
Bell State	Creation Protocol	Bra–ket Representation	Total Gates
$\lvert\Phi^+\rangle$		$\frac{1}{\sqrt{2}}(\lvert\uparrow\rangle\lvert\uparrow\rangle+\lvert\downarrow\rangle\lvert\downarrow\rangle)$	2
$\lvert\Phi^-\rangle$		$\frac{1}{\sqrt{2}}(\lvert\uparrow\rangle\lvert\uparrow\rangle-\lvert\downarrow\rangle\lvert\downarrow\rangle)$	3
$\lvert\Psi^+\rangle$		$\frac{1}{\sqrt{2}}(\lvert\uparrow\rangle\lvert\downarrow\rangle+\lvert\downarrow\rangle\lvert\uparrow\rangle)$	3
$\lvert\Psi^-\rangle$		$\frac{1}{\sqrt{2}}(\lvert\uparrow\rangle\lvert\downarrow\rangle-\lvert\downarrow\rangle\lvert\uparrow\rangle)$	4

Table 1.1 Quantum circuitry needed to obtain each of the four Bell states given two input qubits that are in the computational basis states $\lvert 0\rangle$ (or $\lvert\uparrow\rangle$). The outcome of the circuits is shown under the "Bra–ket Representation" column. The total number of quantum gates needed for each process is also provided.

represents the state of maximal entanglement that a n-particle systems can be in. The quantum circuit that creates a GHZ entanglement state for a three-particle system is shown in Fig. 1.2.

There is another type of three particle entanglement called the W state. The bra-ket notation for the W state is

$$\lvert W\rangle = \frac{1}{\sqrt{3}}(\lvert\uparrow\uparrow\downarrow\rangle+\lvert\uparrow\downarrow\uparrow\rangle+\lvert\downarrow\uparrow\uparrow\rangle). \tag{1.3}$$

Quantum states need not be maximally entangled; they can only be partially entangled with each other.

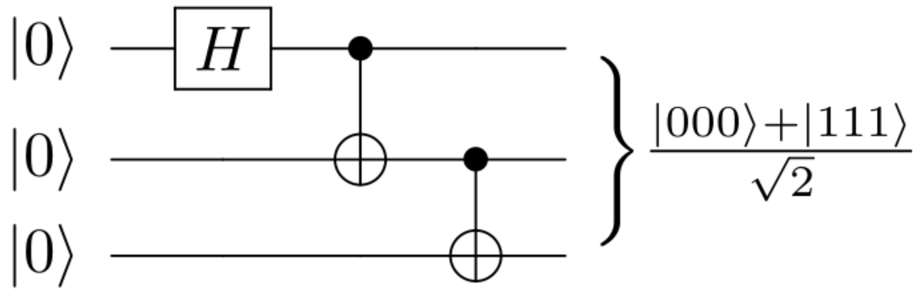


Figure 1.2 Quantum circuit for the creation of a GHZ entanglement state between three qubits.

1.4 Standard Quantum Teleportation

In Section 1.1 qubits were described and compared to classical bits. An important point to remember is that the "No Cloning Theorem" in quantum mechanics forbids the cloning of a quantum state. This is different from classical bits, where it is a trivial matter to make copies of data and distribute it. This begs the question: if quantum information can't be copied, is there any way to share it without having to physically move the particle to which the qubit is attached?

Such a means of communicating quantum information was discovered in 1993 (Bennett et al. 1993), and is known as quantum teleportation. Quantum teleportation allows an unknown quantum state to be communicated from one location to another without physically transporting the particle to which the state is attached. The following resources are required to accomplish quantum teleportation: an unknown qubit to be teleported, a classical communication channel to send two bits of classical information, the ability to create and distribute a Bell state, the ability to perform Bell measurements (Nielsen & Chuang 2011) on the joint state of one half of the Bell pair and the state to be teleported, and the ability to perform quantum gates on the other qubit in the Bell pair.

With these resources, the protocol for quantum teleportation can be broken down into four steps.

1. A Bell state is created between two qubits. One of each of the members of this Bell state are given to the sending party (who we will call Alice) and the receiving party (who we will call Bob).

2. Alice performs a Bell measurement on her half of the Bell pair and her unknown quantum state that she would like to teleport to Bob. The Bell basis has four outcomes to this joint measurement; the result can thus be encoded in two bits of classical information to be sent to Bob. At this point, the measured qubits have no more use and can be discarded.

3. The two bits of information are sent to Bob. This step limits the speed of quantum teleportation to the speed of light.

4. Because of the entanglement of the two Bell state particles before the measurement, after the measurement is performed by Alice, Bob's half of the Bell state is in one of four possible states that are related to the unknown quantum state originally in Alice's possession. One of these possible states is identical to the unknown state Alice had; the other three are related to it by a known transformation. Using the information sent from Alice, Bob can know the identity of the qubit which is in his possession and apply the appropriate transformation to retrieve the state that Alice wished to teleport to him (See Table 2.1).

1.5 Overview

In the remainder of this thesis, I will discuss some of the progress that has been made in the field of quantum teleportation research since Bennett's introduction of the concept in 1993. In Chapter 2, I discuss the details of the teleportation protocol described in Section 1.4. I then discuss alternate teleportation protocols, some of which make use of other quantum resources besides Bell States, which resources are described in Section 1.3. The degrees of freedom (DOF) of teleportation are identified from these protocols.

In Chapter 3, I present a map of teleportation that places each protocol according to five of their degrees of freedom. From this map, I identify a candidate for a yet undiscovered teleportation protocol: the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios.

Chapter 2

Methods

The work of this thesis is to organize teleportation protocols into a map which illustrates trends and reveals spaces in which teleportation could occur which have not been discovered previously. To do this, it is necessary to have a thorough understanding of the quantitative side of the standard teleportation protocol; this is discussed in Section 2.1. In Section 2.2, alternate teleportation protocols will be examined and compared to the standard protocol. From this study, the degrees of freedom(DOF) of teleportation are identified in Section 2.3. These tools are vital in an effort to construct a map of teleportation possibilities.

2.1 Quantitative Description of Standard Teleportation

In Section 1.4, a qualitative description of quantum teleportation is provided. Here I provide a quantitative description of quantum teleportation. Using a quantum circuitry model of quantum computation, teleportation can be described as illustrated in the circuit shown in Fig. 2.1. This teleportation protocol (Bennett et al. 1993) is of such fundamental importance that I will carefully describe the quantum circuit step by step in this section.

The qubit q_1 is the unknown state originally in the possession of Alice, and qubits q_2 and q_3 start

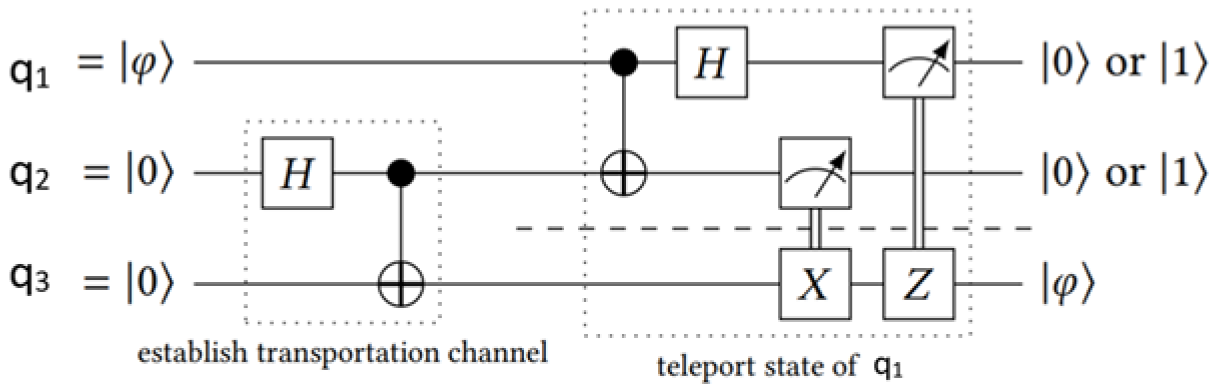


Figure 2.1 Circuit diagram for quantum teleportation. Adapted from (Hillmich et al. 2020).

out in the computational basis state $|0\rangle$, or $|\uparrow\rangle$, as per convention. Initially, the physical systems in which these qubits are placed are close in space; a physical distance short enough that C-NOT gates can be applied between them. The dashed line on the quantum circuit diagram represents a physical separation in space; separated qubits cannot have a C-NOT gate performed between them. The $|\Phi^{+}\rangle$ Bell pair is created by applying the Hadamard gate on q_2 and a C-Not gate between q_2 and q_3 with q_2 as the control qubit (see section 1.4, "Bell states"). If the parties have not yet separated they can each obtain their half of the Bell pair at the source they were created and then travel to their destinations. If the parties have already separated, then a messenger can bring each party half of the Bell pair created. Before the teleportation procedure can continue, Bob and Alice should be separated from each other with half of a Bell state each. Alice has the unknown state to be teleported at this point.

The $|\Phi^{(+)}\rangle$ Bell state for two spin-$\frac{1}{2}$ particles can be expressed as

$$|\Phi_{23}^{(+)}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_2\rangle|\uparrow_3\rangle + |\downarrow_2\rangle|\downarrow_3\rangle). \tag{2.1}$$

The arrow symbols indicate spin up and spin down, which are the two possibilities for the z-projection of spin-$\frac{1}{2}$ systems. The subscripts 2 and 3 refer to qubits q_2 and q_3. The subscript of 1 is reserved for the unknown qubit to be teleported q_1.

Note that (Bennett et al. 1993) choose the $|\Psi_{23}^{(-)}\rangle$ Bell state to start. Any of the four Bell states can be used; it only affects the transformations that Bob must perform on the teleported qubit at the end of the process. I chose the $|\Phi_{23}^{(+)}\rangle$ Bell state because it requires the least gates to be applied in the quantum circuitry model if the input qubits are in the computational basis state of $\big|\uparrow\big\rangle$, which is standard. See Table 1.1 for the gates required to obtain the other three Bell states from two $\big|\uparrow\big\rangle$ input states (recall that $|0\rangle$ can be viewed the same as $|\uparrow\rangle$).

The unknown state of the particle to be teleported $|\varphi_1\rangle$ can be represented as

$$\big|\varphi_1\big\rangle = a\big|\uparrow_1\big\rangle + b\big|\downarrow_1\big\rangle, \tag{2.2}$$

where a and b are complex numbers satisfying the relation $|a|^2 + |b|^2 = 1$.

The complete state of the three particle quantum system after the Bell state is created (after the first C-NOT gate in Fig. 2.1), but before Alice's measurement is

$$\begin{aligned}
\big|\Phi_{123}\big\rangle &= \big|\varphi_1\big\rangle \otimes \big|\Phi_{23}^{(+)}\big\rangle \\
&= (a\big|\uparrow_1\big\rangle + b\big|\downarrow_1\big\rangle) \otimes (\frac{1}{\sqrt{2}}(\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle + \big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle)) \\
&= \frac{1}{\sqrt{2}}(a\big|\uparrow_1\big\rangle + b\big|\downarrow\big\rangle) \otimes (\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle + \big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle) \\
&= \frac{1}{\sqrt{2}}(a\big|\uparrow_1\big\rangle\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle + b\big|\downarrow_1\big\rangle\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle \\
&+ a\big|\uparrow_1\big\rangle\big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle + b\big|\downarrow_1\big\rangle\big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle) \\
&= \frac{a}{\sqrt{2}}(\big|\uparrow_1\big\rangle\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle + \big|\uparrow_1\big\rangle\big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle) \\
&+ \frac{b}{\sqrt{2}}(\big|\downarrow_1\big\rangle\big|\uparrow_2\big\rangle\big|\uparrow_3\big\rangle + \big|\downarrow_1\big\rangle\big|\downarrow_2\big\rangle\big|\downarrow_3\big\rangle)
\end{aligned} \tag{2.3}$$

In Eq. (2.3) each direct product of $\big|_1\big\rangle\big|_2\big\rangle$ can be expressed in terms of the Bell operator basis vectors $\big|\Phi_{12}^{\pm}\big\rangle$ and $\big|\Psi_{12}^{\pm}\big\rangle$ by incorporating the following identities, which are obtained by inverting the definitions of the Bell states in Table. (1.1):

$$
\begin{aligned}
\left|\uparrow_1\right\rangle\left|\uparrow_2\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Phi_{12}^{+}\right\rangle + \left|\Phi_{12}^{-}\right\rangle\right)\\
\left|\uparrow_1\right\rangle\left|\downarrow_2\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Psi_{12}^{+}\right\rangle + \left|\Psi_{12}^{-}\right\rangle\right)\\
\left|\downarrow_1\right\rangle\left|\uparrow_2\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Psi_{12}^{+}\right\rangle - \left|\Psi_{12}^{-}\right\rangle\right)\\
\left|\downarrow_1\right\rangle\left|\downarrow_2\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Phi_{12}^{+}\right\rangle - \left|\Phi_{12}^{-}\right\rangle\right).
\end{aligned} \tag{2.4}
$$

By substituting Eq. (2.4) into Eq. (2.3), we get

$$
\begin{aligned}
\left|\Phi_{123}\right\rangle = \frac{a}{\sqrt{2}}&\left[\frac{1}{\sqrt{2}}\left(\left|\Phi_{12}^{+}\right\rangle + \left|\Phi_{12}^{-}\right\rangle\right)\left|\uparrow_3\right\rangle + \frac{1}{\sqrt{2}}\left(\left|\Psi_{12}^{+}\right\rangle + \left|\Psi_{12}^{-}\right\rangle\right)\left|\downarrow_3\right\rangle\right]\\
+\frac{b}{\sqrt{2}}&\left[\frac{1}{\sqrt{2}}\left(\left|\Psi_{12}^{+}\right\rangle - \left|\Psi_{12}^{-}\right\rangle\right)\left|\uparrow_3\right\rangle + \frac{1}{\sqrt{2}}\left(\left|\Phi_{12}^{+}\right\rangle - \left|\Phi_{12}^{-}\right\rangle\right)\left|\downarrow_3\right\rangle\right].
\end{aligned} \tag{2.5}
$$

Now $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$ are multiplied out and the $\left|_3\right\rangle$ state is distributed:

$$
\begin{aligned}
\left|\Phi_{123}\right\rangle = &\frac{a}{2}\left(\left|\Phi_{12}^{+}\right\rangle\left|\uparrow_3\right\rangle + \left|\Phi_{12}^{-}\right\rangle\left|\uparrow_3\right\rangle\right) + \frac{a}{2}\left(\left|\Psi_{12}^{+}\right\rangle\left|\downarrow_3\right\rangle + \left|\Psi_{12}^{-}\right\rangle\left|\downarrow_3\right\rangle\right)\\
+&\frac{b}{2}\left(\left|\Psi_{12}^{+}\right\rangle\left|\uparrow_3\right\rangle - \left|\Psi_{12}^{-}\right\rangle\left|\uparrow_3\right\rangle\right) + \frac{b}{2}\left(\left|\Phi_{12}^{+}\right\rangle\left|\downarrow_3\right\rangle - \left|\Phi_{12}^{-}\right\rangle\left|\downarrow_3\right\rangle\right).
\end{aligned} \tag{2.6}
$$

Now $1/2$ is factored out of everything, and the *a*s and *b*s are multiplied out:

$$
\begin{aligned}
\left|\Phi_{123}\right\rangle = \frac{1}{2}&\left(a\left|\Phi_{12}^{+}\right\rangle\left|\uparrow_3\right\rangle + a\left|\Phi_{12}^{-}\right\rangle\left|\uparrow_3\right\rangle + a\left|\Psi_{12}^{+}\right\rangle\left|\downarrow_3\right\rangle + a\left|\Psi_{12}^{-}\right\rangle\left|\downarrow_3\right\rangle\right.\\
&\left.+b\left|\Psi_{12}^{+}\right\rangle\left|\uparrow_3\right\rangle - b\left|\Psi_{12}^{-}\right\rangle\left|\uparrow_3\right\rangle + b\left|\Phi_{12}^{+}\right\rangle\left|\downarrow_3\right\rangle - b\left|\Phi_{12}^{-}\right\rangle\left|\downarrow_3\right\rangle\right).
\end{aligned} \tag{2.7}
$$

Identical terms are collected and organized from left to right as $\left|\Psi_{12}^{-}\right\rangle, \left|\Psi_{12}^{+}\right\rangle, \left|\Phi_{12}^{-}\right\rangle, \left|\Phi_{12}^{+}\right\rangle$

$$
\begin{aligned}
\left|\Phi_{123}\right\rangle = \frac{1}{2}&\left[\left|\Psi_{12}^{-}\right\rangle\left(a\left|\downarrow_3\right\rangle - b\left|\uparrow_3\right\rangle\right) + \left|\Psi_{12}^{+}\right\rangle\left(a\left|\downarrow_3\right\rangle + b\left|\uparrow_3\right\rangle\right)\right.\\
&\left.+\left|\Phi_{12}^{-}\right\rangle\left(a\left|\uparrow_3\right\rangle - b\left|\downarrow_3\right\rangle\right) + \left|\Phi_{12}^{+}\right\rangle\left(a\left|\uparrow_3\right\rangle + b\left|\downarrow_3\right\rangle\right)\right].
\end{aligned} \tag{2.8}
$$

From Eq. (2.8), we can see what state Bob will have. The state in Bob's possession q_3 is correlated with the state that Alice measures (the joint Bell state for qubits 1 and 2). For example, if Alice's joint measurement leaves her with $\left|\Phi_{12}^{-}\right\rangle$, then the state of the qubit in Bob's possession will be $\left|\varphi_3\right\rangle = a\left|\uparrow_3\right\rangle - b\left|\downarrow_3\right\rangle$. This differs from the original unknown state just by a negative sign in front of the b, which corresponds to the matrix transformation of

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ since } \begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Table 2.1 shows the transformations needed for all four possible outcomes of the joint measurement. With two bits of classical information, four messages can be sent (00, 01, 10, and 11). Since there are four outcomes, if Alice and Bob agree beforehand which message corresponds to which measurement outcome, Bob can obtain the information needed to perform the appropriate transformation on his state and come into possession of the state $\left|\varphi\right\rangle$ that was initially in Alice's possession with two classical bits of information from Alice. Thus, the state has been transported from Alice to Bob by utilizing both classical and entanglement channels.

Alice's Measurement	**Transformation**
$\|\Psi^-\rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$\|\Phi^-\rangle$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$\|\Phi^-\rangle$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\|\Phi^+\rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Table 2.1 Teleportation transformations for teleportation with a $|\Phi^+\rangle$ Bell state. Alice will measure one of the four Bell states from her joint measurement. The result will correspond to the transformations shown. If decided beforehand what matrix corresponds to what measurement outcome, Alice can inform Bob what transformation to apply to his state with just two classical bits.

2.2 Alternate Teleportation Protocols

From the backbone of the teleportation protocol described in Section 2.1, an increasing number of new teleportation protocols have been proposed. These protocols involve varying numbers of qubits to be teleported and various entanglement resources to enable the teleportation. Each also comes with its own possibilities of circuitry design. The common feature between these protocols is the teleportation of at least one qubit from one physical particle to another without the need to physically move the particle to which the qubit is attached (such as a photonic mode or an electron).

Many of these teleportation protocols involve entanglement resources different from Bell pairs. Teleportation is possible through the use of a GHZ state. I work out the teleportation using both classical and GHZ channels in Appendix A.

A teleportation protocol known as "probabilistic teleportation" (Agrawal & Pati 2002) can teleport an unknown state with 100% fidelity—or without any noise being introduced—using non-maximally entangle measurements at the cost of reduced probability. A way has been found to teleport an arbitrary two-qubit state via a single Bell pair given short-distances(Tan & Han 2021). "Short-distances" in this protocol references how Bob needs to take into his possession a qubit which Alice performs a C-NOT operation on with one of her qubits. Therefore, both Alice and Bob must be able to interact with this qubit, which limits the distance they can be from each other. It's a novel idea with potential applications in minimizing quantum resources in practical settings such as quantum chips. There are still more discoveries concerning unique teleportation protocols. To name a few: W-state teleportation (Joo et al. 2003), perfect teleportation with partially entangled quantum channels (Chen et al. 2021) , continuous variable teleportation(Pirandola et al. 2015), and bidirectional teleportation(Aliloute et al. 2021). Each put their own twist on the standard teleportation protocol.

What makes each of these teleportation protocols unique? Why were they not all uncovered in the first decade after the discovery of quantum teleportation? It is in part the fact that each has particular conditions or circumstances under which they can be incorporated. For example, distance limitations exist for the accomplishment of the teleportation of a two-qubit state with a single Bell pair. In Section 2.3, I organize some of the existing teleportation protocols and account for resources and conditions needed for each.

2.3 Teleportation Degrees of Freedom

Maps and charts play an important role in organizing scientific facts and deriving new properties. In chemistry, the periodic table of the elements was formulated to organize a large amount of data relating to existing atoms. As the periodic table was developed, gaps were found where yet undiscovered elements were suspected to exist. The prime example is the element gallium, which Dmitri Mendeleev predicted in 1871 after publishing the first early prototype of the modern periodic table of the elements.

The table or chart of nuclides was later created to show information relating to the larger set of nuclides that exist but couldn't be shown on the periodic table. The organization of the table also led to discoveries and progress in the field of nuclear physics and chemistry.

An attempt to organize existing teleportation protocols into such a table that might give further insight into what possibilities exist in quantum teleportation has not yet been proposed. In this Section, I present such a map and discuss the method for searching for the most useful representation.

Mendeleev considered what degrees of freedom(DOF)—independent pieces of information—uniquely identified the elements. An understanding of these could motivate the organization of the elements into a table or chart. He first organized the elements by mass. He also found that some of the elements had chemical similarities and reacted similarly to other elements, and that there was a pattern in the occurrence of these trends. He was the first to organize the elements into a table with gaps left for predicted elements that had not yet been discovered. He did so by considering the mass of the elements as one DOF and the reactivity of the elements as the other. By doing so, he helped chemistry advance significantly.

The question at hand to create a map of teleportation is thus "what DOF exist that uniquely identify teleportation protocols"? By identifying these DOF we can construct a table that may illustrate trends and reveal insights to the science of quantum teleportation as a whole in a similar fashion as the periodic table did for chemistry.

The number of qubits teleported is one factor. Each protocol could always be repeated to teleport a multiple number of qubits. But this is simply a matter of repetition, as illustrated in Fig. 2.2. However, some unique methods of teleportation involve the teleportation of multiple qubits, such as

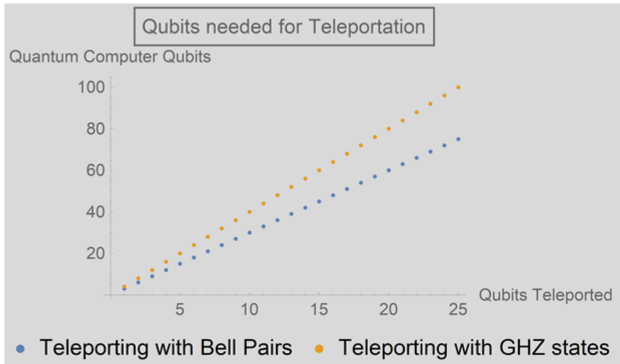


Figure 2.2 Bell vs. GHZ teleportation efficiency. Multiple numbers of qubits can be sent by duplicating protocols for teleportation. Here Bell and GHZ trends are compared. A greater efficiency is observed for the Bell states than for the GHZ states, with the slopes being 1/3 or 1/4 respectively.

the protocol which involves the short-distance teleportation of a two-qubit state with a single Bell pair, and bidirectional teleportation which teleports a total of two qubits. The minimum number of qubits teleported for a unique protocol is therefore the first piece of information that qualifies as a DOF.

Another factor is how many quantum computer qubits are needed to accomplish the teleportation. For example, both Bell state and GHZ state teleportation involve the teleporting of a single qubit, but the GHZ state teleportation involves four quantum computer qubits compared to the three that are needed in the standard teleportation protocol (see Fig. 2.2). Also the short-distance teleportation of a two-qubit state with a single Bell pair protocol teleports two qubits with a total of five quantum computer qubits. Not all teleportation types are as efficient given the amount of qubits they require, but they all have other unique attributes that could make them useful.

The amount of classical information that must be communicated through a classical channel for the teleportation to take place is another important factor that varies from protocol to protocol. In general, for every joint measurement of two qubits that Alice performs, she must convey two bits of classical information to Bob to complete the teleportation. But there is at least one protocol that promises teleportation with less than perfect probability while only conveying of a single bit of classical information(Parakh 2021).

The number of single qubit quantum gates, the number of multi-qubit quantum gates, and their organization in a quantum circuit to accomplish teleportation are all important DOF to consider. Some teleportation protocols belong to the same classification according to the above mentioned DOF (qubits teleported, number of quantum computer qubits, and number of classical bits of information conveyed), but vary considerably in their use of quantum gates. This use and organization of quantum gates can be referred to as a single DOF, which we can call the "specifics of the quantum circuitry" for the protocol.

One of the most important DOF is the type of entanglement resource used in the teleportation protocol. Section 1.3 describes some of these entanglement resources, including Bell states, GHZ states, W states, and partially entangled states. Perfect teleportation has been shown theoretically possible for each of these, with each offering different possibilities ((Bennett et al. 1993),(Xu et al. 2021),(Joo et al. 2003)).

All of the teleportation protocols from Section 2.2 can be uniquely identified and described using the five DOF described in this section. The question becomes: "Having found the degrees of freedom, can we organize or map teleportation protocols to present the information in a way that reveals trends and uncovers undiscovered teleportation protocols?"

Chapter 3

Discussion

3.1 Results

In this Chapter I will present two results. The first is a map of teleportation which categorizes teleportation protocols by their DOF. The second is a protocol for the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios. I will end with a discussion of work that could be done to extend the results of this thesis.

3.1.1 Creating a Map of Teleportation

I present in Fig 3.1 a first attempt of organizing teleportation protocols. I use five of the seven DOF for teleportation described in Section 2.3. The two degrees of freedom that are not included are the number of quantum gates required and the specifics of the quantum circuitry. The number of quantum gates needed seems to me to be the least significant DOF as far as the physics taking place are concerned. However, it might be included by adding a third dimension. The specifics of the quantum circuitry would be impractical to convey on the map, as each space would require a quantum circuit diagram to illustrate. Fig 3.1 is cumbersome to read, as it does contain a lot of information. I walk the reader through the interpretation and use of Fig 3.1 below.

In Fig 3.1 there are two DOF on the x-axis and three on the the y-axis. The primary x-axis records the total number of qubits that will be teleported. The sub x-axis records how many quantum computer qubits are needed to teleport the number of qubits indicated by the primary x-axis. The primary y-axis tells the type of quantum resource used, the sub y-axis tells the number of the quantum resource needed, and the sub-sub y-axis tells the total number of classical bits of information needed. Note again there is no description of the total number of quantum gates needed, or the specifics of the quantum circuitry design. These are important DOF for teleportation, but at present they are not included.

After the layout of this table was designed, some spaces were quickly identified as being illogical spaces for teleportation to take place. The number of qubits teleported must not be more than the total number of quantum computer qubits needed. Also, the number of classical bits of information needed to perform a teleportation must (in general) be greater than or equal to the number of qubits that a measurement is performed on at the end of the protocol. These illogical spaces were filled in with gray.

Next, some existing teleportation protocols were placed in their appropriate positions on the table. "A" placed on the map indicates spaces in which the standard teleportation protocol can take place. The dagger '†' attached to "A" and other letters indicate that the protocol is duplicated more than one time to fill that space on the table. "B" indicates a type of classical teleportation, where part of a Bell pair is teleported. The protocol is the same as for "A", but the result is the teleportation of part of a Bell pair. "C" is where teleportation with a single classical bit exists. In (Parakh 2021), this protocol was proposed to minimize classical resources in quantum teleportation. It comes at a heavy cost of less than perfect odds of teleportation success. The reason for the row of C's is that for each failed attempt, an additional quantum computer qubit will be needed to "try again". Odds become exponentially more likely with an increasing number of attempts. "D" places the protocol from

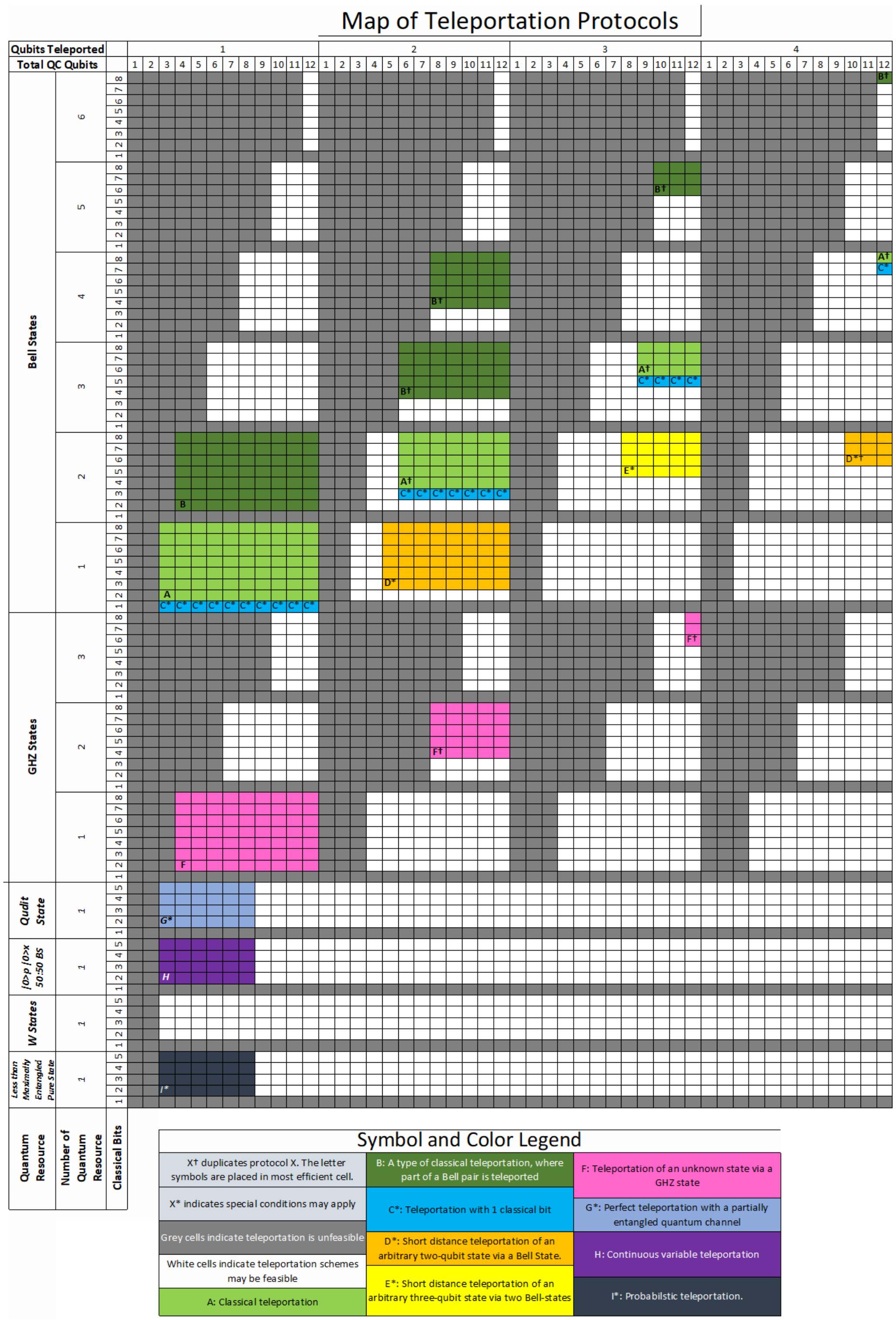


Figure 3.1 Map of protocols from Section 2.2 according to their DOF. The dagger '†' is attached to letters to indicate that the protocol by that letter is duplicated more then once to fill that space on the table. Spaces without a letter that are colored in indicated "inefficient spaces", where teleportation described by the letter of the same color can be done, but where unused resources will be left over. The * indicates special conditions apply.

"Short-Distance Teleportation of an Arbitrary Two-Qubit State via a Bell State" (Tan & Han 2021). "E" is the protocol of "short-distance teleportation of an arbitrary three-qubit state via two Bell states" which is proposed in this thesis and described in Section 3.1.2. "F" indicates teleportation of an unknown state via a GHZ state. The details of this protocol are given in Appendix A. "G" places the protocol for "Perfect teleportation with a partially entangled quantum channel"(Chen et al. 2021).

Here the present form of the table fails to convey all the information of this protocol. The classical information needed is $2log_2(d)$, where d is the dimension of the qudit state used, and two of the "qubits" accounted for on the sub x-axis need to actually be qudits. "H" places continuous variable quantum teleportation as described in (Pirandola et al. 2015). "I" indicates "Probabilistic Quantum Teleportation"(Agrawal & Pati 2002). This protocol is performed similarly to standard teleportation, but Alice performs a non-maximally entangled measurement having the same amount of entanglement as that of the shared quantum resource. Perfect teleportation has a possibility of having unitary—or perfect—fidelity with non-unitary probability.

Colored spaces without a letter represent spaces in which the teleportation protocol of the same color that is marked with a letter can occur there with some leftover resources. These might be called the "inefficient" spaces. For example, standard teleportation can of course be used to teleport a single qubit with three quantum computer qubits, one Bell pair, and three classical bits of information. But one of these bits of classical information is not needed at all to accomplish the task, and so within that space it is possible with left over resources.

One objective of creating this map of teleportation is to identify spaces for teleportation that can be filled but have not yet been discovered. Identifying trends in the map could aid in accomplishing this objective. By considering the trends for "A" (standard teleportation), "B" (standard teleportation of Bell pair members) and "F" (teleportation using a GHZ state), I investigate whether a trend could exist with less triviality amongst any of the other protocols. Protocol "D" (short-distance teleportation of a two-qubit state with a single Bell pair) falls within the domain of the map (Bell state or GHZ quantum resources) where the patterns in "A", "B", and "F" are observed. This protocol teleports a two-qubit state with just one Bell pair, given short-distance scenarios. Duplicating this protocol could be done, but a unique teleportation protocol possibility is also identifiable; the teleportation of a three-qubit state with two Bell pairs. Section 3.1.2 addresses this protocol and where it fits onto the map.

3.1.2 Analyzing Short-Distance Teleportation Protocols

The paper "Short-Distance Teleportation of an Arbitrary Two-Qubit State Via a Bell State"(Tan & Han 2021) showed how under short-distance scenarios a two-qubit state can be teleported by utilizing a single Bell pair. Fig 2 from that paper shows how this is accomplished and is reproduced in Fig 3.2 here. Qubits 1 and 2 will be teleported to locations 4 and 5. Qubits 3 and 4 are initially in a Bell state with each other, and 5 is in close proximity to both Alice and Bob.

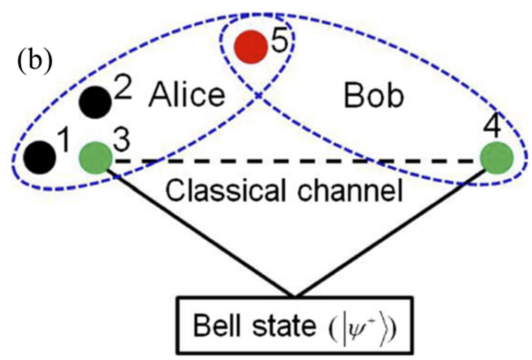


Figure 3.2 Schematic for teleportation of a two-qubit state with a single Bell Pair under short-distance conditions. Qubits 1 and 2 make up the two-qubit state being teleported. Qubits 3 and 4 are a Bell state, and qubit 5 is the auxiliary qubit which is close to both Alice and Bob. Taken from (Tan & Han 2021).

Whether this protocol could be extended to teleport larger numbers of qubits while minimizing quantum resources was not put forth in the Conclusion and Discussion part of the paper. I decided to investigate the question "can a three-qubit state be teleported with two Bell pairs under short-distance conditions", where short-distance indicates the ability for both Alice and Bob to be within reach of certain qubits throughout the entire process. This means they both can perform C-NOT operations on those qubits which are

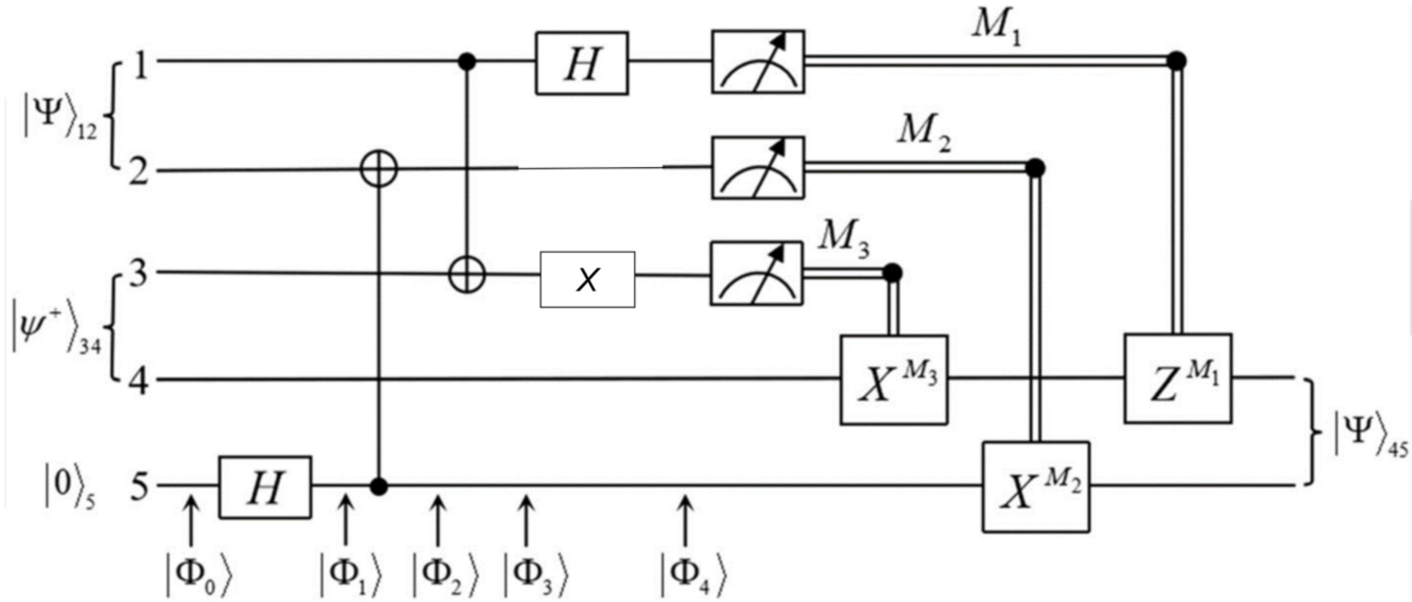


Figure 3.3 Quantum circuit for teleportation of an arbitrary two-qubit state with a single Bell pair under short-distance circumstances, adapted from (Tan & Han 2021). Note that in (Tan & Han 2021), the text by step 4 indicates the X-gate should be applied to qubit 3 and not qubit 2. The text correctly identifies the placement of the X-gate but the circuit diagram is erroneous. The diagram shown above is the corrected circuit diagram.

close to both of them with qubits that are in their own possession. To do this, I considered the figures contained in (Tan & Han 2021) closely. Under the conditions provided in Fig 3.2, the circuit diagram in Fig 3.3 (with the appropriate fix mentioned in its description of moving the X-gate from the second qubit to the third qubit) is able to perform the mentioned teleportation of the two-qubit state with a single Bell pair.

With the map of teleportation I organized in the previous section, I considered whether there are spaces within the map in which a three-qubit state could be teleported with just two Bell pairs under short-distance conditions. To be possible, the total number of quantum computer qubits would have to increase from the five which are needed to teleport a two-qubit state with a single Bell pair to eight. One additional qubit is needed for the 3rd qubit in the three-qubit state, and two more qubits are needed to accommodate the second Bell pair. So within the map of teleportation, one knows to look for this space on the x-axis at the two qubits teleported column and at the eight qubits or more location on the sub x-axis. One also knows that they are looking at a space where two Bell pairs are used. By extrapolating from the circuit diagram for the two-qubit state teleportation, one could guess that five classical bits of information will be needed to complete this teleportation. From this, we could create a schematic for the protocol that teleports a three-qubit state with two Bell pairs. Fig 3.4 illustrates that Alice is in close proximity to the qubits 1,2,3,4,6 and 8. Bob is in close proximity to qubits 5,7 and 8. This is what the two large ovals indicate. They group together the qubits that are close to Alice and those that are close to Bob. The black dots 1,2, and 3 denote the qubits that are to be teleported from Alice to Bob. The green dots denote the Bell pairs (different shades of green for different Bell pairs). The red dot represents the auxiliary qubit which is shared by Alice and Bob. After the teleportation is accomplished, qubits 5, 7, and 8 will now be identical to what qubits 1,2 and 3 originally were.

From this schematic and reasoning from the previous paragraph, one can place this protocol onto the map of quantum teleportation. It is highlighted yellow and marked with the letter "E". Like all the protocols that have come before, this protocol can be described by a quantum circuit diagram, and one should be able to show mathematically that the circuit is physically possible.

The quantum circuit diagram for the short-distance teleportation of a three-qubit state with just two Bell pairs is presented in Fig 3.5. The mathematics involved are lengthy, and a program was created to verify the plausibility of the results. These are both provided in Appendix B.

This is the first instance of the map of teleportation in Fig 3.1 having predictive power to identify yet unidentified spaces where teleportation can take place.

Three-Qubit State Teleportation with Two Bell Pairs Schematic

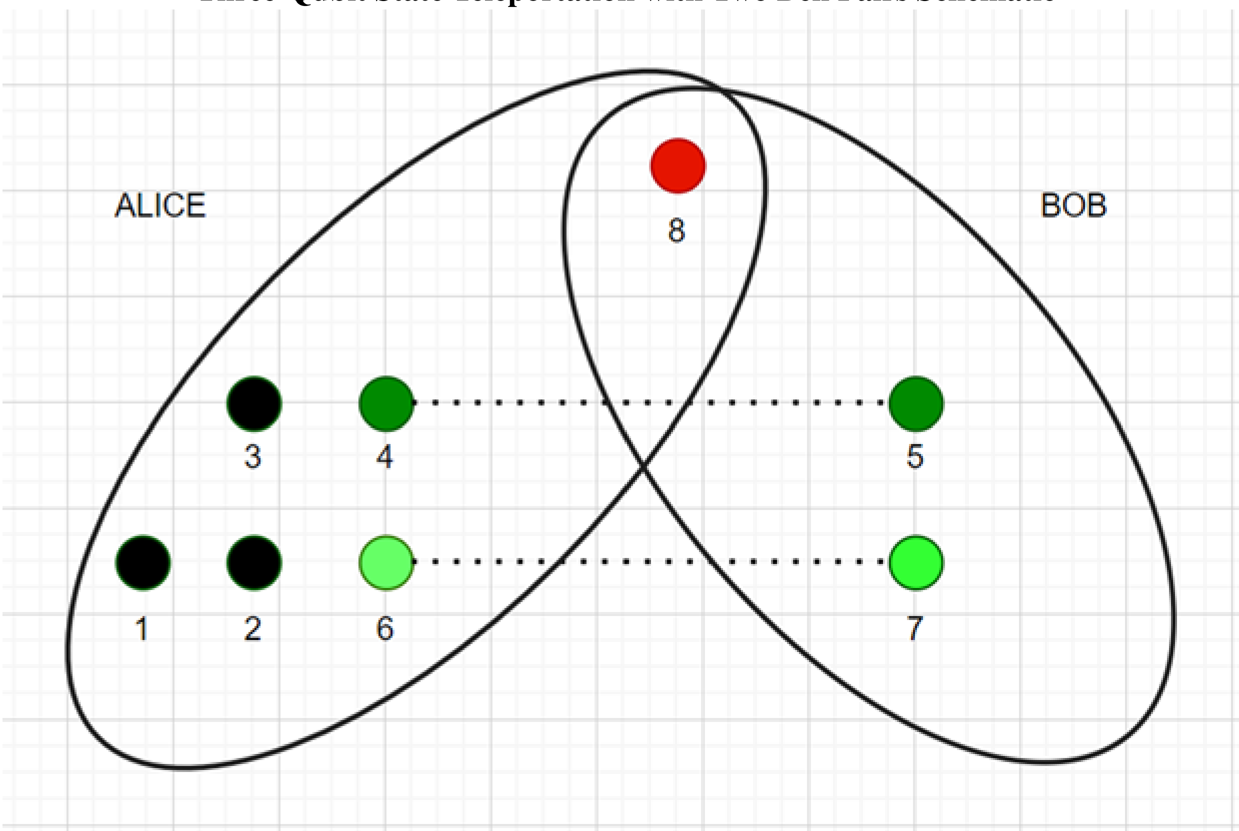


Figure 3.4 Schematic for the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios. Colored circles indicate qubits. The large ovals indicate which qubits are close to the two parties (Alice and Bob). The black qubits indicate the three-qubit state to be teleported from Alice to Bob. The green qubits denote the Bell pairs (different shades of green for different bell pairs). The red qubit represents the auxiliary qubit which is close to both Alice and Bob. After the teleportation is accomplished qubits 5, 7, and 8 will contain the three-qubit state initially held by 1, 2, and 3.

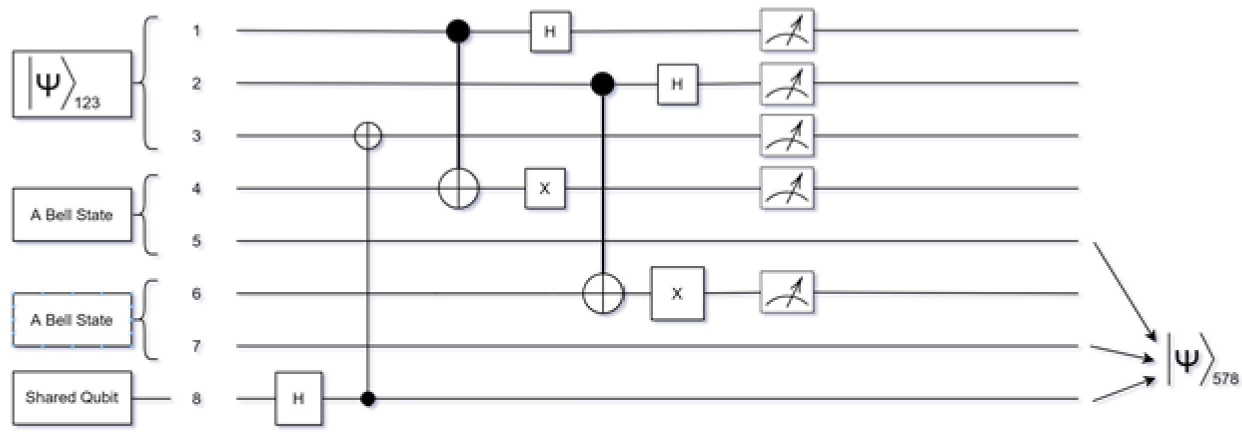


Figure 3.5 The circuit diagram for the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios.

3.2 Conclusion

The map of teleportation presented in Fig 3.1 shows predictive promise. Just as the periodic table of the elements and the table of nuclides aided scientist and students within the field of chemistry, a map of teleportation space could aid those interested in quantum teleportation topics. Patterns are revealed that show resources needed to teleport multiple qubits with varying teleportation protocols. Experimentalists could also see what minimum resources are needed to accomplish different teleportation protocols in the lab. The map also gives a bird's-eye view of the possibilities and extent of teleportation, and may lead to further progress in its study.

The current state of the map is not polished. This represents the first draft, and with more effort will evolve and become easier to read and become more compact. The table has the strength of showing trends for individual teleportation protocols. It has the weakness of containing no information concerning the particulars of each different protocols, including the quantum gates needed for each protocol.

A viable space for the teleportation of a three-qubit state under short-distance situations was discovered, in part by the consideration of its possible placement on the map of teleportation.

A quantum circuit was proposed, and checking the details of the math that the circuit describes gives initially promising results that the circuit is valid for the teleportation of a three-qubit state using two Bell pairs under short-distance scenarios. The discovery of this protocol indicates that by considering the five DOF of teleportation for particular protocols (qubits teleported, type of quantum resource, number of classical bits, number of quantum computer qubits, and number of entanglement resources), new spaces for teleportation can be discovered.

3.3 Further Research

Fig 3.1 gives a first attempt at creating a map of the possibilities of teleportation. Just as the periodic table changed over the course of 100 years as discoveries were made and appropriate changes were implemented, major adjustments to create a more pedagogical and useful map of teleportation could be developed. The form of Fig 3.1 is not very pedagogical or user friendly at present. Additionally, many existing quantum teleportation protocols have not been considered and placed on this map yet. Many more could be included, and as this is done, more patterns may emerge and more intuitive or appealing design schemes may become apparent.

The discovery of the three-qubit state teleportation with two Bell states under short-distance scenarios immediately motivates the question as to whether a four-qubit state could be teleported with three (or even with two) Bell pairs. From the map of teleportation it is obvious that a duplication of the two-qubit state teleportation protocol would result in the teleporting of two two-qubit states with two Bell pairs under short-distance scenarios. But is this different from teleporting a four-qubit state, rather then two two-qubit states? Further research is needed, and could be immediately fruitful.

Lastly, this map describes theoretically perfect protocols. No noise enters through the use of quantum gates and teleportation fidelity is 100%, save the protocols which themselves have a less then unitary probability of occurring (as in probabilistic teleportation and teleportation using a single classical bit). It could be a useful project to consider the real worldly cases where noise exists, and consider whether the map would change in these instances.

Appendix A

Teleportation with a GHZ State

Teleportation can be accomplished with a GHZ state. The unknown state of the particle to be teleported can be represented as

$$\big|\psi\big\rangle = a\big|\uparrow\big\rangle + b\big|\downarrow\big\rangle. \tag{A.1}$$

where a and b are complex numbers satisfying the relation $|a|^2 + |b|^2 = 1$.

A GHZ state is given by

$$\big|GHZ\big\rangle = \frac{1}{\sqrt{2}}(\big|\uparrow_A\big\rangle\big|\uparrow_A|\uparrow_B\big\rangle + \big|\downarrow_A\big\rangle\big|\downarrow_A|\downarrow_B\big\rangle). \tag{A.2}$$

The combined state of $|\phi\rangle$ and $|GHZ\rangle$ is thus

$$\begin{aligned}|\Gamma\rangle = |\psi\rangle \otimes |GHZ\rangle &= (\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes \frac{1}{\sqrt{2}}(\big|\uparrow_A\big\rangle\big|\uparrow_A|\uparrow_B\big\rangle + \big|\downarrow_A\big\rangle\big|\downarrow_A|\downarrow_B\big\rangle) \\ &= \frac{1}{\sqrt{2}}[\alpha(|\uparrow\uparrow_A\uparrow_A\uparrow_B\rangle + |\uparrow\downarrow_A\downarrow_A\downarrow_B\rangle) + \beta(|\downarrow\uparrow_A\uparrow_A\uparrow_B\rangle + |\downarrow\downarrow_A\downarrow_A\downarrow_B\rangle)]\end{aligned} \tag{A.3}$$

Next, a C-NOT gate is applied with the qubit to be teleported as the control and both pieces of the GHZ state in Alice's possession being targets. These come one after the other, and since the state of the qubit to be teleported doesn't change between steps, I can do both steps at the same time and get the same result as if I did them one after the other.

$$|\Gamma*\rangle = \frac{1}{\sqrt{2}}[\alpha(|\uparrow\uparrow_A\uparrow_A\uparrow_B\rangle + |\uparrow\downarrow_A\downarrow_A\downarrow_B\rangle) + \beta(|\downarrow\downarrow_A\downarrow_A\uparrow_B\rangle + |\downarrow\uparrow_A\uparrow_A\downarrow_B\rangle)] \tag{A.4}$$

Now the Hadamard gate is applied to the first qubit. It has the effect that

$$H : |0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{A.5}$$

and

$$H : |1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle). \tag{A.6}$$

I end up with

$$\begin{aligned} |\Gamma*\rangle = \frac{1}{2}[&\alpha(|\uparrow\uparrow_A\uparrow_A\uparrow_B\rangle + |\downarrow\uparrow_A\uparrow_A\uparrow_B\rangle + |\uparrow\downarrow_A\downarrow_A\downarrow_B\rangle + |\downarrow\downarrow_A\downarrow_A\downarrow_B\rangle) \\ &+\beta(|\uparrow\downarrow_A\downarrow_A\uparrow_B\rangle - |\downarrow\downarrow_A\downarrow_A\uparrow_B\rangle + |\uparrow\uparrow_A\uparrow_A\downarrow_B\rangle - |\downarrow\uparrow_A\uparrow_A\downarrow_B\rangle)] \end{aligned} \tag{A.7}$$

or

$$\begin{aligned} |\Gamma*\rangle = \frac{1}{2}[&|\uparrow\uparrow_A\uparrow_A\rangle(\alpha|\uparrow_B\rangle + \beta|\downarrow_B\rangle) + |\uparrow\downarrow_A\downarrow_A\rangle(\alpha|\downarrow_B\rangle + \beta|\uparrow_B\rangle) \\ &+|\downarrow\uparrow_A\uparrow_A\rangle(\alpha|\uparrow_B\rangle - \beta|\downarrow_B\rangle) + |\downarrow\downarrow_A\downarrow_A\rangle(\alpha|\downarrow_B\rangle - \beta|\uparrow_B\rangle). \end{aligned} \tag{A.8}$$

This equation is the equivalent of what Equation 2.8 is except for GHZ states.

From Equation A.8, I can construct the following table that shows what operations should be applied to the state in B to get back the original state $|\psi\rangle$.

GHZ Teleportation Transformations		
States in A	States in B	Operation in B
$\lvert \uparrow\uparrow_A\uparrow_A\rangle$	$\alpha\lvert \uparrow_B\rangle + \beta\lvert \downarrow_B\rangle$	None
$\lvert \uparrow\downarrow_A\downarrow_A\rangle$	$\alpha\lvert \downarrow_B\rangle + \beta\lvert \uparrow_B\rangle$	X gate
$\lvert \downarrow\uparrow_A\uparrow_A\rangle$	$\alpha\lvert \uparrow_B\rangle - \beta\lvert \downarrow_B\rangle$	Z gate
$\lvert \downarrow\downarrow_A\downarrow_A\rangle$	$\alpha\lvert \downarrow_B\rangle - \beta\lvert \uparrow_B\rangle$	Z and X gate

Table A.1 For GHZ teleportation, Alice measures three states. The result of this measurement is shown in the first column. In the second column is the corresponding state of the qubit in Bob's possession. In the third column, the appropriate Pauli matrix transformation(s) are named that Bob must apply to his qubit to get the qubit that Alice teleported to him.

Appendix B

Teleportation of a Three-Qubit State with Two Bell Pairs Under Short-Distance Scenarios

In this Appendix, a record of my work is shown that confirms that the circuit diagram in Fig. 3.5 accomplishes the teleportation of a three-qubit state with two Bell pairs under short-distance scenarios. I used a Mathematica notebook and Python code to do this. I have shown screenshots of all of my work below.

The Python code positively confirms that every expected combination of states exists within my data, confirming that the circuit diagram developed in Fig.3.5 actually accomplished the teleportation of a three-qubit state as desired.

Step by Step Work For the Teleportation of a 3QS with Two Bell Pair Under Short-Distance Sceneries

The initial 3-qubit state to be teleported is:

$|\Psi_{123}\rangle = a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle$

There are eight possible measurement outcomes of measuring a 3-qubit state. The specifics of the 3-qubit state determine the coefficients that come in front of each of the eight outcomes. The norm square of each coefficient gives the probability of measuring that outcome.

By doing the following operations, I expect 8*2*2*2 (or 64) results.

$|\Psi_{123}\rangle \otimes |01\rangle \otimes |01\rangle \otimes |0\rangle +$
$|\Psi_{123}\rangle \otimes |10\rangle \otimes |01\rangle \otimes |0\rangle +$
$|\Psi_{123}\rangle \otimes |01\rangle \otimes |10\rangle \otimes |0\rangle +$
$|\Psi_{123}\rangle \otimes |10\rangle \otimes |10\rangle \otimes |0\rangle +$
$|\Psi_{123}\rangle \otimes |01\rangle \otimes |01\rangle \otimes |1\rangle +$
$|\Psi_{123}\rangle \otimes |10\rangle \otimes |01\rangle \otimes |1\rangle +$
$|\Psi_{123}\rangle \otimes |01\rangle \otimes |10\rangle \otimes |1\rangle +$
$|\Psi_{123}\rangle \otimes |10\rangle \otimes |10\rangle \otimes |1\rangle$

This gives the state of the system after applying the first Hadamard gate. The state of the system is now:

$$
\begin{aligned}
|\phi_1\rangle = \frac{1}{2\sqrt{2}} (&a_{000}|00001010\rangle + a_{001}|00101010\rangle + a_{010}|01001010\rangle + a_{011}|01101010\rangle \\
&+ a_{100}|10001010\rangle + a_{101}|10101010\rangle + a_{110}|11001010\rangle + a_{111}|11101010\rangle \\
\\
&+ a_{000}|00010010\rangle + a_{001}|00110010\rangle + a_{010}|01010010\rangle + a_{011}|01110010\rangle \\
&+ a_{100}|10010010\rangle + a_{101}|10110010\rangle + a_{110}|11010010\rangle + a_{111}|11110010\rangle \\
\\
&+ a_{000}|00001100\rangle + a_{001}|00101100\rangle + a_{010}|01001100\rangle + a_{011}|01101100\rangle \\
&+ a_{100}|10001100\rangle + a_{101}|10101100\rangle + a_{110}|11001100\rangle + a_{111}|11101100\rangle \\
\\
&+ a_{000}|00010100\rangle + a_{001}|00110100\rangle + a_{010}|01010100\rangle + a_{011}|01110100\rangle \\
&+ a_{100}|10010100\rangle + a_{101}|10110100\rangle + a_{110}|11010100\rangle + a_{111}|11110100\rangle \\
\\
&+ a_{000}|00001011\rangle + a_{001}|00101011\rangle + a_{010}|01001011\rangle + a_{011}|01101011\rangle \\
&+ a_{100}|10001011\rangle + a_{101}|10101011\rangle + a_{110}|11001011\rangle + a_{111}|11101011\rangle \\
\\
&+ a_{000}|00010011\rangle + a_{001}|00110011\rangle + a_{010}|01010011\rangle + a_{011}|01110011\rangle \\
&+ a_{100}|10010011\rangle + a_{101}|10110011\rangle + a_{110}|11010011\rangle + a_{111}|11110011\rangle \\
\\
&+ a_{000}|00001101\rangle + a_{001}|00101101\rangle + a_{010}|01001101\rangle + a_{011}|01101101\rangle \\
&+ a_{100}|10001101\rangle + a_{101}|10101101\rangle + a_{110}|11001101\rangle + a_{111}|11101101\rangle \\
\\
&+ a_{000}|00010101\rangle + a_{001}|00110101\rangle + a_{010}|01010101\rangle + a_{011}|01110101\rangle \\
&+ a_{100}|10010101\rangle + a_{101}|10110101\rangle + a_{110}|11010101\rangle + a_{111}|11110101\rangle
\end{aligned}
$$

Okay, now I apply a CNOT gate with q8 as the control and q3 as the target qubits. Red numbers are those whose value changed.

$$|\phi_2\rangle = \frac{1}{2\sqrt{2}}\,(a_{000}|00001010\rangle + a_{001}|00101010\rangle + a_{010}|01001010\rangle + a_{011}|01101010\rangle$$
$$+a_{100}|10001010\rangle + a_{101}|10101010\rangle + a_{110}|11001010\rangle + a_{111}|11101010\rangle$$

$$a_{000}|00010010\rangle + a_{001}|00110010\rangle + a_{010}|01010010\rangle + a_{011}|01110010\rangle$$
$$+a_{100}|10010010\rangle + a_{101}|10110010\rangle + a_{110}|11010010\rangle + a_{111}|11110010\rangle$$

$$a_{000}|00001100\rangle + a_{001}|00101100\rangle + a_{010}|01001100\rangle + a_{011}|01101100\rangle$$
$$+a_{100}|10001100\rangle + a_{101}|10101100\rangle + a_{110}|11001100\rangle + a_{111}|11101100\rangle$$

$$a_{000}|00010100\rangle + a_{001}|00110100\rangle + a_{010}|01010100\rangle + a_{011}|01110100\rangle$$
$$+a_{100}|10010100\rangle + a_{101}|10110100\rangle + a_{110}|11010100\rangle + a_{111}|11110100\rangle$$

$$a_{000}|00101011\rangle + a_{001}|00001011\rangle + a_{010}|01101011\rangle + a_{011}|01001011\rangle$$
$$+a_{100}|10101011\rangle + a_{101}|10001011\rangle + a_{110}|11101011\rangle + a_{111}|11001011\rangle$$

$$a_{000}|00110011\rangle + a_{001}|00010011\rangle + a_{010}|01110011\rangle + a_{011}|01010011\rangle$$
$$+a_{100}|10110011\rangle + a_{101}|10010011\rangle + a_{110}|11110011\rangle + a_{111}|11010011\rangle$$

$$a_{000}|00101101\rangle + a_{001}|00001101\rangle + a_{010}|01101101\rangle + a_{011}|01001101\rangle$$
$$+a_{100}|10101101\rangle + a_{101}|10001101\rangle + a_{110}|11101101\rangle + a_{111}|11001101\rangle$$

$$a_{000}|00110101\rangle + a_{001}|00010101\rangle + a_{010}|01110101\rangle + a_{011}|01010101\rangle$$
$$+a_{100}|10110101\rangle + a_{101}|10010101\rangle + a_{110}|11110101\rangle + a_{111}|11010101\rangle$$

Now I apply a CNOT with q1 being the control and q4 being the target.

$$|\phi_3\rangle = \frac{1}{2\sqrt{2}}\,(a_{000}|00001010\rangle + a_{001}|00101010\rangle + a_{010}|01001010\rangle + a_{011}|01101010\rangle$$
$$+a_{100}|10011010\rangle + a_{101}|10111010\rangle + a_{110}|11011010\rangle + a_{111}|11111010\rangle$$

$$a_{000}|00010010\rangle + a_{001}|00110010\rangle + a_{010}|01010010\rangle + a_{011}|01110010\rangle$$
$$+a_{100}|10000010\rangle + a_{101}|10100010\rangle + a_{110}|11000010\rangle + a_{111}|11100010\rangle$$

$$a_{000}|00001100\rangle + a_{001}|00101100\rangle + a_{010}|01001100\rangle + a_{011}|01101100\rangle$$
$$+a_{100}|10011100\rangle + a_{101}|10111100\rangle + a_{110}|11011100\rangle + a_{111}|11111100\rangle$$

$$a_{000}|00010100\rangle + a_{001}|00110100\rangle + a_{010}|01010100\rangle + a_{011}|01110100\rangle$$
$$+a_{100}|10000100\rangle + a_{101}|10100100\rangle + a_{110}|11000100\rangle + a_{111}|11100100\rangle$$

$$a_{000}|00101011\rangle + a_{001}|00001011\rangle + a_{010}|01101011\rangle + a_{011}|01001011\rangle$$
$$+a_{100}|10111011\rangle + a_{101}|10011011\rangle + a_{110}|11111011\rangle + a_{111}|11011011\rangle$$

$$a_{000}|00110011\rangle + a_{001}|00010011\rangle + a_{010}|01110011\rangle + a_{011}|01010011\rangle$$
$$+a_{100}|10100011\rangle + a_{101}|10000011\rangle + a_{110}|11100011\rangle + a_{111}|11000011\rangle$$

$$a_{000}|00101101\rangle + a_{001}|00001101\rangle + a_{010}|01101101\rangle + a_{011}|01001101\rangle$$
$$+a_{100}|10111101\rangle + a_{101}|10011101\rangle + a_{110}|11111101\rangle + a_{111}|11011101\rangle$$

$$a_{000}|00110101\rangle + a_{001}|00010101\rangle + a_{010}|01110101\rangle + a_{011}|01010101\rangle$$
$$+a_{100}|10100101\rangle + a_{101}|10000101\rangle + a_{110}|11100101\rangle + a_{111}|11000101\rangle$$

Next I apply X-Gate to qubit 4

$$|\phi_4\rangle = \frac{1}{2\sqrt{2}}\ (a_{000}|00011010\rangle+a_{001}|00111010\rangle+a_{010}|01011010\rangle+a_{011}|01111010\rangle$$
$$+a_{100}|10001010\rangle+a_{101}|10101010\rangle+a_{110}|11001010\rangle+a_{111}|11101010\rangle$$

$$a_{000}|00000010\rangle+a_{001}|00100010\rangle+a_{010}|01000010\rangle+a_{011}|01100010\rangle$$
$$+a_{100}|10010010\rangle+a_{101}|10110010\rangle+a_{110}|11010010\rangle+a_{111}|11110010\rangle$$

$$a_{000}|00011100\rangle+a_{001}|00111100\rangle+a_{010}|01011100\rangle+a_{011}|01111100\rangle$$
$$+a_{100}|10001100\rangle+a_{101}|10101100\rangle+a_{110}|11001100\rangle+a_{111}|11101100\rangle$$

$$a_{000}|00000100\rangle+a_{001}|00100100\rangle+a_{010}|01000100\rangle+a_{011}|01100100\rangle$$
$$+a_{100}|10010100\rangle+a_{101}|10110100\rangle+a_{110}|11010100\rangle+a_{111}|11110100\rangle$$

$$a_{000}|00111011\rangle+a_{001}|00011011\rangle+a_{010}|01111011\rangle+a_{011}|01011011\rangle$$
$$+a_{100}|10101011\rangle+a_{101}|10001011\rangle+a_{110}|11101011\rangle+a_{111}|11001011\rangle$$

$$a_{000}|00100011\rangle+a_{001}|00000011\rangle+a_{010}|01100011\rangle+a_{011}|01000011\rangle$$
$$+a_{100}|10110011\rangle+a_{101}|10010011\rangle+a_{110}|11110011\rangle+a_{111}|11010011\rangle$$

$$a_{000}|00111101\rangle+a_{001}|00011101\rangle+a_{010}|01111101\rangle+a_{011}|01011101\rangle$$
$$+a_{100}|10101101\rangle+a_{101}|10001101\rangle+a_{110}|11101101\rangle+a_{111}|11001101\rangle$$

$$a_{000}|00100101\rangle+a_{001}|00000101\rangle+a_{010}|01100101\rangle+a_{011}|01000101\rangle$$
$$+a_{100}|10110101\rangle+a_{101}|10010101\rangle+a_{110}|11110101\rangle+a_{111}|11010101\rangle$$

Next, I apply the Hadamard gate to qubit 1.

$$|\phi_5\rangle = \frac{1}{4}\ (a_{000}|00011010\rangle+a_{000}|10011010\rangle+a_{001}|00111010\rangle+a_{001}|10111010\rangle+a_{010}|01011010\rangle+a_{010}|11011010\rangle+a_{011}|01111010\rangle+a_{011}|11111010\rangle$$
$$+a_{100}|00001010\rangle-a_{100}|10001010\rangle+a_{101}|00101010\rangle-a_{101}|10101010\rangle+a_{110}|01001010\rangle-a_{110}|11001010\rangle+a_{111}|01101010\rangle-a_{111}|11101010\rangle$$

$$+a_{000}|00000010\rangle+a_{000}|10000010\rangle+a_{001}|00100010\rangle+a_{001}|10100010\rangle+a_{010}|01000010\rangle+a_{010}|11000010\rangle+a_{011}|01100010\rangle+a_{011}|11100010\rangle$$
$$+a_{100}|00010010\rangle-a_{100}|10010010\rangle+a_{101}|00110010\rangle-a_{101}|10110010\rangle+a_{110}|01010010\rangle-a_{110}|11010010\rangle+a_{111}|01110010\rangle-a_{111}|11110010\rangle$$

$$+a_{000}|00011100\rangle+a_{000}|10011100\rangle+a_{001}|00111100\rangle+a_{001}|10111100\rangle+a_{010}|01011100\rangle+a_{010}|11011100\rangle+a_{011}|01111100\rangle+a_{011}|11111100\rangle$$
$$+a_{100}|00001100\rangle-a_{100}|10001100\rangle+a_{101}|00101100\rangle-a_{101}|10101100\rangle+a_{110}|01001100\rangle-a_{110}|11001100\rangle+a_{111}|01101100\rangle-a_{111}|11101100\rangle$$

$$+a_{000}|00000100\rangle+a_{000}|10000100\rangle+a_{001}|00100100\rangle+a_{001}|10100100\rangle+a_{010}|01000100\rangle+a_{010}|11000100\rangle+a_{011}|01100100\rangle+a_{011}|11100100\rangle$$
$$+a_{100}|00010100\rangle-a_{100}|10010100\rangle+a_{101}|00110100\rangle-a_{101}|10110100\rangle+a_{110}|01010100\rangle-a_{110}|11010100\rangle+a_{111}|01110100\rangle-a_{111}|11110100\rangle$$

$$+a_{000}|00111011\rangle+a_{000}|10111011\rangle+a_{001}|00011011\rangle+a_{001}|10011011\rangle+a_{010}|01111011\rangle+a_{010}|11111011\rangle+a_{011}|01011011\rangle+a_{011}|11011011\rangle$$
$$+a_{100}|00101011\rangle-a_{100}|10101011\rangle+a_{101}|00001011\rangle-a_{101}|10001011\rangle+a_{110}|01101011\rangle-a_{110}|11101011\rangle+a_{111}|01001011\rangle-a_{111}|11001011\rangle$$

$$+a_{000}|00100011\rangle+a_{000}|10100011\rangle+a_{001}|00000011\rangle+a_{001}|10000011\rangle+a_{010}|01100011\rangle+a_{010}|11100011\rangle+a_{011}|01000011\rangle+a_{011}|11000011\rangle$$
$$+a_{100}|00110011\rangle-a_{100}|10110011\rangle+a_{101}|00010011\rangle-a_{101}|10010011\rangle+a_{110}|01110011\rangle-a_{110}|11110011\rangle+a_{111}|01010011\rangle-a_{111}|11010011\rangle$$

$$+a_{000}|00111101\rangle+a_{000}|10111101\rangle+a_{001}|00011101\rangle+a_{001}|10011101\rangle+a_{010}|01111101\rangle+a_{010}|11111101\rangle+a_{011}|01011101\rangle+a_{011}|11011101\rangle$$
$$+a_{100}|00101101\rangle-a_{100}|10101101\rangle+a_{101}|00001101\rangle-a_{101}|10001101\rangle+a_{110}|01101101\rangle-a_{110}|11101101\rangle+a_{111}|01001101\rangle-a_{111}|11001101\rangle$$

$$+a_{000}|00100101\rangle+a_{000}|10100101\rangle+a_{001}|00000101\rangle+a_{001}|10000101\rangle+a_{010}|01100101\rangle+a_{010}|11100101\rangle+a_{011}|01000101\rangle+a_{011}|11000101\rangle$$
$$+a_{100}|00110101\rangle-a_{100}|10110101\rangle+a_{101}|00010101\rangle-a_{101}|10010101\rangle+a_{110}|01110101\rangle-a_{110}|11110101\rangle+a_{111}|01010101\rangle-a_{111}|11010101\rangle$$

Now I apply a CNOT gate with qubit 2 as the control and qubit 6 as the target

$$|\phi_6\rangle = \frac{1}{2\sqrt{2}} (a_{000}|00011010\rangle + a_{000}|10011010\rangle + a_{001}|00111010\rangle + a_{001}|10111010\rangle + a_{010}|01011110\rangle + a_{010}|11011110\rangle + a_{011}|01111110\rangle + a_{011}|11111110\rangle$$
$$+a_{100}|00001010\rangle - a_{100}|10001010\rangle + a_{101}|00101010\rangle - a_{101}|10101010\rangle + a_{110}|01001110\rangle - a_{110}|11001110\rangle + a_{111}|01101110\rangle - a_{111}|11101110\rangle$$

$$+a_{000}|00000010\rangle + a_{000}|10000010\rangle + a_{001}|00100010\rangle + a_{001}|10100010\rangle + a_{010}|01000110\rangle + a_{010}|11000110\rangle + a_{011}|01100110\rangle + a_{011}|11100110\rangle$$
$$+a_{100}|00010010\rangle - a_{100}|10010010\rangle + a_{101}|00110010\rangle - a_{101}|10110010\rangle + a_{110}|01010110\rangle - a_{110}|11010110\rangle + a_{111}|01110110\rangle - a_{111}|11110110\rangle$$

$$+a_{000}|00011100\rangle + a_{000}|10011100\rangle + a_{001}|00111100\rangle + a_{001}|10111100\rangle + a_{010}|01011000\rangle + a_{010}|11011000\rangle + a_{011}|01111000\rangle + a_{011}|11111000\rangle$$
$$+a_{100}|00001100\rangle - a_{100}|10001100\rangle + a_{101}|00101100\rangle - a_{101}|10101100\rangle + a_{110}|01001000\rangle - a_{110}|11001000\rangle + a_{111}|01101000\rangle - a_{111}|11101000\rangle$$

$$+a_{000}|00000100\rangle + a_{000}|10000100\rangle + a_{001}|00100100\rangle + a_{001}|10100100\rangle + a_{010}|01000000\rangle + a_{010}|11000000\rangle + a_{011}|01100000\rangle + a_{011}|11100000\rangle$$
$$+a_{100}|00010100\rangle - a_{100}|10010100\rangle + a_{101}|00110100\rangle - a_{101}|10110100\rangle + a_{110}|01010000\rangle - a_{110}|11010000\rangle + a_{111}|01110000\rangle - a_{111}|11110000\rangle$$

$$+a_{000}|00111011\rangle + a_{000}|10111011\rangle + a_{001}|00011011\rangle + a_{001}|10011011\rangle + a_{010}|01111111\rangle + a_{010}|11111111\rangle + a_{011}|01011111\rangle + a_{011}|11011111\rangle$$
$$+a_{100}|00101011\rangle - a_{100}|10101011\rangle + a_{101}|00001011\rangle - a_{101}|10001011\rangle + a_{110}|01101111\rangle - a_{110}|11101111\rangle + a_{111}|01001111\rangle - a_{111}|11001111\rangle$$

$$+a_{000}|00100011\rangle + a_{000}|10100011\rangle + a_{001}|00000011\rangle + a_{001}|10000011\rangle + a_{010}|01100111\rangle + a_{010}|11100111\rangle + a_{011}|01000111\rangle + a_{011}|11000111\rangle$$
$$+a_{100}|00110011\rangle - a_{100}|10110011\rangle + a_{101}|00010011\rangle - a_{101}|10010011\rangle + a_{110}|01110111\rangle - a_{110}|11110111\rangle + a_{111}|01010111\rangle - a_{111}|11010111\rangle$$

$$+a_{000}|00111101\rangle + a_{000}|10111101\rangle + a_{001}|00011101\rangle + a_{001}|10011101\rangle + a_{010}|01111001\rangle + a_{010}|11111001\rangle + a_{011}|01011001\rangle + a_{011}|11011001\rangle$$
$$+a_{100}|00101101\rangle - a_{100}|10101101\rangle + a_{101}|00001101\rangle - a_{101}|10001101\rangle + a_{110}|01101001\rangle - a_{110}|11101001\rangle + a_{111}|01001001\rangle - a_{111}|11001001\rangle$$

$$+a_{000}|00100101\rangle + a_{000}|10100101\rangle + a_{001}|00000101\rangle + a_{001}|10000101\rangle + a_{010}|01100001\rangle + a_{010}|11100001\rangle + a_{011}|01000001\rangle + a_{011}|11000001\rangle$$
$$+a_{100}|00110101\rangle - a_{100}|10110101\rangle + a_{101}|00010101\rangle - a_{101}|10010101\rangle + a_{110}|01110001\rangle - a_{110}|11110001\rangle + a_{111}|01010001\rangle - a_{111}|11010001\rangle$$

Now I apply an X gate to qubit 6

$$|\phi_7\rangle = \frac{1}{2\sqrt{2}} (a_{000}|00011110\rangle + a_{000}|10011110\rangle + a_{001}|00111110\rangle + a_{001}|10111110\rangle + a_{010}|01011010\rangle + a_{010}|11011010\rangle + a_{011}|01111010\rangle + a_{011}|11111010\rangle$$
$$+a_{100}|00001110\rangle - a_{100}|10001110\rangle + a_{101}|00101110\rangle - a_{101}|10101110\rangle + a_{110}|01001010\rangle - a_{110}|11001010\rangle + a_{111}|01101010\rangle - a_{111}|11101010\rangle$$

$$+a_{000}|00000110\rangle + a_{000}|10000110\rangle + a_{001}|00100110\rangle + a_{001}|10100110\rangle + a_{010}|01000010\rangle + a_{010}|11000010\rangle + a_{011}|01100010\rangle + a_{011}|11100010\rangle$$
$$+a_{100}|00010110\rangle - a_{100}|10010110\rangle + a_{101}|00110110\rangle - a_{101}|10110110\rangle + a_{110}|01010010\rangle - a_{110}|11010010\rangle + a_{111}|01110010\rangle - a_{111}|11110010\rangle$$

$$+a_{000}|00011000\rangle + a_{000}|10011000\rangle + a_{001}|00111000\rangle + a_{001}|10111000\rangle + a_{010}|01011100\rangle + a_{010}|11011100\rangle + a_{011}|01111100\rangle + a_{011}|11111100\rangle$$
$$+a_{100}|00001000\rangle - a_{100}|10001000\rangle + a_{101}|00101000\rangle - a_{101}|10101000\rangle + a_{110}|01001100\rangle - a_{110}|11001100\rangle + a_{111}|01101100\rangle - a_{111}|11101100\rangle$$

$$+a_{000}|00000000\rangle + a_{000}|10000000\rangle + a_{001}|00100000\rangle + a_{001}|10100000\rangle + a_{010}|01000100\rangle + a_{010}|11000100\rangle + a_{011}|01100100\rangle + a_{011}|11100100\rangle$$
$$+a_{100}|00010000\rangle - a_{100}|10010000\rangle + a_{101}|00110000\rangle - a_{101}|10110000\rangle + a_{110}|01010100\rangle - a_{110}|11010100\rangle + a_{111}|01110100\rangle - a_{111}|11110100\rangle$$

$$+a_{000}|00111111\rangle + a_{000}|10111111\rangle + a_{001}|00011111\rangle + a_{001}|10011111\rangle + a_{010}|01111011\rangle + a_{010}|11111011\rangle + a_{011}|01011011\rangle + a_{011}|11011011\rangle$$
$$+a_{100}|00101111\rangle - a_{100}|10101111\rangle + a_{101}|00001111\rangle - a_{101}|10001111\rangle + a_{110}|01101011\rangle - a_{110}|11101011\rangle + a_{111}|01001011\rangle - a_{111}|11001011\rangle$$

$$+a_{000}|00100111\rangle + a_{000}|10100111\rangle + a_{001}|00000111\rangle + a_{001}|10000111\rangle + a_{010}|01100011\rangle + a_{010}|11100011\rangle + a_{011}|01000011\rangle + a_{011}|11000011\rangle$$
$$+a_{100}|00110111\rangle - a_{100}|10110111\rangle + a_{101}|00010111\rangle - a_{101}|10010111\rangle + a_{110}|01110011\rangle - a_{110}|11110011\rangle + a_{111}|01010011\rangle - a_{111}|11010011\rangle$$

$$+a_{000}|00111001\rangle + a_{000}|10111001\rangle + a_{001}|00011001\rangle + a_{001}|10011001\rangle + a_{010}|01111101\rangle + a_{010}|11111101\rangle + a_{011}|01011101\rangle + a_{011}|11011101\rangle$$
$$+a_{100}|00101001\rangle - a_{100}|10101001\rangle + a_{101}|00001001\rangle - a_{101}|10001001\rangle + a_{110}|01101101\rangle - a_{110}|11101101\rangle + a_{111}|01001101\rangle - a_{111}|11001101\rangle$$

$$+a_{000}|00100001\rangle + a_{000}|10100001\rangle + a_{001}|00000001\rangle + a_{001}|10000001\rangle + a_{010}|01100101\rangle + a_{010}|11100101\rangle + a_{011}|01000101\rangle + a_{011}|11000101\rangle$$
$$+a_{100}|00110001\rangle - a_{100}|10110001\rangle + a_{101}|00010001\rangle - a_{101}|10010001\rangle + a_{110}|01110101\rangle - a_{110}|11110101\rangle + a_{111}|01010101\rangle - a_{111}|11010101\rangle$$

The last step is applying the Hadamard gate to q2. This is grueling labor I'm not excited to do, but it must be done.

$$|\phi_8\rangle = \frac{1}{\sqrt{32}} (a_{000}|00011110\rangle+a_{000}|01011110\rangle+a_{000}|10011110\rangle+a_{000}|11011110\rangle+a_{001}|00111110\rangle+a_{001}|01111110\rangle+a_{001}|10111110\rangle+a_{001}|11111110\rangle$$
$$+a_{010}|00011010\rangle-a_{010}|01011010\rangle+a_{010}|10011010\rangle-a_{010}|11011010\rangle+a_{011}|00111010\rangle-a_{011}|01111010\rangle+a_{011}|10111010\rangle-a_{011}|11111010\rangle$$
$$+a_{100}|00001110\rangle+a_{100}|01001110\rangle-a_{100}|10001110\rangle-a_{100}|11001110\rangle+a_{101}|00101110\rangle+a_{101}|01101110\rangle-a_{101}|10101110\rangle-a_{101}|11101110\rangle$$
$$+a_{110}|00001010\rangle-a_{110}|01001010\rangle-a_{110}|10001010\rangle+a_{110}|11001010\rangle+a_{111}|00101010\rangle-a_{111}|01101010\rangle-a_{111}|10101010\rangle+a_{111}|11101010\rangle$$

$$+a_{000}|00000110\rangle+a_{000}|01000110\rangle+a_{000}|10000110\rangle+a_{000}|11000110\rangle+a_{001}|00100110\rangle+a_{001}|01100110\rangle+a_{001}|10100110\rangle+a_{001}|11100110\rangle$$
$$+a_{010}|00000010\rangle-a_{010}|01000010\rangle+a_{010}|10000010\rangle-a_{010}|11000010\rangle+a_{011}|00100010\rangle-a_{011}|01100010\rangle+a_{011}|10100010\rangle-a_{011}|11100010\rangle$$
$$+a_{100}|00010110\rangle+a_{100}|01010110\rangle-a_{100}|10010110\rangle-a_{100}|11010110\rangle+a_{101}|00110110\rangle+a_{101}|01110110\rangle-a_{101}|10110110\rangle-a_{101}|11110110\rangle$$
$$+a_{110}|00010010\rangle-a_{110}|01010010\rangle-a_{110}|10010010\rangle+a_{110}|11010010\rangle+a_{111}|00110010\rangle-a_{111}|01110010\rangle-a_{111}|10110010\rangle+a_{111}|11110010\rangle$$

$$+a_{000}|00011000\rangle+a_{000}|01011000\rangle+a_{000}|10011000\rangle+a_{000}|11011000\rangle+a_{001}|00111000\rangle+a_{001}|01111000\rangle+a_{001}|10111000\rangle+a_{001}|11111000\rangle$$
$$+a_{010}|00011100\rangle-a_{010}|01011100\rangle+a_{010}|10011100\rangle-a_{010}|11011100\rangle+a_{011}|00111100\rangle-a_{011}|01111100\rangle+a_{011}|10111100\rangle-a_{011}|11111100\rangle$$
$$+a_{100}|00001000\rangle+a_{100}|01001000\rangle-a_{100}|10001000\rangle-a_{100}|11001000\rangle+a_{101}|00101000\rangle+a_{101}|01101000\rangle-a_{101}|10101000\rangle-a_{101}|11101000\rangle$$
$$+a_{110}|00001100\rangle-a_{110}|01001100\rangle-a_{110}|10001100\rangle+a_{110}|11001100\rangle+a_{111}|00101100\rangle-a_{111}|01101100\rangle-a_{111}|10101100\rangle+a_{111}|11101100\rangle$$

$$+a_{000}|00000000\rangle+a_{000}|01000000\rangle+a_{000}|10000000\rangle+a_{000}|11000000\rangle+a_{001}|00100000\rangle+a_{001}|01100000\rangle+a_{001}|10100000\rangle+a_{001}|11100000\rangle$$
$$+a_{010}|00000100\rangle-a_{010}|01000100\rangle+a_{010}|10000100\rangle-a_{010}|11000100\rangle+a_{011}|00100100\rangle-a_{011}|01100100\rangle+a_{011}|10100100\rangle-a_{011}|11100100\rangle$$
$$+a_{100}|00010000\rangle+a_{100}|01010000\rangle-a_{100}|10010000\rangle-a_{100}|11010000\rangle+a_{101}|00110000\rangle+a_{101}|01110000\rangle-a_{101}|10110000\rangle-a_{101}|11110000\rangle$$
$$+a_{110}|00010100\rangle-a_{110}|01010100\rangle-a_{110}|10010100\rangle+a_{110}|11010100\rangle+a_{111}|00110100\rangle-a_{111}|01110100\rangle-a_{111}|10110100\rangle+a_{111}|11110100\rangle$$

$$+a_{000}|00111111\rangle+a_{000}|01111111\rangle+a_{000}|10111111\rangle+a_{000}|11111111\rangle+a_{001}|00011111\rangle+a_{001}|01011111\rangle+a_{001}|10011111\rangle+a_{001}|11011111\rangle$$
$$+a_{010}|00111011\rangle-a_{010}|01111011\rangle+a_{010}|10111011\rangle-a_{010}|11111011\rangle+a_{011}|00011011\rangle-a_{011}|01011011\rangle+a_{011}|10011011\rangle-a_{011}|11011011\rangle$$
$$+a_{100}|00101111\rangle+a_{100}|01101111\rangle-a_{100}|10101111\rangle-a_{100}|11101111\rangle+a_{101}|00001111\rangle+a_{101}|01001111\rangle-a_{101}|10001111\rangle-a_{101}|11001111\rangle$$
$$+a_{110}|00101011\rangle-a_{110}|01101011\rangle-a_{110}|10101011\rangle+a_{110}|11101011\rangle+a_{111}|00001011\rangle-a_{111}|01001011\rangle-a_{111}|10001011\rangle+a_{111}|11001011\rangle$$

$$+a_{000}|00100111\rangle+a_{000}|01100111\rangle+a_{000}|10100111\rangle+a_{000}|11100111\rangle+a_{001}|00000111\rangle+a_{001}|01000111\rangle+a_{001}|10000111\rangle+a_{001}|11000111\rangle$$
$$+a_{010}|00100011\rangle-a_{010}|01100011\rangle+a_{010}|10100011\rangle-a_{010}|11100011\rangle+a_{011}|00000011\rangle-a_{011}|01000011\rangle+a_{011}|10000011\rangle-a_{011}|11000011\rangle$$
$$+a_{100}|00110111\rangle+a_{100}|01110111\rangle-a_{100}|10110111\rangle-a_{100}|11110111\rangle+a_{101}|00010111\rangle+a_{101}|01010111\rangle-a_{101}|10010111\rangle-a_{101}|11010111\rangle$$
$$+a_{110}|00110011\rangle-a_{110}|01110011\rangle-a_{110}|10110011\rangle+a_{110}|11110011\rangle+a_{111}|00010011\rangle-a_{111}|01010011\rangle-a_{111}|10010011\rangle+a_{111}|11010011\rangle$$

$$+a_{000}|00111001\rangle+a_{000}|01111001\rangle+a_{000}|10111001\rangle+a_{000}|11111001\rangle+a_{001}|00011001\rangle+a_{001}|01011001\rangle+a_{001}|10011001\rangle+a_{001}|11011001\rangle$$
$$+a_{010}|00111101\rangle-a_{010}|01111101\rangle+a_{010}|10111101\rangle-a_{010}|11111101\rangle+a_{011}|00011101\rangle-a_{011}|01011101\rangle+a_{011}|10011101\rangle-a_{011}|11011101\rangle$$
$$+a_{100}|00101001\rangle+a_{100}|01101001\rangle-a_{100}|10101001\rangle-a_{100}|11101001\rangle+a_{101}|00001001\rangle+a_{101}|01001001\rangle-a_{101}|10001001\rangle-a_{101}|11001001\rangle$$
$$+a_{110}|00101101\rangle-a_{110}|01101101\rangle-a_{110}|10101101\rangle+a_{110}|11101101\rangle+a_{111}|00001101\rangle-a_{111}|01001101\rangle-a_{111}|10001101\rangle+a_{111}|11001101\rangle$$

$$+a_{000}|00100001\rangle+a_{000}|01100001\rangle+a_{000}|10100001\rangle+a_{000}|11100001\rangle+a_{001}|00000001\rangle+a_{001}|01000001\rangle+a_{001}|10000001\rangle+a_{001}|11000001\rangle$$
$$+a_{010}|00100101\rangle-a_{010}|01100101\rangle+a_{010}|10100101\rangle-a_{010}|11100101\rangle+a_{011}|00000101\rangle-a_{011}|01000101\rangle+a_{011}|10000101\rangle-a_{011}|11000101\rangle$$
$$+a_{100}|00110001\rangle+a_{100}|01110001\rangle-a_{100}|10110001\rangle-a_{100}|11110001\rangle+a_{101}|00010001\rangle+a_{101}|01010001\rangle-a_{101}|10010001\rangle-a_{101}|11010001\rangle$$
$$+a_{110}|00110101\rangle-a_{110}|01110101\rangle-a_{110}|10110101\rangle+a_{110}|11110101\rangle+a_{111}|00010101\rangle-a_{111}|01010101\rangle-a_{111}|10010101\rangle+a_{111}|11010101\rangle$$

If I examine the quantum circuit diagram, I can see that qubits 5, 7, and 8 are the ones that should be separable. I don't like the gap between 5 and 7, so I am going to switch the positions of qubits 5 and 6 now for my own clarity. The state remains $|\phi_8\rangle$.

$$|\phi_8\rangle = (1/(\sqrt{32}))\ (a_{000}|00011110\rangle+a_{000}|01011110\rangle+a_{000}|10011110\rangle+a_{000}|11011110\rangle+a_{001}|00111110\rangle+a_{001}|01111110\rangle+a_{001}|10111110\rangle+a_{001}|11111110\rangle$$
$$+a_{010}|00010110\rangle-a_{010}|01010110\rangle+a_{010}|10010110\rangle-a_{010}|11010110\rangle+a_{011}|00110110\rangle-a_{011}|01110110\rangle+a_{011}|10110110\rangle-a_{011}|11110110\rangle$$
$$+a_{100}|00001110\rangle+a_{100}|01001110\rangle-a_{100}|10001110\rangle-a_{100}|11001110\rangle+a_{101}|00101110\rangle+a_{101}|01101110\rangle-a_{101}|10101110\rangle-a_{101}|11101110\rangle$$
$$+a_{110}|00000110\rangle-a_{110}|01000110\rangle-a_{110}|10000110\rangle+a_{110}|11000110\rangle+a_{111}|00100110\rangle-a_{111}|01100110\rangle-a_{111}|10100110\rangle+a_{111}|11100110\rangle$$

$$+a_{000}|00001010\rangle+a_{000}|01001010\rangle+a_{000}|10001010\rangle+a_{000}|11001010\rangle+a_{001}|00101010\rangle+a_{001}|01101010\rangle+a_{001}|10101010\rangle+a_{001}|11101010\rangle$$
$$+a_{010}|00000010\rangle-a_{010}|01000010\rangle+a_{010}|10000010\rangle-a_{010}|11000010\rangle+a_{011}|00100010\rangle-a_{011}|01100010\rangle+a_{011}|10100010\rangle-a_{011}|11100010\rangle$$
$$+a_{100}|00011010\rangle+a_{100}|01011010\rangle-a_{100}|10011010\rangle-a_{100}|11011010\rangle+a_{101}|00111010\rangle+a_{101}|01111010\rangle-a_{101}|10111010\rangle-a_{101}|11111010\rangle$$
$$+a_{110}|00010010\rangle-a_{110}|01010010\rangle-a_{110}|10010010\rangle+a_{110}|11010010\rangle+a_{111}|00110010\rangle-a_{111}|01110010\rangle-a_{111}|10110010\rangle+a_{111}|11110010\rangle$$

$$+a_{000}|00010100\rangle+a_{000}|01010100\rangle+a_{000}|10010100\rangle+a_{000}|11010100\rangle+a_{001}|00110100\rangle+a_{001}|01110100\rangle+a_{001}|10110100\rangle+a_{001}|11110100\rangle$$
$$+a_{010}|00011100\rangle-a_{010}|01011100\rangle+a_{010}|10011100\rangle-a_{010}|11011100\rangle+a_{011}|00111100\rangle-a_{011}|01111100\rangle+a_{011}|10111100\rangle-a_{011}|11111100\rangle$$
$$+a_{100}|00000100\rangle+a_{100}|01000100\rangle-a_{100}|10000100\rangle-a_{100}|11000100\rangle+a_{101}|00100100\rangle+a_{101}|01100100\rangle-a_{101}|10100100\rangle-a_{101}|11100100\rangle$$
$$+a_{110}|00001100\rangle-a_{110}|01001100\rangle-a_{110}|10001100\rangle+a_{110}|11001100\rangle+a_{111}|00101100\rangle-a_{111}|01101100\rangle-a_{111}|10101100\rangle+a_{111}|11101100\rangle$$

$$+a_{000}|00000000\rangle+a_{000}|01000000\rangle+a_{000}|10000000\rangle+a_{000}|11000000\rangle+a_{001}|00100000\rangle+a_{001}|01100000\rangle+a_{001}|10100000\rangle+a_{001}|11100000\rangle$$
$$+a_{010}|00001000\rangle-a_{010}|01001000\rangle+a_{010}|10001000\rangle-a_{010}|11001000\rangle+a_{011}|00101000\rangle-a_{011}|01101000\rangle+a_{011}|10101000\rangle-a_{011}|11101000\rangle$$
$$+a_{100}|00010000\rangle+a_{100}|01010000\rangle-a_{100}|10010000\rangle-a_{100}|11010000\rangle+a_{101}|00110000\rangle+a_{101}|01110000\rangle-a_{101}|10110000\rangle-a_{101}|11110000\rangle$$
$$+a_{110}|00011000\rangle-a_{110}|01011000\rangle-a_{110}|10011000\rangle+a_{110}|11011000\rangle+a_{111}|00111000\rangle-a_{111}|01111000\rangle-a_{111}|10111000\rangle+a_{111}|11111000\rangle$$

$$+a_{000}|00111111\rangle+a_{000}|01111111\rangle+a_{000}|10111111\rangle+a_{000}|11111111\rangle+a_{001}|00011111\rangle+a_{001}|01011111\rangle+a_{001}|10011111\rangle+a_{001}|11011111\rangle$$
$$+a_{010}|00110111\rangle-a_{010}|01110111\rangle+a_{010}|10110111\rangle-a_{010}|11110111\rangle+a_{011}|00010111\rangle-a_{011}|01010111\rangle+a_{011}|10010111\rangle-a_{011}|11010111\rangle$$
$$+a_{100}|00101111\rangle+a_{100}|01101111\rangle-a_{100}|10101111\rangle-a_{100}|11101111\rangle+a_{101}|00001111\rangle+a_{101}|01001111\rangle-a_{101}|10001111\rangle-a_{101}|11001111\rangle$$
$$+a_{110}|00100111\rangle-a_{110}|01100111\rangle-a_{110}|10100111\rangle+a_{110}|11100111\rangle+a_{111}|00000111\rangle-a_{111}|01000111\rangle-a_{111}|10000111\rangle+a_{111}|11000111\rangle$$

$$+a_{000}|00101011\rangle+a_{000}|01101011\rangle+a_{000}|10101011\rangle+a_{000}|11101011\rangle+a_{001}|00001011\rangle+a_{001}|01001011\rangle+a_{001}|10001011\rangle+a_{001}|11001011\rangle$$
$$+a_{010}|00100011\rangle-a_{010}|01100011\rangle+a_{010}|10100011\rangle-a_{010}|11100011\rangle+a_{011}|00000011\rangle-a_{011}|01000011\rangle+a_{011}|10000011\rangle-a_{011}|11000011\rangle$$
$$+a_{100}|00111011\rangle+a_{100}|01111011\rangle-a_{100}|10111011\rangle-a_{100}|11111011\rangle+a_{101}|00011011\rangle+a_{101}|01011011\rangle-a_{101}|10011011\rangle-a_{101}|11011011\rangle$$
$$+a_{110}|00110011\rangle-a_{110}|01110011\rangle-a_{110}|10110011\rangle+a_{110}|11110011\rangle+a_{111}|00010011\rangle-a_{111}|01010011\rangle-a_{111}|10010011\rangle+a_{111}|11010011\rangle$$

$$+a_{000}|00110101\rangle+a_{000}|01110101\rangle+a_{000}|10110101\rangle+a_{000}|11110101\rangle+a_{001}|00010101\rangle+a_{001}|01010101\rangle+a_{001}|10010101\rangle+a_{001}|11010101\rangle$$
$$+a_{010}|00111101\rangle-a_{010}|01111101\rangle+a_{010}|10111101\rangle-a_{010}|11111101\rangle+a_{011}|00011101\rangle-a_{011}|01011101\rangle+a_{011}|10011101\rangle-a_{011}|11011101\rangle$$
$$+a_{100}|00100101\rangle+a_{100}|01100101\rangle-a_{100}|10100101\rangle-a_{100}|11100101\rangle+a_{101}|00000101\rangle+a_{101}|01000101\rangle-a_{101}|10000101\rangle-a_{101}|11000101\rangle$$
$$+a_{110}|00101101\rangle-a_{110}|01101101\rangle-a_{110}|10101101\rangle+a_{110}|11101101\rangle+a_{111}|00001101\rangle-a_{111}|01001101\rangle-a_{111}|10001101\rangle+a_{111}|11001101\rangle$$

$$+a_{000}|00100001\rangle+a_{000}|01100001\rangle+a_{000}|10100001\rangle+a_{000}|11100001\rangle+a_{001}|00000001\rangle+a_{001}|01000001\rangle+a_{001}|10000001\rangle+a_{001}|11000001\rangle$$
$$+a_{010}|00101001\rangle-a_{010}|01101001\rangle+a_{010}|10101001\rangle-a_{010}|11101001\rangle+a_{011}|00001001\rangle-a_{011}|01001001\rangle+a_{011}|10001001\rangle-a_{011}|11001001\rangle$$
$$+a_{100}|00110001\rangle+a_{100}|01110001\rangle-a_{100}|10110001\rangle-a_{100}|11110001\rangle+a_{101}|00010001\rangle+a_{101}|01010001\rangle-a_{101}|10010001\rangle-a_{101}|11010001\rangle$$
$$+a_{110}|00111001\rangle-a_{110}|01111001\rangle-a_{110}|10111001\rangle+a_{110}|11111001\rangle+a_{111}|00011001\rangle-a_{111}|01011001\rangle-a_{111}|10011001\rangle+a_{111}|11011001\rangle)$$

This is the final state after the circuit for the teleportation of a 3-qubit state with two Bell pairs under short-distance sceneries terminates.

Now I will check a few values.

If this worked, I should have 8 of each of the combinations of 5 digits with 0 or 1 in them as the first 5 entries in each 8 qubit state above. Each of these eight copies should have a unique combination of a 3-qubit state attached to the end.

That is, there should be 32 different measurement outcomes, which would be in line with the $\frac{1}{\sqrt{32}}$ I have out front, which when squared would give each a 1/32 probability of occurring.

First, I will check if there are eight $|00000\rangle$s, and that each is followed by a unique combination of 3 qubits ($|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$).

I find with control-f all the values that start with 5 0's.

$a_{110}|00000110\rangle + a_{010}|00000010\rangle + a_{100}|00000100\rangle + a_{000}|00000000\rangle + a_{111}|00000111\rangle + a_{011}|00000011\rangle + a_{101}|00000101\rangle + a_{001}|00000001\rangle$

This can be factored as

$|00000\rangle \otimes (a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle)$

This is exactly what I hoped to find!!

Next, I will make a code to check the rest. Daniel Buchanan assisted in creating a code in Python to check that within the list of numbers there is each unique combination of the 256 ways that 8 binary numbers can be filled with 0s and 1s. That code follows.

```
/*
 * Courtesy of Daniel Buchanan with guidance from Aidan Gillam.
 * 1 Feb 2022
 */

package QS3Parser;

import java.io.IOException;
import java.nio.file.Files;
import java.nio.file.Paths;
import java.util.*;

public class QubitParser {
    private List<String> qubitsList;
    private final Map<String, Set<String>> combos;

    public QubitParser(String qubitsPath) {
        this.combos = new HashMap<>();

        try {
            String content = new String (Files.readAllBytes(Paths.get(qubitsPath)));
            this.qubitsList = (Arrays.asList(content.split(" ")));
        }
        catch (IOException e) { e.printStackTrace(); }
    }

    /**
     * The purpose is to verify if the teleportation scheme is valid.
     * If so, then we would expect to see 32 possible outcomes of equal
     * probability. The measurement outcome is defined by the first 5 qubits.
     * For each outcome, we would expect to see 8 unique combinations of
     * the remaining 3 qubits.
     * @return valid: validity of the teleportation scheme
     */
```

```
    public boolean verifyValidity() {
        Set<String> qubitsSet = new HashSet<>(qubitsList);
        if (qubitsSet.size() != qubitsList.size()) return false;
        qubitsSet.clear();

        for (String s : qubitsList) {
            if (s.length() != 8) return false;
            String substr = s.substring(0, 5);
            if (combos.containsKey(substr)) combos.get(substr).add(s.substring(5));
            else {
                HashSet<String> set = new HashSet<>();
                set.add(s.substring(5));
                combos.put(substr, set);
            }
        }

        for (Map.Entry<String, Set<String>> entry : combos.entrySet()) {
            if (entry.getValue().size() != 8) return false;
        }

        return true;
    }

    /**
     * The purpose is to verify that no duplicates exist in the list of qubits.
     * As a Set in Java cannot contain duplicates, the contents of the initial list
     * is copied to a Set, and if the sizes of the List and Set differ, there was a
     * duplicate somewhere in the initial list of qubits. The qubits in the Set are
     * then removed from the List, and the remaining items are the qubits that had
     * a duplicate in the list.
     */
    public void checkDuplicateValues() {
        Set<String> qubitsSet = new HashSet<>(qubitsList);
        List<String> listCopy = new ArrayList<>(qubitsList);

        for (String s: qubitsSet) listCopy.remove(s);

        System.out.println("Duplicate values:");
        for (String s : listCopy) System.out.println(qubitsList.indexOf(s) + ": " + s);
        System.out.println();
    }

    /**
     * The purpose is to see which values, if any, are missing from the initial list.
     * This is based on the hypothesis that there are 256 unique combinations of 0s and 1s.
     * This creates every single combination of 0s and 1s, and if a combination is not
     * present in the list, then that qubit is missing from the initial list.
     */
    public void checkMissingValues() {
        final int POW = 8;
        List<String> missing = new ArrayList<>();
        for (int i = 0; i < Math.pow(2, POW); ++i) {
            StringBuilder s = new StringBuilder(Integer.toBinaryString(i));
            for (int j = s.length(); j < POW; ++j) s.insert(0, '0');

            if (!qubitsList.contains(s.toString())) missing.add(s.toString());
        }

        System.out.println("Missing values:");
        for (String s : missing) System.out.println(s);
        System.out.println();
    }
}
```

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Index